A group of quadrature formulae is presented applicable to both non-singular functions and functions with end-point singularities, generalizing the classical end-point corrected trapezoidal quadrature rules. We present an algorithm for the construction of very high-order end-point corrected trapezoidal rules, taking advantage of functional information outside the interval of integration. The scheme applies not only to non-singular functions, but also for a wide class of functions with monotonic singularities. Numerical experiments are presented demonstrating the practical usefulness of the new class of quadratures. Tables of quadrature weights are included for singularities of the form $s(x) = log(|x|)$, $s(x) = |x|^{\lambda}$ for a variety of values of $\lambda$.

# High-Order Corrected Trapezoidal Rules for Singular Functions

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# 1 Introduction

The trapezoidal rule is known to be an easy and numerically stable means for numerical integration. If a function is periodic and analytic on the interval of integration, the trapezoidal rule converges exponentially fast (see, for example, [7]). However, for non-periodic functions the trapezoidal rule is second order convergent, and end-point corrections are often used to improve the convergence rate. A standard end-point corrected trapezoidal rule is fourth order convergent, and is given by the formula

$$\int_a^b f(x)dx = \sum_{i=1}^{n-2} f(x_i) + \frac{f(x_0)+f(x_{n-1})}{2} + \frac{h}{24}(-f(x_{-1})+f(x_1)+f(x_{n-2})-f(x_n)), \tag{1}$$

where, $h = (b-a)/(n-1)$ and $x_i = a + ih$ for $i = 0, 1, 2, ..., n-1$ (see, for example [1]).

More recently, the Euler-Maclaurin formula is used in [4] to obtain a high-order end-point corrected trapezoidal rule of the form

$$T_\alpha^n(f) = \sum_{i=1}^{n-2} f(x_i) + \frac{f(x_0)+f(x_{n-1})}{2} + h\sum_{j=1}^{m} \alpha_j(f(x_{n-j}) - f(x_j)), \tag{2}$$

where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_m)$ are coefficients such that

$$| T_\alpha^n(f) - \int_a^b f(x)dx |< \frac{c}{n^m}. \tag{3}$$

for some $c > 0$.

The scheme of [4] provides satisfactory quadratures upto order 12; for higher orders, the coefficients $\alpha$ grow rapidly, rendering the scheme useless. In this paper we develop a different class of end-point corrected trapezoidal rules, whereby the growth of correction weights is suppressed, enabling the construction of end-point corrected trapezoidal rules of arbitrarily high order for non-singular functions.

In [5], end-point corrected quadrature formulae are developed to approximate definite integrals of singular functions $f : [a, b] \to R^1$ of the form

$$f(x) = \phi(x)s(x) + \psi(x), \tag{4}$$

and

$$f(x) = \phi(x)s(x), \tag{5}$$

where $a \leq 0 \leq b$, $\phi(x), \psi(x) \in c^k[a, b]$, and $s(x) \in c[a, b]$ is an integrable function with a singularity at 0. The procedure developed in [5] provides satisfactory quadratures only upto order 4; for higher orders, the quadrature weights grow rapidly, also rendering the scheme useless. In this paper we construct a different class of end-point corrected trapezoidal rules, whereby the growth of quadrature weights is partially suppressed for functions of the form (4), obtaining useful quadratures of order upto 12; and completely suppressed for functions of the form (5), providing quadratures of arbitrarily high order. Moreover, we obviate the programming inconvenience associated with the procedure developed in [5], which requires that functional information be tabulated on a grid finer than that required for the uncorrected trapezoidal rule.

**Remark 1.1** The approach of this paper is somewhat related to that of [9]. However, [9] constructs quadratures in higher dimensions, and these quadratures are of relatively low order. In this paper, we construct one-dimesional rules of very high order. Furthermore, most rules of this paper are "standard" in the sense that the correction coefficients do not depend on the number of nodes in the trapezoidal rule being corrected, or on the sampling interval.

## 2 Mathematical Preliminaries

In this section we summarize some well-known results to be used in the reminder of the paper. Lemmas 2.1, 2.2 and 2.3 can be found, for example, in [1].

**Definition 2.1** *Suppose that $a, b$ are a pair of real numbers such that $a < b$, and that $n \geq 2$ is an integer. For a function $f : [a, b] \to R^1$, we define the n-point trapezoidal rule $T_n(f)$ by the formula*

$$T_n(f) = h(\sum_{i=0}^{n-1} f(a+ih) - (\frac{f(a)+f(b)}{2})), \tag{6}$$

*with*

$$h = (b-a)/(n-1). \tag{7}$$

The following lemma provides an error estimate for the approximation to the integral given by the trapezoidal rule. Along with Lemma 2.2, it can be found, for example in [1].

**Lemma 2.1** *(Euler-Maclaurin formula) Suppose that $a, b$ are a pair of real numbers such that $a < b$, and that $m \geq 1$ is an integer. Further, let $B_k$ denote the Bernoulli numbers*

$$B_2 = \frac{1}{6}, B_4 = \frac{-1}{30}, B_6 = \frac{1}{42}, \ldots, . \tag{8}$$

*If $f \in c^{2m+2}[a, b]$ (i.e., $f$ has $2m+2$ continuous derivatives on $[a, b]$), then there exists a real number $\xi$, with $a < \xi < b$, such that*

$$\int_a^b f(x)dx = T_n(f) + \sum_{l=1}^{m} \frac{h^{2l} B_{2l}}{(2l)!}(f^{(2l-1)}(b) - f^{(2l-1)}(a)) - \frac{h^{2m+2} B_{2m+2}}{(2m+2)!} f^{2m+2}(\xi). \tag{9}$$

The following well-known lemma provides an error estimate for Lagrange interpolation.

**Lemma 2.2** *(Lagrange interpolation formula) Suppose that $a, b$ are a pair of real numbers such that $a < b$, $m \geq 3$ be an odd integer, and $f \in c^m[a, b]$. Furthermore, let $h$ be defined in (7), and $f$ be tabulated at equispaced points, $x_k = \frac{b-a}{2} + kh$. Then for any real number $p$ there exists a real number $\xi$, $-mh < \xi < mh$, such that*

$$f(x_0 + ph) = \sum_{k=\frac{-(m-1)}{2}}^{\frac{(m-1)}{2}} A_k^m(p) f(x_k) + R_{m-1}, \tag{10}$$

*with*

$$A_k^m(p) = \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!(p-k)} \prod_{t=0}^{m-1} (p + \frac{m-1}{2} - t), \tag{11}$$

*and*

$$R_{m-1} = \frac{1}{m!} \prod_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} (p-k) h^m f^{(m)}(\xi). \tag{12}$$

**Lemma 2.3** *If $f : [a, b] \to R^1$ is a function satisfying the conditions of Lemma 2.2, and the coefficients $D_{i,k}^m$ are given by the formula*

$$D_{i,k}^m = \frac{\partial^{(2i-1)}}{\partial p^{(2i-1)}} (A_k^m(p)) \mid^{p=0}, \tag{13}$$

*then*

$$f^{(2i-1)}(x_0) = \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{D_{i,k}^m}{h^{2i-1}} f(x_k) + O(h^m), \tag{14}$$

*for any $m, i$ such that $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$, and $1 \leq k \leq \frac{m-1}{2}$.*

**Proof.** The proof is as an immediate consequence of (10) and (13). □

**Lemma 2.4** *Suppose that $m, l, k$ are integers, and the coefficients $a_{k,l}^m$ are defined by the recurrence relation*

$$a_{1,1}^3 = 1, \tag{15-a}$$

$$a_{1,2}^3 = 1, \tag{15-b}$$

$$a_{k,l}^{2k+1} = (k - k^2) a_{k-1,l}^{2k-1} + a_{k-1,l-1}^{2k-1} + a_{k-1,l-2}^{2k-1}, \tag{15-c}$$

$$a_{k,l}^{m+2} = a_{k,l-2}^m - (\frac{m+1}{2})^2 a_{k,l}^m, \tag{15-d}$$

*with* $a^m_{k,l} = 0$, *for all* $k \leq 0$, *or* $l \leq 0$, *or* $m \leq 1$. *Then*

$$A^m_k(p) = \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!} \sum_{l=1}^{\frac{m-1}{2}} a^m_{k,l} p^l, \tag{16}$$

*for any odd* $m \geq 3$, $1 \leq k \leq \frac{m-1}{2}$ *and* $A^m_k(p)$ *is defined by (11).*

**Proof.** Due to (11),

$$A^m_k(p) = \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!} C^m_k(p), \tag{17}$$

where

$$C^m_k(p) = \frac{1}{(p-k)} \prod_{t=0}^{m-1} (p + \frac{m-1}{2} - t). \tag{18}$$

Thus it is sufficient to show that

$$C^m_k(p) = \sum_{l=1}^{\frac{m-1}{2}} a^m_{k,l} p^l. \tag{19}$$

This will be shown by induction. Indeed, if $m = 3$ then, due to (18),

$$C^3_1(p) = p^2 + p, \tag{20}$$

which is equivalent to (15-a),(15-b).
Assume now that for some $m, k$ such that $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$,

$$C^m_k(p) = \sum_{l=1}^{\frac{m-1}{2}} a^m_{k,l} p^l. \tag{21}$$

Combining (18) and (21), we have

$$\begin{aligned} C^{m+2}_k(p) &= (p + \frac{m+1}{2})(p - \frac{m+1}{2}) \sum_{l=1}^{\frac{m-1}{2}} a^m_{k,l} p^l \\ &= (p^2 - (\frac{m+1}{2})^2) \sum_{l=1}^{\frac{m-1}{2}} a^m_{k,l} p^l \\ &= \sum_{l=1}^{\frac{m-1}{2}} a^m_{k,l} p^{l+2} - (\frac{m+1}{2})^2 \sum_{l=1}^{\frac{m-1}{2}} a^m_{k,l} p^l, \end{aligned} \tag{22}$$

which is equivalent to (15-d).
Now, assume that for some $k$

$$C^{2k+1}_k(p) = \sum_{l=1}^{k} a^{2k+1}_{k,l} p^l. \tag{23}$$

Combining (23) and (18), we have

$$\begin{aligned} C_k^{2k+3}(p) &= (p-k)(p-(k+1))\sum_{l=1}^{k} a_{k,l}^{2k+1} p^l \\ &= (p^2+p-(k^2+k))\sum_{l=1}^{k} a_{k,l}^{2k+1} p^l \\ &= \sum_{l=1}^{k} a_{k,l}^{2k+1} p^{l+2} + \sum_{l=1}^{k} a_{k,l}^{2k+1} p^{l+1} - (k^2+k)\sum_{l=1}^{k} a_{k,l}^{2k+1} p^l, \end{aligned} \tag{24}$$

which is equivalent to (15-c). □

**Lemma 2.5** *Suppose that $m \geq 3$ is odd. Then,*

$$D_{i,k}^m = \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!} a_{k,2i-1}^m (2i-1)!, \tag{25}$$

*for any $k, i$ such that $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$, and $1 \leq i \leq \frac{m-1}{2}$,*

with the coefficients $a_{k,l}^m$ defined by the recurrence relation in Lemma 2.4.
**Proof.** Substituting (16) into (13), we immediately obtain

$$\begin{aligned} D_{i,k}^m &= \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!} \frac{\partial^{(2i-1)}}{\partial p^{(2i-1)}} \sum_{l=1}^{m-1} a_{k,l}^m p^l \, |^{p=0} \\ &= \frac{(-1)^{\frac{m-1}{2}+k}}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!} a_{k,2i-1}^m (2i-1)!. \end{aligned} \tag{26}$$

□

The following six lemmas provide identities which are used in the proof of Theorem 3.1.

**Lemma 2.6** *If $k \geq 2$ is an integer and $a_{k,l}^m$ is defined in Lemma 2.4.*

$$| (l)! \cdot a_{k,l}^{2k+1} | \quad < \quad | (l+2)! \cdot a_{k,l+2}^{2k+1} |, \tag{27}$$

*for all $l = 1, 2, ..., 2k-3$.*

**Proof.**
If $k = 2$, and $l = 1$ then $| (1)! \cdot a_{2,1}^5 | = 2$, $| (3)! \cdot a_{2,3}^5 | = 12$, and therefore (27) is obviously true.
Now, assume that

$$| (l)! \cdot a_{k,l}^{2k+1} | \quad < \quad | (l+2)! \cdot a_{k,l+2}^{2k+1} |, \tag{28}$$

for some $k \geq 2$ and all $l = 1, 2, ..., 2k-3$.
Now, due to (15-a), (15-b), (15-c), and (15-d),

$$(l)! \cdot a_{k+1,l}^{2k+3} = (((k+1)-(k+1)^2)a_{k,l}^{2k+1} + a_{k,l-1}^{2k+1} + a_{k,l-2}^{2k+1}) \cdot (l!), \tag{29}$$

and

$$(l+2)!\cdot a^{2k+3}_{k+1,l+2} = (((k+1)-(k+1)^2)a^{2k+1}_{k,l+2} + a^{2k+1}_{k,l+1} + a^{2k+1}_{k,l})\cdot(l+2)!. \tag{30}$$

Finally, combining (28), (29), and (30) we easily obtain

$$|\,(l)!\cdot a^{2k+3}_{k+1,l}\,| \quad < \quad |\,(l+2)!\cdot a^{2k+3}_{k+1,l+2}\,|, \tag{31}$$

for all $l = 1, 2, ..., 2k-1$. □

**Lemma 2.7** *If $k \geq 2$ is an integer, and $a^m_{k,l}$ is defined in Lemma 2.4 then*

$$|\,(l)!\cdot a^{m}_{k,l}\,| \quad < \quad |\,(l+2)!\cdot a^{m}_{k,l+2}\,|, \tag{32}$$

*for all $m \geq 2k+1$ and $l = 1, 2, ..., 2k-3$.*

**Proof.** Lemma (2.6) establishes the base case, i.e., that (32) is true when $m = 2k+1$. Now, assume that

$$|\,(l)!\cdot a^{m}_{k,l}\,| \quad < \quad |\,(l+2)!\cdot a^{m}_{k,l+2}\,|, \tag{33}$$

for some odd $m \geq 2k+1$, and all $l = 1, 2, ..., 2k-3$.
Now, due to (15-a), (15-b), (15-c), and (15-d),

$$(l)!\cdot a^{m+2}_{k,l} = (a^m_{k,l-2} - (\frac{m+1}{2})^2 a^m_{k,l})\cdot(l!), \tag{34}$$

and

$$(l+2)!\cdot a^{m+2}_{k,l+2} = (a^m_{k,l} - (\frac{m+1}{2})^2 a^m_{k,l+2})\cdot(l+2)!. \tag{35}$$

Finally, combining (33), (34), and (35) we easily obtain

$$(l)!\cdot|\,a^{m+2}_{k,l}\,| \quad < \quad |\,(l+2)!\cdot a^{m+2}_{k+1,l+2}\,|, \tag{36}$$

for all $l = 1, 2, ..., 2k-3$. □

**Lemma 2.8** *If $m, k$ are integers such that $m \geq 3$ is odd, and $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$, then*

$$|\,(1)!\cdot a^m_{k,1}\,|<|\,(3)!\cdot a^m_{k,3}\,|<|\,(5)!\cdot a^m_{k,5}\,|< ... <|\,(m-2)!\cdot a^m_{k,m-2}\,|\,. \tag{37}$$

**Proof.** This Lemma follows directly from Lemma 2.6 and Lemma 2.7 □

**Lemma 2.9** *If $m \geq 3$ is odd, then*

$$\frac{(m-1)(m-2)!}{2((\frac{m-1}{2})!)^2} < \frac{(2\pi)^{m-1}}{4}. \tag{38}$$

**Proof.** If $m = 3$ then obviously $1 < \frac{(2\pi)^2}{4}$.
Now, assume that for some odd $m \geq 3$,

$$\frac{(m-1)(m-2)!}{2((\frac{m-1}{2})!)^2} < \frac{(2\pi)^{m-1}}{4}. \tag{39}$$

Obviously,

$$\frac{(m+1)(m)}{(\frac{(m+1)}{2})^2} = \frac{4m}{(m+1)} < (2\pi)^2, \tag{40}$$

and combining (39) and (40) we obtain

$$\frac{(m+1)(m)}{(\frac{(m+1)}{2})^2}\frac{(m-1)(m-2)!}{2((\frac{m-1}{2})!)^2} < \frac{(2\pi)^{m-1}}{4}(2\pi)^2, \tag{41}$$

which is equivalent to

$$\frac{(m+1)(m)!}{2((\frac{m+1}{2})!)^2} < \frac{(2\pi)^{m+1}}{4}. \tag{42}$$

Now, the conclusion of the lemma is an immediate consequence of (39) and (42). $\square$

**Lemma 2.10** *If $m \geq 3$ is odd then*

$$| D^m_{i,k} | < \frac{(2\pi)^{m-1}}{4}, \tag{43}$$

*for any $k, i$ such that $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$, and $1 \leq i \leq \frac{m-1}{2}$.*

**Proof.** Combining Lemmas 2.5, 2.6, 2.7, and 2.8, it is easy to see that

$$| D^m_{1,k} | < | D^m_{2,k} | < \ldots < | D^m_{\frac{m-1}{2},k} | . \tag{44}$$

Consequently, it is sufficient to show that

$$| D^m_{\frac{m-1}{2},k} | < \frac{(2\pi)^{m-1}}{4}. \tag{45}$$

First we observe that (obviously) for any $k$ such that $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$,

$$\frac{k(m-2)!}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!} < \frac{(m-1)(m-2)!}{2((\frac{m-1}{2})!)^2}. \tag{46}$$

Then, we combine (15-a), (15-b), (15-c), (15-d), and (25) to obtain

$$| D^m_{\frac{m-1}{2},k} | = \frac{k(m-2)!}{(\frac{m-1}{2}+k)!(\frac{m-1}{2}-k)!}. \tag{47}$$

Now, (45) follows immediately from the combination of (46), (47), and Lemma 2.9. $\square$

**Lemma 2.11** *For any $l \geq 1$ the Bernoulli number $B_{2l}$ satisfies the inequality*

$$| \frac{B_{2l}}{(2l)!} | < \frac{4}{(2\pi)^{2l}}. \tag{48}$$

**Proof.** As is well known (see for example, [1]), for any $l \geq 1$

$$B_{2l} = \frac{(-1)^{l-1} 2 (2l)!}{(2\pi)^{2l}} \sum_{k=1}^{\infty} \frac{1}{k^{2l}}, \tag{49}$$

and

$$\sum_{k=1}^{\infty} \frac{1}{k^{2l}} < 2. \tag{50}$$

Now, the conclusion of the lemma is an immediate consequence of (49) and (50). □
The proof of the following lemma can be found in [5].

**Lemma 2.12** *Suppose that $m \geq 1$, $s \in c^m(0,1]$ possesses a finite integral on the interval $[0,1]$, and that $s^{(m)}(x)$ is monotonic in some neighborhood of $0$. Then the product $x \cdot s(x)$ is bounded on $[0,1]$. Suppose further that $w \in c^m[0,1]$ is such that $w(0) = w'(0) = w''(0) = ... = w^{(m)}(0) = 0$. Then the function $\psi(x) = s(x) \cdot w(x)$ is defined on the closed interval $[0,1]$, and $\psi(0) = \psi'(0) = \psi''(0) = ... = \psi^{(m)}(0) = 0$.*

# 3 End-point Corrections for Non-singular Functions

## 3.1 End-point corrected trapezoidal rules

While the authors have failed to find the contents of this section in the literature, it is an immediate consequence of well-known facts from classical analysis. We present it here for completeness, and because we found the resulting high-order quadrature rules quite useful (see Section 7.1).

Suppose that $n, m$, are a pair of integers with $m \geq 3$ and odd, and $n \geq 2$. Further, suppose that $a, b$ are a pair of real numbers such that $a < b$, $h = (b-a)/(n-1)$, and $f : [a-mh, b+mh] \to R^1$ is an integrable function. We define the corrected trapezoidal rule $T^n_{\beta^m}$ for non-singular functions by the formula

$$T^n_{\beta^m}(f) = T_n(f) + h \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} (f(b+kh) - f(a+kh))\beta^m_k. \tag{51}$$

The real coefficients $\beta^m_k$ are given by the formula

$$\beta^m_k = \sum_{l=1}^{\frac{m-1}{2}} \frac{D^m_{l,k} B_{2l}}{(2l)!}, \tag{52}$$

where $D^m_{l,k}$ are defined in (13) (also, see (25)) and $B_{2l}$ are the Bernoulli numbers.

We will say that the rule $T^n_{\beta^m}$ is of order $m$ if for any $f \in c^m[a-mh, b+mh]$, there exists a real number $c > 0$ such that

$$| T^n_{\beta^m}(f) - \int_a^b f(x)dx | < \frac{c}{n^m}. \quad (53)$$

**Theorem 3.1** *If $m \geq 3$ is an odd integer then for any $k$ such that $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$,*

$$| \beta^m_k | < \frac{m-1}{2}, \quad (54)$$

*where the coefficients $\beta^m_k$ are defined in (52).*

**Proof.** Combining Lemma 2.10 and Lemma 2.11 we immediately observe that

$$| \frac{D^m_{l,k} B_{2l}}{(2l)!} | < 1, \quad (55)$$

and hence

$$| \beta^m_k | = \sum_{l=1}^{\frac{m-1}{2}} \frac{D^m_{l,k} B_{2l}}{(2l)!} < \frac{m-1}{2}. \quad (56)$$

□

**Remark 3.1** A somewhat more involved argument shows that in fact $|\beta^m_k| < 1$ for all $k, m$; empirically this can also be seen from the tables in Section 7.1 below. However, for the purposes of this paper (56) is sufficient.

**Theorem 3.2** *Suppose that $m, n$ are a pair of integers with $m \geq 3$ and odd, and $n \geq 2$. Further, suppose that $a, b$ are a pair of real numbers such that $a < b$. Then, the end-point corrected trapezoidal rule $T^n_{\beta^m}$ is of order $m$, i.e., for any $f : [a-mh, b+mh] \rightarrow R^1$ such that $f[a-mh, b+mh] \in c^m[a-mh, b+mh]$, there exists a real number $c > 0$ such that*

$$| T^n_{\beta^m}(f) - \int_a^b f(x)dx | < \frac{c}{n^m}. \quad (57)$$

**Proof.** Combining (52) and (51), we obtain

$$\begin{aligned} T^n_{\beta^m}(f) &= T_n(f) + h \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} (f(b+kh) - f(a+kh)) \sum_{l=1}^{\frac{m-1}{2}} \frac{D^m_{l,k} B_{2l}}{(2l)!} \\ &= T_n(f) + \sum_{l=1}^{\frac{m-1}{2}} \frac{h^{2l} B_{2l}}{(2l)!} ( \sum_{k=-\frac{m-1}{2}}^{\frac{m-1}{2}} \frac{D^m_{l,k}(f(b+kh) - f(a+kh))}{h^{2l-1}} ). \end{aligned} \quad (58)$$

Combining (14) and (58), we have

$$T^n_{\beta^m}(f) = T_n(f) + \sum_{l=1}^{\frac{m-1}{2}} \frac{h^{2l}B_{2l}}{(2l)!}(f^{(2l-1)}(b) - f^{(2l-1)}(a) - 2R^{(2l-1)}_{m-1}). \tag{59}$$

Finally, combining (59) with Lemma 2.1, we observe that for some $a < \xi < b$,

$$T^n_{\beta^m}(f) = \int_a^b f(x)dx + 2R^{(2l-1)}_{m-1} + \frac{h^m B_m}{m!} f^m(\xi), \tag{60}$$

and the theorem immediately follows from (60). □

**Remark 3.2** It is easy to see that for $m \geq 3$ and odd, and any $k$ such that $-\frac{m-1}{2} \leq k \leq \frac{m-1}{2}$, $D^m_{i,-k} = -D^m_{i,k}$, and $D^m_{i,0} = 0$ (due to (13)), and hence $\beta^m_{-k} = -\beta^m_k$ and $\beta^m_0 = 0$ (due to 52). Now, instead of (51) one could define the end-point corrected trapezoidal rule by the formula

$$T^n_{\beta^m}(f) = T_n(f) + h \sum_{k=1}^{\frac{m-1}{2}} (f(b+kh) - f(b-kh) - f(a+kh) + f(a-kh))\beta^m_k. \tag{61}$$

# 4 End-point Corrections for Singular Functions

In this section we construct a group of quadrature formulae for end-point singular functions, generalizing the classical end-point corrected trapezoidal rules. The actual values of end-point corrections are obtained for each singularity as a solution of a system of linear algebraic equations. All the rules developed in this section are simple extensions of the corrected trapezoidal rule $T^n_{\beta^m}$ developed in the preceding section.

A right-end corrected trapezoidal rule $T^n_{R\beta^m}$ is defined by the formula

$$T^n_{R\beta^m}(f) = h(\frac{f(x_{n-1})}{2} + \sum_{i=1}^{n-2} f(x_i)) \quad + \quad h \sum_{k=1}^{\frac{m-1}{2}} (f(b+kh) - f(b-kh))\beta^m_k, \tag{62}$$

where $f(0, b+mh] \rightarrow R^1$ is an integrable function, $n, m$ are a pair of natural numbers with $m \geq 3$ and odd, the coefficients $\beta^m_k$ are given by (52), and

$$\begin{aligned} h &= \frac{b}{n-1}, \\ x_i &= ih. \end{aligned} \tag{63}$$

We will say that the rule $T^n_{R\beta^m}$ is of right-end order $m \geq 3$ if for any $f \in c^{m+1}[0, b+mh]$ such that $f(0)[214z = f'(0) = ... = f^{(m)}(0) = 0$, there exists $c > 0$ such that

$$| T^n_{R\beta^m}(f) - \int_0^b f(x)dx |< \frac{c}{n^m}. \tag{64}$$

It easily follows from Theorem 3.2 that $T^n_{R\beta^m}$ is of right-end order $m$.
Similarly, a left-end corrected trapezoidal rule $T^n_{L\beta^m}$ is defined by the formula

$$T^n_{L\beta^m}(f) = h(\frac{f(x_{-(n-1)})}{2} + \sum_{i=1}^{n-2} f(x_{-i})) \quad + \quad h\sum_{k=1}^{\frac{m-1}{2}}(-f(-b+kh)+f(-b-kh))\beta^m_k, \tag{65}$$

where $f[-b-mh,0) \to R^1$ is an integrable function, $n,m$ are a pair of natural numbers with $m \geq 3$ and odd, the coefficients $\beta^m_k$ are given by (52), and $h, x_i$ are defined by (63). We will say that the rule $T^n_{L\beta^m}$ is of left-end order $m \geq 3$ if for any $f \in c^{m+1}[-b-mh,0)$ such that $f(0) = f'(0) = ... = f^{(m)}(0) = 0$, there exists $c > 0$ such that

$$\mid T^n_{L\beta^m}(f) - \int_{-b}^{0} f(x)dx \mid < \frac{c}{n^m}. \tag{66}$$

It also easily follows from Theorem 3.2 that $T^n_{L\beta^m}$ is of left-end order $m$.

Suppose now that the function $f(-kh, b+mh] \to R^1$ is of the form

$$f(x) = \phi(x)s(x) + \psi(x), \tag{67}$$

with $\phi, \psi \in c^k(-kh, b+mh]$, and $s \in c(-kh, b+mh]$ an integrable function with a singularity at 0. For a finite sequence $\alpha = (\alpha_{-k}, \alpha_{-(k-1)}, \alpha_{-1}, \alpha_1, ..., \alpha_k)$ and $T^n_{R\beta^m}$ defined in (62), we define the end-point corrected rule $T^n_{\alpha\beta^m}$ by the formula

$$T^n_{\alpha\beta^m}(f) = T^n_{R\beta^m}(f) + h \sum_{j=-k, j\neq 0}^{k} \alpha_j f(x_j), \tag{68}$$

with $h = b/(n-1)$, $x_j = jh$.
We will use the expression $T^n_{\alpha\beta^m}$ with appropriately chosen $\alpha$ as quadrature formulae for functions of the form (67), and the following construction provides a tool for finding $\alpha$ once $\beta^m = (\beta^m_1, \beta^m_2, ..., \beta^m_{\frac{m-1}{2}})$ is given, so that the rule is of order $k$, i.e., there exists a $c > 0$ such that

$$\mid T^n_{\alpha\beta^m}(f) - \int_0^b f(x)dx \mid < \frac{c}{n^k}. \tag{69}$$

For a pair of natural numbers $k, m$, with $k \geq 1$ and $m \geq 3$ and odd, we will consider the following system of linear algebraic equations with respect to the unknowns $\alpha^n_j$, with $j = 0, \pm 1, \pm 2, ..., \pm k$:

$$\sum_{j=-k, j\neq 0}^{k} x_j^{i-1}\alpha^n_j = \frac{1}{h}\int_0^b x_j^{i-1}dx - T^n_{R\beta^m}(x^{i-1}), \tag{70}$$

for $i = 1, 2, ..., k$, and

$$\sum_{j=-k, j\neq 0}^{k} x_j^{i-k-1}s(x_j)\alpha^n_j = \frac{1}{h}\int_0^b x_j^{i-k-1}s(x)dx - T^n_{R\beta^m}(x^{i-k-1}s(x)), \tag{71}$$

for $i = k+1, k+2, ..., 2k$, with $h = b/(n-1)$, $x_j = jh$ and $T^n_{R\beta^m}$ defined by (62). We denote the matrix of the system (70), (71) by $A^{nk}_s$, its right-hand side by $Y^{nk}_s$ and its solution by $\alpha_n = (\alpha^n_{-k}, \alpha^n_{-(k-1)}, ..., \alpha^n_{-1}, \alpha^n_1, ..., \alpha^n_k)$. The use of expressions $T^n_{\alpha^n\beta^m}$ as quadrature formulae for functions of the form (67) is based on the following theorem.

**Theorem 4.1** *Suppose that a function $s : (-kh, b+mh] \to R^1$ is such that $s \in c^k(-kh, b+mh]$ and $s^k$ is monotonic on either side of 0. Suppose further that the systems (70), (71) have solutions $(\alpha^n_{-k}, \alpha^n_{-(k-1)}, \alpha^n_{-1}, \alpha^n_1, ..., \alpha^n_k)$ for all sufficiently large $n$, and that the sums*

$$\sum_{j=-k, j\neq 0}^{k} (\alpha^n_j)^2 \tag{72}$$

*are bounded uniformly with respect to $n$. Finally, suppose that the function $f : (-kh, b+mh] \to R^1$ is defined by (67). Then, there exists a real $c > 0$ such that*

$$\mid T^n_{\alpha^n\beta^m}(f) - \int_a^b f(x)dx \mid < \frac{c}{n^k} \tag{73}$$

*for all sufficiently large $n$.*

**Proof.** Applying the Taylor expansion to the function $f$ at $x = 0$ we obtain

$$f(x) = P(f)(x) + R_k(\phi)(x)s(x) + R_k(\psi), \tag{74}$$

where

$$P(f)(x) = s(x)\sum_{i=0}^{k} \frac{\phi^{(i)}(0)}{(i!)}x^i + \sum_{i=0}^{k} \frac{\psi^{(i)}(0)}{i!}x^i, \tag{75}$$

and $R_k(\phi), R_k(\psi)$ are such functions $[-kh, b+mh] \to R^1$ that

$$R_k^{'}(\phi)(0) = R_k^{''}(\phi)(0) = ... = R_k^{(k)}(\phi)(0) = 0, \tag{76}$$

$$R_k^{'}(\psi)(0) = R_k^{''}(\psi)(0) = ... = R_k^{(k)}(\psi)(0) = 0. \tag{77}$$

Substituting (74) into (73), we obtain

$$\begin{aligned} \mid T^n_{\alpha^n\beta_m}(f) - \int_0^b f(x)dx \mid \quad &\leq \quad \mid T^n_{\alpha^n\beta_m}(P(f)) - \int_0^1 P(f)(x)dx \mid + \\ \mid T^n_{\alpha^n\beta_m}((R_k(\phi)\cdot s) + R_k(\psi)) - &\int_0^b ((R_k(\phi(x))s(x))(x) + R_k(\psi(x)))dx \mid . \end{aligned} \tag{78}$$

Due to (70), (71)

$$T^n_{\alpha^n\beta_m}(P(f)) - \int_0^1 P(f)(x)dx = 0, \tag{79}$$

and we have

$$| T^n_{\alpha^n\beta_m}(f) - \int_0^b f(x)dx | \leq$$

$$\begin{aligned} & | RT^n_{\beta_m k}(s \cdot R_k(\phi)) - \textstyle\int_0^b (s \cdot R_k(\phi))(x)dx | \\ + & | RT^n_{\beta_m k}(R_k(\psi)) - \textstyle\int_0^b (R_k(\psi))(x)dx | \\ + & | \textstyle\sum_{j=1}^{2k} (R_k(\phi)(jh)s(jh)\alpha^n_j) + (R_k(\psi)(jh)\alpha^n_j) | . \end{aligned} \tag{80}$$

Due to (77) and (64), there exists $c_1 > 0$ such that

$$| RT^n_{\beta_m k}(R_k(\psi)) - \int_0^b (R_k(\psi))(x)dx | < \frac{c_1}{n^k}. \tag{81}$$

Combining (76), (64), and Lemma 2.12 we conclude that for some $c_2 > 0$

$$| RT^n_{\beta_m k}(s \cdot R_k(\phi)) - \int_0^b (s \cdot R_k(\phi))(x)dx | < \frac{c_2}{n^k}. \tag{82}$$

Finally, combining (76), (77) and Lemma 2.12 we conclude that for some $c_3 > 0$,

$$| \sum_{j=-k, j\neq 0}^{k} (R_k(\phi)(jh)s(jh)\alpha^n_j) + (R_k(\psi)(jh)\alpha^n_j) | < \frac{c_3}{n^k}. \tag{83}$$

Now, the conclusion of the theorem follows from the combination of (81), (82), and (83). □

## 4.1 Convergence Rates for Singularites of the forms $|x|^\lambda$ and $log(|x|)$

For the remainder of the paper, $\phi_1, \phi_2, ..., \phi_{2k}$ will denote functions $(-kh, b+mh] \to R^1$ defined by the formulae

$$\phi_i(x) = x^{i-1}, \tag{84}$$

for $i = 1, 2, ..., k$, and

$$\phi_i(x) = x^{i-k-1}s(x), \tag{85}$$

for $i = k+1, k+2, ..., 2k$. The following lemma is a particular case of a well-known general fact proven, for example, in [8].

**Lemma 4.2** *If $s(x) = x^\lambda$ with $\lambda$ a real number such that $0 < |\lambda| < 1$, then the functions $\phi_1, \phi_2, ..., \phi_{2k}$ constitute a Chebyshev system on the interval $(-kh, b+mh]$ (i.e., the determinant of the $2k \times 2k$ matrix $B_{ij}$ defined by the formula $B_{ij} = \phi_i(t_j)$ is non-zero for any $2k$ distinct points on the interval $(-kh, b+mh]$).*

**Theorem 4.3** *If $s(x) = |x|^\lambda$ with $0 < |\lambda| < 1$, then the convergence rate of the quadrature rule $T^n_{\alpha^n\beta_m}$ is at least $k$.*

**Proof.** It immediately follows from Lemma 4.2 that the matrix of the system (70), (71) is non-singular. We rescale the system (70), (71) by multiplying its $i$th equation by $\frac{1}{h^{i-1}}$, for $i = 1, 2, ..., k$, and by $\frac{1}{h^{i-1-k+\lambda}}$, for $i = k+1, k+2, ..., 2k$, obtaining the system of equations

$$\sum_{j=-k, j\neq 0}^{k} j^{i-1}\alpha_k^n = \frac{1}{h^i}\left(\int_0^b x^{i-1}dx - T_{R\beta^m}^n(x^{i-1})\right), \qquad (86)$$

$i = 1, 2, ..., k$, and

$$\sum_{j=-k, j\neq 0}^{k} j^{i-k-1+\lambda}\alpha_k^n = \frac{1}{h^{i-k+\lambda}}\left(\int_0^b x^{i-k-1+\lambda}dx - T_{R\beta^m}^n(x^{i-k-1+\lambda})\right), \qquad (87)$$

for $i = k+1, k+2, ..., 2k$.
We will denote the matrix of the system (86), (87) by $B_k$, and its right hand side by $Z_k^n$. Obviously, $B_k$ is independent of $n$, and using Theorem 3.2 we observe that if $m > k$ then $|Z_k^n|$ is bounded uniformly with respect to $n$. Now, due to Theorem 4.1, the convergence rate of $T_{\alpha^n\beta^m}^n$ is at least $k$. □

The proof of Theorem 4.3 can be repeated almost verbatim with $s(x) = log(|x|)$, instead of $s(x) = |x|^\lambda$, resulting in the following theorem.

**Theorem 4.4** *If $s(x) = log(|x|)$ then the convergence rate of the quadrature rule $T_{\alpha^n\beta_m}^n$ is at least $k$.*

## 4.2 Asymptotic behaviour of correction coefficients as $n \to \infty$

An obvious drawback of the expressions $T_{\alpha\beta^m}^n$ as practical quadrature rules is the fact that the weights $\alpha^n = (\alpha_{-k}^n, ..., \alpha_{-1}^n, \alpha_1^n, ..., \alpha_k^n)$ have to be determined for each value of $n$ by solving a system of linear algebraic equations. For singularities of the form $s(x) = log(|x|)$, $s(x) = |x|^\lambda$ we eliminate this problem by constructing a new set of quadrature weights $\gamma = (\gamma_{-k}, \gamma_{-(k-1)}, ..., \gamma_{-1}, \gamma_1, ..., \gamma_k)$, independent of $n$, and such that the quadrature rules $T_{\gamma^k\beta^m}^n$ are still of order not less than $k$.

**Lemma 4.5** *Suppose that $\beta = (\beta_1^m, \beta_2^m, ..., \beta_{\frac{m-1}{2}}^m)$ is such that the right-hand order of the quadrature formula $T_{R\beta^m}^n$ is $m$. Further, let $z > 0$ be some real number. Then for any integers $p, q$ such that $p < q$,*

$$\left| \frac{1}{h_p^{z+1}}\left(T_{R\beta^m}^p(x^z) - \int_0^b x^z dx\right) - \frac{1}{h_q^{z+1}}\left(T_{R\beta^m}^q(x^z) - \int_0^b x^z dx\right) \right| = O(h_p^{m-z-1}), \qquad (88)$$

*where $h_p = b/(p-1)$, and $h_q = b/(q-1)$.*

**Proof.** Due to Theorem 3.2, there exist real $c_1, c_2 > 0$ such that

$$\left(T_{R\beta^m}^p(x^z) - \int_0^b x^z dx\right) = c_1 h_p^m - h_p \sum_{j=-k}^{k} (jh_p)^z, \qquad (89)$$

and

$$(T^q_{R\beta^m}(x^z) - \int_0^b x^z dx) = c_2 h_q^m - h_q \sum_{j=-k}^{k} (jh_q)^z. \tag{90}$$

Now, combining (89), (90) we obtain

$$\begin{aligned}
& \left| \tfrac{1}{h_p^{z+1}}(T^p_{R\beta^m}(x^z) - \textstyle\int_0^b x^z dx) - \tfrac{1}{h_q^{z+1}}(T^q_{R\beta^m}(x^z) - \int_0^b x^z dx) \right| \\
= & \frac{1}{h_p^{z+1}}(c_1 h_p^m - h_p \sum_{j=-k}^{k} (jh_p)^z) - \frac{1}{h_q^{z+1}}(c_2 h_q^m - h_q \sum_{j=-k}^{k} (jh_q)^z) \\
= & (c_1 h_p^{m-z-1} - \sum_{j=-k}^{k} (j)^z) - (c_2 h_q^{m-z-1} - \sum_{j=-k}^{k} (j)^z) \\
= & c_1 h_p^{m-z-1} - c_2 h_q^{m-z-1} \\
= & O(h_p^{m-z-1}).
\end{aligned} \tag{91}$$

□

**Theorem 4.6** *Suppose that $k, m$ are two natural numbers such that $k \leq m-1$ and that $\beta = (\beta_1^m, \beta_2^m, ..., \beta^m_{\frac{m-1}{2}})$ is such that the right-hand order of the quadrature $T_{R\beta^m}$ is $m$. Suppose further that $s(x) = |x|^\lambda$ with $0 \leq |\lambda| \leq 1$, and that the coefficients $(\alpha^n_{-k}, \alpha^n_{-(k-1)}, \alpha^n_{-1}, \alpha^n_1, ..., \alpha^n_k)$ are the solutions of the system (70),(71).Then*

*1) There exists a limit*

$$lim_{n\to\infty} \alpha_i^n = \gamma_i, \tag{92}$$

*for each $i = 1, 2, ..., 2k$.*
*2) For all $i = 1, 2, ..., 2k$,*

$$| \alpha_i^n - \gamma_i | = O(\frac{1}{n^{m-k}}). \tag{93}$$

*3) $\gamma_i$ do not depend on $m$, as long as $m \geq k+1$.*
*4) The quadrature formulae $T^n_{\gamma\beta^m}$ are of order at least $k$.*

**Proof.** Suppose that $p, q$ are two natural numbers, and $p < q$. Obviously,

$$\begin{aligned}
\alpha^p &= (B_k)^{-1} Z_k^p, \\
\alpha^q &= (B_k)^{-1} Z_k^q, \\
\alpha^p - \alpha^q &= (B_k)^{-1}(Z_k^p - Z_k^q).
\end{aligned} \tag{94}$$

Due to Lemma 4.5, there exists $c > 0$ such that

$$||Z_k^p - Z_k^q|| < \frac{c}{p^{m-k}}. \tag{95}$$

and by combining (94), (95), we see that for some $d > 0$

$$||\alpha^p - \alpha^q|| < \frac{d}{p^{m-k}}. \tag{96}$$

Since the weights $\alpha^n$ constitute a Cauchy sequence, they converge to some limit $\gamma = (\gamma_{-k}, \gamma_{-(k-1)}, ..., \gamma_{-1}, \gamma_1, ..., \gamma_k)$, which proves 1, and 2, 3, 4 follow easily. □

The proof of the following theorem is a repetition, almost verbatim, of the proofs of the Lemma 4.5 and Theorem 4.6

**Theorem 4.7** *If under the conditions of Theorem 4.6 we replace $s(x) = |x|^\lambda$ with $s(x) = log(|x|)$, conclusions 1-4 remain correct.*

For singularities of the form $|x|^\lambda$ and $log(|x|)$, Theorem 4.6 and 4.7 reduce the quadratures $T^n_{\alpha\beta^m}$ to the more "conventional" form

$$\int_0^b f(x)dx \approx T^n_{\gamma^k\beta^m}(f) = T^n_{R\beta^m}(f) + h \sum_{j=-k, j\neq 0}^{k} \gamma_j f(x_j). \tag{97}$$

**Remark 4.1** The whole theory in sections 4.1-4.2 has been constructed for functions with a singularity at the left end of the interval. Obviously, an identical theory holds for functions with a singularity at the right end of the interval. However, in all formulae the expression $T^n_{R\beta^m}$ has to be replaced with $T^n_{L\beta^m}$ (see (62), (65)).

### 4.3 Central Corrections for Singular Functions

In this section, we will be considering functions $f[-b-mh, 0) \cup (0, b+mh] \to R^1$ of the form

$$f(x) = \phi(x)s(x) + \psi(x), \tag{98}$$

with $\phi, \psi \in c^l[-b-mh, b+mh]$, and $s \in c[-b-mh, 0) \cup (0, b+mh]$ an integrable function with a singularity at 0. We will define the central-point corrected trapezoidal rule

$$T^n_{\mu^n\beta^m}(f) = T^n_{R\beta^m}(f) + T^n_{L\beta^m}(f) + h\sum_{j=1}^{l} \mu^n_j (f(x_j) + f(x_{-j})), \tag{99}$$

with $h, x_j$ defined by(63), $\beta^m_i$ defined by (52), $T^n_{R\beta^m}, T^n_{L\beta^m}$ defined by (62) and (65) respectively, and $\mu^n = (\mu^n_1, \mu^n_2, ..., \mu^n_l)$ an arbitrary sequence of length $l$.

We will use the expression $T^n_{\mu^n\beta^m}$ with appropriately chosen $\mu^n$ as quadrature formulae for functions of the form (98), and the following construction provides a tool for finding $\mu^n$ once $\beta^m$ is given, so that the rule is of order $2l$, i.e., there exists some $c > 0$ such that

$$| T^n_{\mu^n\beta^m}(f) - \int_{-b}^{b} f(x)dx | < \frac{c}{n^{2l}}. \tag{100}$$

For a pair of natural numbers $l, m$, we will consider the following system of linear algebraic equations with respect to the unknowns $\mu_j^n$:

$$\sum_{j=1}^{l} x_j^{2i-2}\mu_j^n = \int_{-b}^{b} x^{2i-2}dx - T_{R\beta^m}^n(x^{2i-2}) - T_{L\beta^m}^n(x^{2i-2}), \tag{101}$$

for $i = 1, 2, ..., l$, and

$$\sum_{j=1}^{l} x_j^{2i-2-2l}s(x_j)\mu_j^n = \int_{-b}^{b} x^{2i-2-2l}s(x)dx - T_{R\beta^m}^n(x^{2i-2-2l}s(x)) - T_{L\beta^m}^n(x^{2i-2-2l}s(x)), \tag{102}$$

for $i = l+1, l+2, ..., 2l$, with $h = b/(n-1)$, $x_j = jh$.
The proofs of Theorem 4.8, 4.9, and 4.10 are almost identical to those of Theorems 4.1, 4.3, and 4.6 respectively, and are thus stated below without proof.

**Theorem 4.8** *Suppose that a function $s : [-b-mh, 0) \cup (0, b+mh] \to R^1$ is such that $s \in c^l[-b-mh, 0) \cup (0, b+mh]$ and $s^l$ is monotonic on either side of $0$. Suppose further that the systems (101), (102) have solutions $(\mu_{-l}^n, \mu_{-(l-1)}^n, \mu_{-1}^n, \mu_1^n, ..., \mu_l^n)$ for all sufficiently large $n$, and that the sums*

$$\sum_{j=-l, j\neq 0}^{l} (\mu_l^n)^2 \tag{103}$$

*are bounded uniformly with respect to $n$. Finally, suppose that the function $f : [-b-mh, 0) \cup (0, b+mh] \to R^1$ is defined by (98). Then, there exists such $c > 0$ that*

$$| T_{\mu^n\beta^m}^n(f) - \int_a^b f(x)dx |< \frac{c}{n^{2l}} \tag{104}$$

*for all sufficiently large $n$.*

**Theorem 4.9** *If $s(x) = |x|^\lambda$ with $0 < |\lambda| < 1$, or $s(x) = log(|x|)$, then the convergence rate of the quadrature rule $T_{\mu^n\beta_m}^n$ is at least $2l$.*

**Theorem 4.10** *Suppose that $k, m$ are two natural numbers such that $k \leq m-1$ and that $\beta = (\beta_1^m, \beta_2^m, ..., \beta_{\frac{m-1}{2}}^m)$ is such that the right-end order of the quadrature $T_{R\beta^m}$ is $m$, and the left-end order of the quadrature $T_{L\beta^m}$ is $m$. Suppose further that $s(x) = |x|^\lambda$, $0 \leq| \lambda |\leq 1$, or $s(x) = log(|x|)$, and that the coefficients $(\mu_{-k}^n, \mu_{-(k-1)}^n, \mu_{-1}^n, \mu_1^n, ..., \mu_k^n)$ are the solutions of the system (70),(71). Then*

*1) There exists a limit*

$$lim_{n\to\infty}\mu_i^n = \mu_i, \tag{105}$$

*for each $i = 1, 2, ..., 2k$.*
*2) For all $i = 1, 2, ..., 2k$,*

$$| \mu_i^n - \mu_i |= O(\frac{1}{n^{m-l}}). \tag{106}$$

*3) $\mu_i$ do not depend on $m$, as long as $m \geq l+1$.*
*4) The quadrature formulae $T_{\mu\beta^m}^n$ are of order at least $2l$.*

For singularities of the form $|x|^\lambda$ and $log(|x|)$, the Theorem 4.10 reduces the quadrature to the more "conventional" form

$$\int_{-b}^{b} f(x)dx \approx T^n_{\mu\beta^m}(f) = T^n_{R\beta^m}(f) + T^n_{L\beta^m}(f) + \sum_{j=1}^{l} \mu_j(f(x_j) + f(x_{-j})). \tag{107}$$

## 4.4 Central Corrections for Singular Functions $f(x) = \phi(x)s(x)$

In this section we construct a quadrature formula specifically for the purpose of approximating definite integrals of functions of the form

$$f(x) = \phi(x)s(x), \tag{108}$$

where $\phi(x) : c^p[-b-mh, b+mh] \to R^1$, and $s \in c[-b-mh, 0) \cup (0, b+mh]$ an integrable function with a singularity at 0. For a finite sequence $\rho = (\rho_0, \rho_1, \rho_2, ..., \rho_p)$, and $T_{R\beta^m}$, $T_{L\beta^m}$ defined in (62) and (65) respectively, we define the corrected trapezoidal rule $T^n_{\rho^n\beta^m}$ by the formula

$$T^n_{\rho^n\beta^m}(f) = T^n_{R\beta^m}(f) + T^n_{L\beta^m}(f) + h\sum_{j=0}^{p} \rho^n_j(\phi(jh) + \phi(-jh))). \tag{109}$$

For integers $n, m, p$ where $n \geq 2$, $p \geq 1$, $m \geq 3$ and odd, we will consider the following system of equations with respect to the unknowns $\rho^n = (\rho^n_0, \rho^n_1, \rho^n_2, ..., \rho^n_p)$:

$$\sum_{j=0}^{p} x_j^{2i-2}\rho^n_j = \frac{1}{h}\int_{-b}^{b}(x^{2i-2}s(x))dx - T^n_{R\beta^m}(x^{2i-2}s(x)) - T^n_{L\beta^m}(x^{2i-2}s(x)), \tag{110}$$

where, $h = b/(n-1)$, $x_j = jh$, and $i = 1, 2, ..., p+1$.
The proof of the following theorem is almost identical to the proof of Theorem 4.1.

**Theorem 4.11** *Suppose that $n > 2$ is an integer, and $h, x_i$ are defined by (63). Further, suppose that $f(x) = \phi(x)s(x)$ where $\phi : [-b-mh, b+mh] \to R^1$, and $s \in c[-b-mh, 0) \cup (0, b+mh]$ is an integrable function with a singularity at 0. Finally, suppose that the system of equations (110) has a solution $(\rho^n_0, \rho^n_1, ..., \rho^n_p)$ for any sufficiently large n and that the sums $\sum_{j=0}^{p}(\rho^n_j)^2)$ are bounded uniformly with respect to n. Then there exists a real $c > 0$ such that*

$$| T^n_{\rho^k h}(f) - \int_{-b}^{b} f(x)dx |< \frac{c}{n^{2p}}. \tag{111}$$

The proof of the following theorem is almost identical to that of Theorem 4.6, and is omitted.

**Theorem 4.12** *Suppose that $s(x) = log(|x|)$. Then for all $n \geq 2p$, the system (110) has a solution $\rho^n = (\rho^n_0, \rho^n_1, \rho^n_2, ..., \rho^n_p)$, and*

$$\rho_0 = \frac{1}{h}\int_{-b}^{b} log(|x|)dx - T^n_{R\beta^m}log(|x|) - T^n_{L\beta^m}log(|x|) - \sum_{j=1}^{p}\rho_j. \tag{112}$$

*Furthermore, there exist such real numbers* $\rho_1, \rho_2, ..., \rho_p$ *and a real* $d > 0$ *such that*

$$lim_{n\to\infty}\rho_j^n = \rho_j, \tag{113}$$

*and*

$$\mid \rho_j^n - \rho_j \mid < d \cdot h^{m-p} \tag{114}$$

*for all* $j = 1, 2, ..., p$. *Finally, there exists a real* $c_0$ *such that*

$$\mid \rho_0^n - (c_0 + 0.5log(h) - \sum_{j=1}^{p} \rho_j \mid < d \cdot h^{m-p} \tag{115}$$

*for all* $n \geq 2p$.

**Remark 4.2** Formulae (114), (115) indicate that for sufficiently large $m$, the convergence of $\rho_1^n, \rho_2^n, ..., \rho_p^n$ to $\rho_1, \rho_2, ..., \rho_p$ is virtually instantaneous, and that (115) is a nearly perfect approximation to $\rho_0^n$. The numerical values of $\rho_1, \rho_2, ..., \rho_p$ can be found for various values of $p$ in Section 7.4. Also, note that $c_0$ does not depend on $p$, and its numerical value (to 16 digits) is $-.9189385332046727$.

The proof of the following theorem is similar to the proof of Theorem 4.6.

**Theorem 4.13** *Suppose that* $s(x) = |x|^\lambda$, *with* $\lambda$ *a real number such that* $0 < |\lambda| < 1$, *and (110) has a solution* $\rho^n = (\rho_0^n, \rho_1^n, \rho_2^n, ..., \rho_p^n)$. *Then for all* $n > 2p$, *the quadrature weights* $\rho_0^n, \rho_1^n, \rho_2^n, ..., \rho_p^n$ *are independent of* $n$.

### 4.5 Corrected trapezoidal rules for other singularities

In the preceding sections quadrature formulae are provided for singular functions of the form

$$f(x) = \phi(x)s(x) + \psi(x), \tag{116}$$

and

$$f(x) = \phi(x)s(x), \tag{117}$$

where the singularity $s(x)$ is of the form $log(|x|)$, or $x^\lambda$ $(0 < |\lambda| < 1)$. Obviously the procedure developed in the preceding sections can be applied to other singularities. As an example, we construct a quadrature formula to approximate the definite integral,

$$\int_{-a}^{a} f(x)dx, \tag{118}$$

where $f$ is of the form (117),

$$s(x) = \frac{1}{\sqrt{a^2 - x^2}}, \tag{119}$$

with $a > 0$, and $\phi(x) \in c^k[-a - kh, a + kh]$ and even (i.e., $\phi(-x) = \phi(x)$).

**Remark 4.3** The choice of the singularity (119) is dictated by the frequency with which it is encountered in the numerical solution of partial differential equations, in signal processing, and other areas. Otherwise, almost any integrable, monotone singularity could have been chosen.

We define the corrected trapezoidal rule $T^n_{\nu^n}$ by the formula

$$T^n_{\nu^n}(f) = \sum_{j=-(n-2)}^{n-2} f(x_j) + h \sum_{i=1}^{k} \nu_i^n f(y_i), \tag{120}$$

where $h = a/(n-1)$, $x_j = jh$, $y_i = a - hi$ for $1 \le i \le k/2$, and $y_i = a + h(i - k/2)$ for $k/2 + 1 \le i \le k$. We will use the expression $T^n_{\nu^n}$ with appropriately chosen $\nu^n$ as quadrature formulae for functions of the form (117), and the following construction provides a tool for finding $\nu^n$, so that the rule is of order $2k-2$, i.e., there exists a real $c > 0$ such that

$$\mid T^n_{\nu^n}(f) - \int_{-a}^{a} f(x)dx \mid < \frac{c}{n^{2k-2}}. \tag{121}$$

For an even integer $k \ge 2$, we will consider the following system of linear algebraic equations with respect to the unknowns $\nu_j^n$, with $j = 1, 2, ..., k$:

$$\sum_{j=1}^{k} \frac{y_j^{2(i-1)}}{\sqrt{a^2 - x^2}} \nu_j^n = \int_{-a}^{a} \frac{x^{2(i-1)}}{\sqrt{a^2 - x^2}} dx - \sum_{l=-(n-2)}^{n-2} \left(\frac{x_l^{2(i-1)}}{\sqrt{a^2 - x_l^2}}\right), \tag{122}$$

with $h = \frac{a}{n-1}$, $x_j = jh$, $y_j = a - hj$ for all $1 \le j \le k/2$, and $y_j = a + h(j - k/2)$ for all $k/2 + 1 \le j \le k$. It is easy to see that the linear system (122) is independent of the length of the interval $a$, and the unknowns $\nu_1^n, \nu_2^n, ..., \nu_k^n$ can be determined by solving the system of equations

$$\sum_{j=1}^{k} \frac{y_j^{2(i-1)}}{\sqrt{1 - x^2}} \nu_j^n = \int_{-1}^{1} \frac{x^{2(i-1)}}{\sqrt{1 - x^2}} dx - \sum_{l=-(n-2)}^{n-2} \left(\frac{x_l^{2(i-1)}}{1 - x_l^2}\right), \tag{123}$$

with $h = \frac{1}{n-1}$, $x_j = jh$, $y_j = 1 - hj$ for all $1 \le j \le k/2$, and $y_j = 1 + h(j - k/2)$ for all $k/2 + 1 \le j \le k$.

The proof of the following theorem is quite similar to the proof of Theorem 4.1, and is omitted.

**Theorem 4.14** *Suppose that for some $a > 0$, $f(x) = \frac{\phi(x)}{\sqrt{(a^2 - x^2)}}$ with $\phi \in c^k[-a - kh, a + kh]$. Then there exists such $c > 0$ that*

$$\mid T^n_{\nu^n}(f) - \int_{-a}^{a} f(x)dx \mid < \frac{c}{n^{2k-2}} \tag{124}$$

*for all sufficiently large $n$.*

The authors have been unable to construct a quadrature rule for singularities of the form (119), which is independent of the number $n$ of points used in the uncorrected trapezoidal rule. However, this is a relatively minor deficiency since the weights in such cases can be precomputed and stored.

# 5 Numerical Results

Algorithms have been implemented for the construction of the quadratures $T^n_{\beta^m}$, $T^n_{\gamma^k\beta^m}$, $T^n_{\mu^k\beta^m}$, $T^n_{\rho^k\beta^m}$, and $T^n_{\nu^n}$.
The correction coefficients $\beta^m$ are calculated using (25), and (52). In the tables in Section 7.1 the correction coefficients for orders of convergence upto 43 are tabulated. In Table 1, convergence results are presented for some of the rules $T^n_{\beta^m}$. Column 1 of this table contains the number of nodes discretizing the interval $[0,1]$ was discretized. In column 2 are the relative errors of the standard 1-sided 4th order corrected trapezoidal rule, given here for comparison. Columns 3-9 contain the relative errors for the rule $T^n_{\beta^m}$ for various orders of convergence $m$. In all cases the integrand was of the form

$$f(x) = \sin(200x) + \cos(201x). \tag{125}$$

The quadrature weights for the rules $T^n_{\gamma^k\beta^m}$, $T^n_{\mu^k\beta^m}$, $T^n_{\rho^k\beta^m}$, and $T^n_{\nu^n}$ are all obtained as solutions of linear systems, and it is easy to see that the linear systems used for determining these weights (see, for example (70), (71)) are very ill-conditioned. In order to combat the high condition number, all systems were solved using the mathematical package MAPLE using 200 significant digits.
In order to evaluate the coefficients $\gamma$ for singularities of the form $s(x) = |x|^\lambda$ or $s(x) = log(|x|)$, we start with the right-end corrected trapezoidal rule $T^n_{\beta^m}$ of order 40. Under these conditions,

$$| \alpha_i^n - \gamma_i | < O(\frac{1}{n^{40-k}}) \tag{126}$$

for all $-k \leq i \leq k, k \neq 0$ (see Theorem 4.6) and for reasonable $k$, the convergence of $\alpha_i^n$ to $\gamma_i$ is almost instantaneous. The construction of the quadrature weights $\mu_i$ is performed in a similar manner. In Section 7.2 the coefficients $\gamma_i$ are listed for the singularities $log(|x|)$, $|x|^{\frac{1}{2}}$, $|x|^{-\frac{1}{2}}$, $|x|^{\frac{1}{3}}$, $|x|^{-\frac{1}{3}}$, $|x|^{-\frac{1}{9}}$ and for the same singularities, the quadrature weights $\mu_i$ are listed in Section 7.3. In Table 2, convergence results are presented for some of the rules $T^n_{\gamma^k\beta^m}$ for various singularities. Column 1 of this table contains the number of nodes in the discretization of the interval $[0,1]$. In Table 3, convergence results are presented for some of quadrature rules $T^n_{\mu^k\beta^m}$ for various singularities. Column 1 of this table contains the number of nodes in the discretization of the interval $[-1,1]$. In all cases the integrand was of the form

$$f(x) = (\sin(20x) + \cos(21x)) + (\sin(23x) + \cos(22x))s(x), \tag{127}$$

and the order of convergence used was 10.
Finally, algorithms have been implemented for evaluating quadratures $T^n_{\rho^p h}$, to integrate functions of the form

$$f(x) = \phi(x)log(|x|). \tag{128}$$

The quadrature weights are obtained by solving the linear system (110). Note that the quadrature weights are independent of the discretization $h$, except for the first weight $\rho_0$ which is calculated using the formula (115). Presented in Table 4 are convergence results for integrating functions of the form (128) where,

$$\phi(x) = \sin(200x) + \cos(201x). \tag{129}$$

Column 1 shows the number of nodes in the discretization of the interval $[-1, 1]$. Columns 3-6 show the relative errors for the various orders of convergence $m$ as shown.

Table 1: Convergence of quadrature rules $T^n_{\beta m}$ for non-singular functions

| N | k=4 | m=3 | m=9 | m=15 | m=21 | m=27 | m=33 | m=39 |
|---|---|---|---|---|---|---|---|---|
| 20 | .230E-01 | .112E-01 | .131E-01 | .136E-01 | .138E-01 | .138E-01 | .138E-01 | .138E-01 |
| 40 | .132E-01 | .120E-01 | .122E-01 | .122E-01 | .122E-01 | .122E-01 | .122E-01 | .122E-01 |
| 80 | .457E-02 | .108E-02 | .654E-03 | .430E-03 | .292E-03 | .202E-03 | .142E-03 | .100E-03 |
| 160 | .216E-03 | .804E-04 | .223E-05 | .743E-07 | .264E-08 | .972E-10 | .365E-11 | .139E-12 |
| 320 | .310E-05 | .522E-05 | .292E-08 | .199E-11 | .116E-14 | .304E-15 | .306E-15 | .306E-15 |
| 640 | .191E-06 | .328E-06 | .304E-11 | .105E-15 | .703E-16 | .703E-16 | .703E-16 | .703E-16 |
| 1280 | .239E-07 | .205E-07 | .266E-14 | .345E-15 | .346E-15 | .346E-15 | .346E-15 | .346E-15 |

Table 2: Convergence of quadrature rules $T^n_{\gamma^k \beta m}$ for singular functions (10th order)

| N | $log(\|x\|)$ | $\|x\|^{\frac{1}{2}}$ | $\|x\|^{\frac{-1}{2}}$ | $\|x\|^{\frac{1}{3}}$ | $\|x\|^{\frac{-1}{3}}$ |
|---|---|---|---|---|---|
| 40 | 0.29128E-03 | 0.25056E-04 | 0.11650E-02 | 0.42510E-04 | 0.53715E-03 |
| 80 | 0.72599E-07 | 0.30493E-07 | 0.98819E-06 | 0.53217E-07 | 0.52449E-06 |
| 160 | 0.56928E-10 | 0.17499E-10 | 0.10903E-08 | 0.32715E-10 | 0.49582E-09 |
| 320 | 0.65586E-13 | 0.59119E-14 | 0.76827E-12 | 0.12962E-13 | 0.31491E-12 |
| 640 | 0.18596E-14 | 0.16376E-14 | 0.66613E-15 | 0.17208E-14 | 0.13878E-14 |

Table 3: Convergence of quadrature rules $T^n_{\mu^k \beta m}$ for singular functions (10th order)

| N | $log(\|x\|)$ | $\|x\|^{\frac{1}{2}}$ | $\|x\|^{\frac{-1}{2}}$ | $\|x\|^{\frac{1}{3}}$ | $\|x\|^{\frac{-1}{3}}$ |
|---|---|---|---|---|---|
| 40 | 0.57489E-03 | 0.49592E-04 | 0.23137E-02 | 0.84150E-04 | 0.10655E-02 |
| 80 | 0.14438E-06 | 0.60500E-07 | 0.19680E-05 | 0.10563E-06 | 0.10436E-05 |
| 160 | 0.11348E-09 | 0.34867E-10 | 0.21762E-08 | 0.65197E-10 | 0.98921E-09 |
| 320 | 0.13357E-12 | 0.13614E-13 | 0.15360E-11 | 0.28103E-13 | 0.62927E-12 |
| 640 | 0.61062E-15 | 0.16237E-14 | 0.42188E-14 | 0.16237E-14 | 0.50515E-14 |

Table 4: Convergence of the quadrature rule $T^n_{\rho^k \beta m}$ for functions $f(x) = \phi(x)log(|x|)$

| N | m=3 | m=9 | m=15 | m=21 | m=27 | m=33 | m=39 |
|---|---|---|---|---|---|---|---|
| 40 | .546E-01 | .536E-01 | .536E-01 | .536E-01 | .536E-01 | .536E-01 | .536E-01 |
| 80 | .291E-03 | .764E-03 | .265E-03 | .129E-03 | .640E-04 | .300E-04 | .107E-04 |
| 160 | .282E-03 | .241E-04 | .209E-05 | .255E-08 | .482E-09 | .125E-11 | .143E-13 |
| 320 | .437E-04 | .190E-04 | .912E-06 | .392E-09 | .162E-09 | .294E-12 | .147E-14 |
| 640 | .573E-05 | .315E-05 | .468E-06 | .166E-07 | .108E-09 | .583E-13 | .119E-14 |

## 6 Generalizations and Conclusions

A group of algorithms has been presented for the construction of high-order corrected trapezoidal rules for functions with various types of singularities, both end-point and in the middle of the interval of integration. In many cases, the corrected rule can have effectively an arbitrarily high order, without the attendent growths of correction weights. The drawback of the approach is the need for the integrand to be available in a small area outside the interval of integration, whenever the singularity being corrected is on one of the ends of that interval.
The algorithm of the present paper admits several straightforward generalizations.

1. There are classes of singularities not covered by this paper for which some versions of Theorem 4.1 can be fairly easily proven.
2. The quadratures can be easily modified to handle functions of the form

$$f(x) = \psi(x) + \sum_{i=1}^{m} \phi_i(x) \cdot s_i(x), \tag{130}$$

where $\psi$, $\phi_1$,$\phi_2$,...,$\phi_m$ are smooth functions, and $s_1$, $s_2$, ..., $s_m$ are several different singularities.
3. Quadrature rules developed of this paper have fairly obvious analogues in two and three dimensions. However, the proofs of the multidimensional versions of the theorems in this paper are somewhat more involved than those of their one dimesional counterparts. These results will be reported at a later date.
4. High-order corrected trapezoidal rules can be used to approximate integrals

$$\int_0^{\pi} cos(a \cos\theta) d\theta \tag{131}$$

by rewriting the integral as

$$\int_{-a}^{a} \frac{cos(x)}{\sqrt{a^2 - x^2}} dx \tag{132}$$

and using the quadrature rule $T_{\nu^n}$ defined in (120). This rule proves to be of fundamental importance in the development of the Fast Bessel Transform (see, for example [10]).
5. Integral equations of the form

$$\int_0^L \sigma(w) log(|z - w|) ds_w = C \tag{133}$$

are encountered in the study of partial differential equations (see, for example [11]). In order to apply the Nystrom algorithm to the integral equation (133), the left-hand side is decomposed into a sum

$$\int_0^L \sigma(w) log(|z - w|) ds_w = I(z) + J(z), \tag{134}$$

where the integral operators $I$ and $J$ are defined by the formulae

$$I(z) = \int_0^L \sigma(w) log(|\frac{z - w}{\gamma^{-1}(z) - \gamma^{-1}(w)}|) ds_w, \tag{135}$$

$$J(z) = \int_0^L \sigma(w) log(| \gamma^{-1}(z) - \gamma^{-1}(w) |) ds_w. \tag{136}$$

Now, the integral operator $I$ can be discretized by the uncorrected trapezoidal rule and the operator $J$ can be discretized by the corrected trapezoidal rule $T^n_{\rho h}$ defined in (109) to a rapidly convergent finite-dimesional approximation to (133).

# 7 Correction weights for Non-singular and Singular functions

## 7.1 Quadrature Weights $\beta_k^m$ for Non-singular Functions

$$\begin{aligned}\int_a^b f(x)dx &\approx T^n_{\beta^m}(f) \\ &= T_n(f) + h\sum_{k=1}^{\frac{m-1}{2}} (f(b+kh) - f(b-kh) - f(a+kh) + f(a-kh))\beta_k^m.\end{aligned}$$

| m = 3 | m = 5 | m = 7 |
|---|---|---|
| 0.4166666666666667D-01 | 0.5694444444444444D-01 | 0.6483961640211640D-01 |
| | -0.7638888888888889D-02 | -0.1395502645502646D-01 |
| | | 0.1579034391534392D-02 |

| m = 9 | m = 11 | m = 13 |
|---|---|---|
| 0.6965636022927690D-01 | 0.7289995064734647D-01 | 0.7523240913673701D-01 |
| -0.1877177028218695D-01 | -0.2247873075998076D-01 | -0.2539430387171893D-01 |
| 0.3643353174603175D-02 | 0.5728518443362193D-02 | 0.7672233851187638D-02 |
| -0.3440531305114638D-03 | -0.9618798768104324D-03 | -0.1739366039940610D-02 |
| | 0.7722834328737106D-04 | 0.2539297439987751D-03 |
| | | -0.1767014007114040D-04 |

| m = 15 | m = 17 | m = 19 |
|---|---|---|
| 0.7699017460749256D-01 | 0.7836226334784643D-01 | 0.7946301859082432D-01 |
| -0.2773799116605967D-01 | -0.2965891540255508D-01 | -0.3126001393779562D-01 |
| 0.9429999321943197D-02 | 0.1100166460634853D-01 | 0.1240262582468400D-01 |
| -0.2591615965155427D-02 | -0.3464763345380610D-02 | -0.4326893325894750D-02 |
| 0.5202578456284055D-03 | 0.8560837610996297D-03 | 0.1240963216686299D-02 |
| -0.6683840498737985D-04 | -0.1531936403942661D-03 | -0.2763550661820004D-03 |
| 0.4097355409686621D-05 | 0.1753039202853559D-04 | 0.4447195391960246D-04 |
| | -0.9595026156320693D-06 | -0.4581897491741901D-05 |
| | | 0.2263996797568645D-06 |

| m = 21 | m = 23 | m = 25 |
|---|---|---|
| 0.8036566134581083D-01 | 0.8111924751518991D-01 | 0.8175787507251367D-01 |
| -0.3261397807027540D-01 | -0.3377334140778168D-01 | -0.3477689899786187D-01 |
| 0.1365243887004996D-01 | 0.1477039637407387D-01 | 0.1577395396415406D-01 |
| -0.5160102022805384D-02 | -0.5955094025666833D-02 | -0.6707762218226974D-02 |
| 0.1657567565141616D-02 | 0.2092328816706471D-02 | 0.2535074812330083D-02 |
| -0.4325816968527443D-03 | -0.6167158739860944D-03 | -0.8233306719437802D-03 |
| 0.8735769567235570D-04 | 0.1470308086322377D-03 | 0.2231520499850693D-03 |
| -0.1275061020655204D-04 | -0.2710805091870410D-04 | -0.4885697701951313D-04 |
| 0.1193747238089644D-05 | 0.3616565358265304D-05 | 0.8277049522724384D-05 |
| -0.5374153101848776D-07 | -0.3101244008783459D-06 | -0.1016258365190328D-05 |
| | 0.1281914349299291D-07 | 0.8036239225326941D-07 |
| | | -0.3070147670921659D-08 |

| m = 27 | m = 29 | m = 31 |
|---|---|---|
| 0.8230598039728972D-01 | 0.8278153337505391D-01 | 0.8319804338077547D-01 |
| -0.3565386751750355D-01 | -0.3642664110637037D-01 | -0.3711265758638233D-01 |
| 0.1667832775003454D-01 | 0.1749655860883470D-01 | 0.1823974312884766D-01 |
| -0.7417074991466566D-02 | -0.8083781617155592D-02 | -0.8709621212955971D-02 |
| 0.2978395295604828D-02 | 0.3417018075663398D-02 | 0.3847282797776159D-02 |
| -0.1047324179282599D-02 | -0.1284180480514226D-02 | -0.1530046036007233D-02 |
| 0.3146160654817536D-03 | 0.4198855326958103D-03 | 0.5372304569083815D-03 |
| -0.7872277799802227D-04 | -0.1170025842576793D-03 | -0.1636490137583287D-03 |
| 0.1591319181836592D-04 | 0.2714748278587396D-04 | 0.4245334246577455D-04 |
| -0.2491841417488210D-05 | -0.5092371734040995D-05 | -0.9173934315347822D-05 |
| 0.2832550619442283D-06 | 0.7409483976575184D-06 | 0.1604355866780116D-05 |
| -0.2077714429849625D-07 | -0.7838889284981948D-07 | -0.2179294939201383D-06 |
| | 0.5360956533353892D-08 | 0.2155763344330162D-07 |
| | -0.1778140387386520D-09 | -0.1380750254862090D-08 |
| | | 0.4296200771869423D-10 |

| m = 33 | m = 35 | m = 37 |
|---|---|---|
| 0.8356586223906441D-01 | 0.8389305571446765D-01 | 0.8418600148964681D-01 |
| -0.3772568901686392D-01 | -0.3827675171227989D-01 | -0.3877475953008444D-01 |
| 0.1891730418359046D-01 | 0.1953724971593344D-01 | 0.2010640150771007D-01 |
| -0.9296840793733073D-02 | -0.9847903489149048D-02 | -0.1036531420894598D-01 |
| 0.4266725355474089D-02 | 0.4673760300951798D-02 | 0.5067442370362510D-02 |
| -0.1781711570625991D-02 | -0.2036550840838121D-02 | -0.2292444185955085D-02 |
| 0.6648868875120992D-03 | 0.8011551083894191D-03 | 0.9444553816549185D-03 |
| -0.2183589125884934D-03 | -0.2806529564181254D-03 | -0.3499410006344108D-03 |
| 0.6214890604463385D-04 | 0.8640764426675015D-04 | 0.1152776626902024D-03 |
| -0.1506576957398094D-04 | -0.2305218544957479D-04 | -0.3336290631509345D-04 |
| 0.3044582263334879D-05 | 0.5240846629123186D-05 | 0.8369617098659884D-05 |
| -0.4984930776645727D-06 | -0.9942016492531560D-06 | -0.1790615950589770D-05 |
| 0.6348092756603319D-07 | 0.1529838641028607D-06 | 0.3199739595444088D-06 |
| -0.5895566545002414D-08 | -0.1833269916550451D-07 | -0.4643199407153423D-07 |
| 0.3550460830740161D-09 | 0.1604311636472664D-08 | 0.5253570715177823D-08 |
| -0.1040280251184406D-10 | -0.9116340394367584D-10 | -0.4346230819394555D-09 |
| | 0.2523768794744743D-11 | 0.2337667781591708D-10 |
| | | -0.6133208535638922D-12 |

| m = 39 | m = 41 | m = 43 |
|---|---|---|
| 0.8444980879146044D-01 | 0.8468861878191560D-01 | 0.8490582345073519D-01 |
| -0.3922700061890783D-01 | -0.3963949060242128D-01 | -0.4001723785254232D-01 |
| 0.2063059004248263D-01 | 0.2111481741443320D-01 | 0.2156339227395194D-01 |
| -0.1085151806728575D-01 | -0.1130884391857241D-01 | -0.1173947578371039D-01 |
| 0.5447289134690455D-02 | 0.5813149815719777D-02 | 0.6165108551649863D-02 |
| -0.2547701211583463D-02 | -0.2800989375372994D-02 | -0.3051271143145499D-02 |
| 0.1093355313271473D-02 | 0.1246579017292300D-02 | 0.1403005122150116D-02 |
| -0.4255727119317083D-03 | -0.5068750854937796D-03 | -0.5931791433463679D-03 |
| 0.1487041779510615D-03 | 0.1865518346092672D-03 | 0.2286250628124040D-03 |
| -0.4617000028477128D-04 | -0.6158941596033656D-04 | -0.7968542809071795D-04 |
| 0.1259595810865357D-04 | 0.1806736367095093D-04 | 0.2490991825775139D-04 |
| -0.2980436293579194D-05 | -0.4659163000193157D-05 | -0.6921164516490830D-05 |
| 0.6019365929090900D-06 | 0.1042814313838009D-05 | 0.1691476513364548D-05 |
| -0.1016414607443390D-06 | -0.1993926296380813D-06 | -0.3590633249061523D-06 |
| 0.1395254130438025D-07 | 0.3190683763180230D-07 | 0.6517156581265044D-07 |
| -0.1495069020432704D-08 | -0.4154964772643378D-08 | -0.9908863701222516D-08 |
| 0.1172703286200068D-09 | 0.4227988947523140D-09 | 0.1227209106806963D-08 |
| -0.5987202298631028D-11 | -0.3152674188244618D-10 | -0.1188835069924533D-09 |
| 0.1492744845851982D-12 | 0.1531756684278896D-11 | 0.8447500588821128D-11 |
| | -0.3638111051825521D-13 | -0.3914899117784468D-12 |
| | | 0.8877720031504791D-14 |

## 7.2 Quadrature Weights $\gamma_j^k$ for Singular Functions

$$\int_0^b f(x)dx \approx T^n_{\gamma^k\beta^m}(f) = T^n_{R\beta^m}(f) + h \sum_{j=-k,j\neq 0}^{k} \gamma_j f(x_j)$$

| | $s(x) = log(x)$ | $s(x) = x^{\frac{1}{2}}$ | $s(x) = x^{-\frac{1}{2}}$ |
|---|---|---|---|
| | | k= 2 | |
| -1 | 0.7518812338640025D+00 | 0.4911169802967502D+00 | 0.1635135941723353D+01 |
| -2 | -0.6032109664493744D+00 | -0.3176980828356269D+00 | -0.1533115151360971D+01 |
| 1 | 0.1073866830872157D+01 | 0.7141080571189234D+00 | 0.2143719446940490D+01 |
| 2 | -0.7225370982867850D+00 | -0.3875269545800468D+00 | -0.1745740237302873D+01 |
| | | k= 4 | |
| -1 | 0.1420113571035790D+01 | 0.8951854542876017D+00 | 0.3192416400365587D+01 |
| -2 | -0.3125287797178819D+01 | -0.1631355661694529D+01 | -0.8349519005997507D+01 |
| -3 | 0.2592853861401367D+01 | 0.1216528022899115D+01 | 0.7653118908743808D+01 |
| -4 | -0.7648698789584314D+00 | -0.3318968291168987D+00 | -0.2415721426013858D+01 |
| 1 | 0.2027726083620572D+01 | 0.1323278097869649D+01 | 0.4127731944814846D+01 |
| 2 | -0.3730238148796624D+01 | -0.1996997843341944D+01 | -0.9431538570036398D+01 |
| 3 | 0.2914105643150046D+01 | 0.1392513231112159D+01 | 0.8285519053356245D+01 |
| 4 | -0.8344033342739005D+00 | -0.3672544720151524D+00 | -0.2562007305232722D+01 |

| k= 6 | | | |
|---|---|---|---|
| -1 | 0.2051970990601252D+01 | 0.1265469280121926D+01 | 0.4710262208645700D+01 |
| -2 | -0.7407035584542865D+01 | -0.3802563634358600D+01 | -0.2025763995934342D+02 |
| -3 | 0.1219590847580216D+02 | 0.5639024206133662D+01 | 0.3690977699143199D+02 |
| -4 | -0.1064623987147282D+02 | -0.4569107975444730D+01 | -0.3458675005305701D+02 |
| -5 | 0.4799117710681772D+01 | 0.1943368974038607D+01 | 0.1646218520818186D+02 |
| -6 | -0.8837770983721025D+00 | -0.3411137981342110D+00 | -0.3167334195084358D+01 |
| 1 | 0.2915391987686506D+01 | 0.1878261417316043D+01 | 0.6026290938505443D+01 |
| 2 | -0.8797979464048396D+01 | -0.4649333971499730D+01 | -0.2274216675280301D+02 |
| 3 | 0.1365562914252423D+02 | 0.6444550155059975D+01 | 0.3978973181300623D+02 |
| 4 | -0.1157975479644601D+02 | -0.5048462684259424D+01 | -0.3656337403895339D+02 |
| 5 | 0.5130987287355766D+01 | 0.2104363245869803D+01 | 0.1720419649716102D+02 |
| 6 | -0.9342187797694916D+00 | -0.3644552148433214D+00 | -0.3285178657691059D+01 |

| k= 8 | | | |
|---|---|---|---|
| -1 | 0.2661829001135098D+01 | 0.1616169645940613D+01 | 0.6202998068889192D+01 |
| -2 | -0.1336900704886964D+02 | -0.6771767050468779D+01 | -0.3714709770899691D+02 |
| -3 | 0.3292331764210170D+02 | 0.1503196947284841D+02 | 0.1012860584122768D+03 |
| -4 | -0.4773939140223472D+02 | -0.2024989176835058D+02 | -0.1577736812789053D+03 |
| -5 | 0.4288580615706955D+02 | 0.1717995995110646D+02 | 0.1497778690096803D+03 |
| -6 | -0.2359187584186291D+02 | -0.9018058251167396D+01 | -0.8617211496827355D+02 |
| -7 | 0.7312948709041004D+01 | 0.2686335493243228D+01 | 0.2773685303768452D+02 |
| -8 | -0.9817367313018633D+00 | -0.3483500116200692D+00 | -0.3846246456428401D+01 |
| 1 | 0.3760781014023317D+01 | 0.2398992474897278D+01 | 0.7870429343373961D+01 |
| 2 | -0.1580903864167977D+02 | -0.8260181779465771D+01 | -0.4150717430533848D+02 |
| 3 | 0.3674321491528176D+02 | 0.1714292235263991D+02 | 0.1088399244984859D+03 |
| 4 | -0.5179306469244793D+02 | -0.2233476105127601D+02 | -0.1663887812447046D+03 |
| 5 | 0.4575621781632506D+02 | 0.1857536706216344D+02 | 0.1562272759566466D+03 |
| 6 | -0.2489478606121209D+02 | -0.9622728690582360D+01 | -0.8923488368760573D+02 |
| 7 | 0.7656685336983747D+01 | 0.2839683305088209D+01 | 0.2857613653609836D+02 |
| 8 | -0.1021900172352320D+01 | -0.3656611549965858D+00 | -0.3947565212882627D+01 |

| k= 10 | | | |
|---|---|---|---|
| -1 | 0.3256353919777872D+01 | 0.1953545360705999D+01 | 0.7677722423353747D+01 |
| -2 | -0.2096116396850468D+02 | -0.1050311310076629D+02 | -0.5894517227637276D+02 |
| -3 | 0.6872858265408605D+02 | 0.3105516048922884D+02 | 0.2140398605114418D+03 |
| -4 | -0.1393153744796911D+03 | -0.5850644296241638D+02 | -0.4662332548976578D+03 |
| -5 | 0.1874446431742073D+03 | 0.7437254291687940D+02 | 0.6631353162140867D+03 |
| -6 | -0.1715855846429547D+03 | -0.6498918498319249D+02 | -0.6351002576675097D+03 |
| -7 | 0.1061953812152787D+03 | 0.3866979933460322D+02 | 0.4083227672169233D+03 |
| -8 | -0.4269031893958787D+02 | -0.1502289586232686D+02 | -0.1696285390723725D+03 |
| -9 | 0.1009036069527147D+02 | 0.3445119980743215D+01 | 0.4126838241810020D+02 |
| -10 | -0.1066655310499552D+01 | -0.3544413204640886D+00 | -0.4476202232026015D+01 |
| 1 | 0.4576078100790908D+01 | 0.2895451608911961D+01 | 0.9675787330957780D+01 |
| 2 | -0.2469045273524281D+02 | -0.1277820188943208D+02 | -0.6561769910673283D+02 |
| 3 | 0.7648830198138171D+02 | 0.3534092272477722D+02 | 0.2294242274362024D+03 |
| 4 | -0.1508194558089468D+03 | -0.6441908403427060D+02 | -0.4907643918974356D+03 |
| 5 | 0.1996415730837827D+03 | 0.8029833065236247D+02 | 0.6906485447124722D+03 |
| 6 | -0.1807965537141134D+03 | -0.6926226351772149D+02 | -0.6568499770824342D+03 |
| 7 | 0.1110467735366555D+03 | 0.4083390088012690D+02 | 0.4202275815793937D+03 |
| 8 | -0.4438764193424203D+02 | -0.1575467189373152D+02 | -0.1739340651258045D+03 |
| 9 | 0.1044548196545488D+02 | 0.3593677332216888D+01 | 0.4219582451243715D+02 |
| 10 | -0.1100328792904271D+01 | -0.3681517162342983D+00 | -0.4566454997023116D+01 |

| | $s(x) = x^{\frac{1}{3}}$ | $s(x) = x^{-\frac{1}{3}}$ | $s(x) = x^{-\frac{9}{10}}$ |
|---|---|---|---|
| | k= 2 | | |
| -1 | 0.5534091724301567D+00 | 0.1181425202719417D+01 | 0.9469239981678674D+01 |
| -2 | -0.3866961728429464D+00 | -0.1060178333186577D+01 | -0.9440762908621185D+01 |
| 1 | 0.8032238407479816D+00 | 0.1613104391254726D+01 | 0.1027199835611538D+02 |
| 2 | -0.4699368403351921D+00 | -0.1234351260787565D+01 | -0.9800475429172870D+01 |

| | | | |
|---|---|---|---|
| | k= 4 | | |
| -1 | 0.1020832071388625D+01 | 0.2282486199885223D+01 | 0.1887299140127902D+02 |
| -2 | -0.1983186102544885D+01 | -0.5650813876770368D+01 | -0.5533585657332243D+02 |
| -3 | 0.1533381243224831D+01 | 0.5015176492677874D+01 | 0.5446669568113802D+02 |
| -4 | -0.4298392181270347D+00 | -0.1549698701260824D+01 | -0.1798256931439028D+02 |
| 1 | 0.1498331817034082D+01 | 0.3083643341213459D+01 | 0.2031702799746152D+02 |
| 2 | -0.2412699293646369D+01 | -0.6532393248536034D+01 | -0.5722348457297536D+02 |
| 3 | 0.1747071103625803D+01 | 0.5514329562635926D+01 | 0.5565641900092846D+02 |
| 4 | -0.4738916209550530D+00 | -0.1662729769845256D+01 | -0.1827122362011895D+02 |

| | | | |
|---|---|---|---|
| | k= 6 | | |
| -1 | 0.1453673785846622D+01 | 0.3345150279872830D+01 | 0.2823540877500425D+02 |
| -2 | -0.4645097217879781D+01 | -0.1359274618599862D+02 | -0.1374572775395727D+03 |
| -3 | 0.7134681085431341D+01 | 0.2395745376553084D+02 | 0.2701409207262959D+03 |
| -4 | -0.5928340311544518D+01 | -0.2193610831631847D+02 | -0.2668390934875709D+03 |
| -5 | 0.2571788299099303D+01 | 0.1025817941642386D+02 | 0.1321842198719930D+03 |
| -6 | -0.4588965281604129D+00 | -0.1946039989983884D+01 | -0.2624585302793210D+02 |
| 1 | 0.2135823382632839D+01 | 0.4476264293482232D+01 | 0.3025125337714398D+02 |
| 2 | -0.5636852769577275D+01 | -0.1561643865091390D+02 | -0.1418090842954654D+03 |
| 3 | 0.8109443112743051D+01 | 0.2622632514046215D+02 | 0.2756109143783281D+03 |
| 4 | -0.6522597930471829D+01 | -0.2345727234258157D+02 | -0.2708086499560764D+03 |
| 5 | 0.2775248991033700D+01 | 0.1081909216811830D+02 | 0.1337381187138368D+03 |
| 6 | -0.4888738991530396D+00 | -0.2033859578093758D+01 | -0.2650087753598454D+02 |

| | $s(x) = x^{\frac{1}{3}}$ | $s(x) = x^{-\frac{1}{3}}$ | $s(x) = x^{-\frac{9}{10}}$ |
|---|---|---|---|
| | k= 8 | | |
| -1 | 0.1866196808675184D+01 | 0.4383819645513359D+01 | 0.3756991225931813D+02 |
| -2 | -0.8307135206229368D+01 | -0.2478753951112947D+02 | -0.2556568200490798D+03 |
| -3 | 0.1909144688191794D+02 | 0.6535630997043997D+02 | 0.7531560640068682D+03 |
| -4 | -0.2636153756127239D+02 | -0.9943323854718145D+02 | -0.1239060653686887D+04 |
| -5 | 0.2279974850816623D+02 | 0.9269762740745608D+02 | 0.1226707735965091D+04 |
| -6 | -0.1215894117169713D+02 | -0.5255448820422732D+02 | -0.7300983324043684D+03 |
| -7 | 0.3671126621978929D+01 | 0.1670932578919686D+02 | 0.2417364444480294D+03 |
| -8 | -0.4816958588438264D+00 | -0.2292774564229746D+01 | -0.3433771074231191D+02 |
| 1 | 0.2736714477854559D+01 | 0.5819143597164960D+01 | 0.4011575033483495D+02 |
| 2 | -0.1004886340865174D+02 | -0.2833732911493431D+02 | -0.2633133438634569D+03 |
| 3 | 0.2164374286136325D+02 | 0.7130006863921132D+02 | 0.7675751360752552D+03 |
| 4 | -0.2894336017656429D+02 | -0.1060508801269256D+03 | -0.1256487025699375D+04 |
| 5 | 0.2456068654847544D+02 | 0.9756101768474124D+02 | 0.1240341295810031D+04 |
| 6 | -0.1293400965739323D+02 | -0.5482984235409941D+02 | -0.7368055436813910D+03 |
| 7 | 0.3870327870200545D+01 | 0.1732510108672746D+02 | 0.2436290214530279D+03 |
| 8 | -0.5044475379801205D+00 | -0.2366321397723911D+01 | -0.3457193022558553D+02 |

| k= 10 | | | |
|---|---|---|---|
| -1 | 0.2264788460960479D+01 | 0.5405454633516052D+01 | 0.4688376828974556D+02 |
| -2 | -0.1292939169279749D+02 | -0.3917131575943378D+02 | -0.4098330186123370D+03 |
| -3 | 0.3957101757244672D+02 | 0.1375220761115852D+03 | 0.1609252630383255D+04 |
| -4 | -0.7639785056816069D+02 | -0.2925218664728695D+03 | -0.3705300977598581D+04 |
| -5 | 0.9898237584503627D+02 | 0.4085024389395215D+03 | 0.5500852053333245D+04 |
| -6 | -0.8785457780822613D+02 | -0.3854463863750540D+03 | -0.5454835999290140D+04 |
| -7 | 0.5297228202020211D+02 | 0.2447281804852179D+03 | 0.3611006247703252D+04 |
| -8 | -0.2081786990425939D+02 | -0.1005754937354166D+03 | -0.1538237727142339D+04 |
| -9 | 0.4823072180507942D+01 | 0.2423820803863685D+02 | 0.3825346496620566D+03 |
| -10 | -0.5007874588446741D+00 | -0.2606987282704575D+01 | -0.4230611481851793D+02 |
| 1 | 0.3311620888787288D+01 | 0.7126578020279918D+01 | 0.4993063308066224D+02 |
| 2 | -0.1559126529305869D+02 | -0.4460041915515531D+02 | -0.4215785445878369D+03 |
| 3 | 0.4475277794263923D+02 | 0.1496138805763186D+03 | 0.1638730173971623D+04 |
| 4 | -0.8371954827689160D+02 | -0.3113382992063869D+03 | -0.3755162938068600D+04 |
| 5 | 0.1064593309138400D+03 | 0.4292142311074982D+03 | 0.5559352058138100D+04 |
| 6 | -0.9333016061951656D+02 | -0.4015725721227424D+03 | -0.5502789929350287D+04 |
| 7 | 0.5578209928784346D+02 | 0.2534431424358416D+03 | 0.3638060903851112D+04 |
| 8 | -0.2177894078450347D+02 | -0.1036930201058000D+03 | -0.1548279140995316D+04 |
| 9 | 0.5020172381264958D+01 | 0.2490333391795860D+02 | 0.3847470160018897D+03 |
| 10 | -0.5191450872697767D+00 | -0.2671164050811178D+01 | -0.4252574395098780D+02 |

## 7.3 Quadrature Weights $\mu_j^k$ for Singular Functions

$$\int_{-b}^{b} f(x)dx \approx T^n_{\mu_l \beta^m}(f)$$
$$= T^n_{R\beta^m}(f) + T^n_{L\beta^m}(f) + h\sum_{j=1}^{l} \mu_j(f(x_j) + f(x_{-j}))$$

| | $s(x) = log(x)$ | $s(x) = x^{\frac{1}{2}}$ | $s(x) = x^{-\frac{1}{2}}$ |
|---|---|---|---|
| k= 1 | | | |
| 1 | 0.1825748064736159D+01 | 0.1205225037415674D+01 | 0.3778855388663843D+01 |
| 2 | -0.1325748064736159D+01 | -0.7052250374156737D+00 | -0.3278855388663843D+01 |
| k= 2 | | | |
| 1 | 0.3447839654656362D+01 | 0.2218463552157251D+01 | 0.7320148345180434D+01 |
| 2 | -0.6855525945975443D+01 | -0.3628353505036473D+01 | -0.1778105757603391D+02 |
| 3 | 0.5506959504551413D+01 | 0.2609041254011273D+01 | 0.1593863796210005D+02 |
| 4 | -0.1599273213232332D+01 | -0.6991513011320512D+00 | -0.4977728731246580D+01 |
| k= 3 | | | |
| 1 | 0.4967362978287758D+01 | 0.3143730697437969D+01 | 0.1073655314715114D+02 |
| 2 | -0.1620501504859126D+02 | -0.8451897605858329D+01 | -0.4299980671214643D+02 |
| 3 | 0.2585153761832639D+02 | 0.1208357436119364D+02 | 0.7669950880443822D+02 |
| 4 | -0.2222599466791883D+02 | -0.9617570659704153D+01 | -0.7115012409201039D+02 |
| 5 | 0.9930104998037539D+01 | 0.4047732219908410D+01 | 0.3366638170534288D+02 |
| 6 | -0.1817995878141594D+01 | -0.7055690129775324D+00 | -0.6452512852775417D+01 |

| k= 4 | | | |
|---|---|---|---|
| 1 | 0.6422610015158415D+01 | 0.4015162120837891D+01 | 0.1407342741226315D+02 |
| 2 | -0.2917804569054941D+02 | -0.1503194882993455D+02 | -0.7865427201433540D+02 |
| 3 | 0.6966653255738346D+02 | 0.3217489182548832D+02 | 0.2101259829107628D+03 |
| 4 | -0.9953245609468264D+02 | -0.4258465281962659D+02 | -0.3241624625236100D+03 |
| 5 | 0.8864202397339461D+02 | 0.3575532701326990D+02 | 0.3060051449663269D+03 |
| 6 | -0.4848666190307500D+02 | -0.1864078694174975D+02 | -0.1754069986558793D+03 |
| 7 | 0.1496963404602475D+02 | 0.5526018798331437D+01 | 0.5631298957378288D+02 |
| 8 | -0.2003636903654183D+01 | -0.7140111666166550D+00 | -0.7793811669311027D+01 |

| k= 5 | | | |
|---|---|---|---|
| 1 | 0.7832432020568779D+01 | 0.4848996969617959D+01 | 0.1735350975431153D+02 |
| 2 | -0.4565161670374749D+02 | -0.2328131499019837D+02 | -0.1245628713831056D+03 |
| 3 | 0.1452168846354678D+03 | 0.6639608321400605D+02 | 0.4434640879476441D+03 |
| 4 | -0.2901348302886379D+03 | -0.1229255269966870D+03 | -0.9569976467950934D+03 |
| 5 | 0.3870862162579900D+03 | 0.1546708735692419D+03 | 0.1353783860926559D+04 |
| 6 | -0.3523821383570680D+03 | -0.1342514485009140D+03 | -0.1291950234749944D+04 |
| 7 | 0.2172421547519342D+03 | 0.7950370021473013D+02 | 0.8285503487963169D+03 |
| 8 | -0.8707796087382989D+02 | -0.3077756775605837D+02 | -0.3435626041981771D+03 |
| 9 | 0.2053584266072635D+02 | 0.7038797312960103D+01 | 0.8346420693053734D+02 |
| 10 | -0.2166984103403823D+01 | -0.7225930366983869D+00 | -0.9042657229049132D+01 |

| | $s(x) = x^{\frac{1}{3}}$ | $s(x) = x^{-\frac{1}{3}}$ |
|---|---|---|
| k= 1 | | |
| 1 | 0.1356633013178138D+01 | 0.2794529593974142D+01 |
| 2 | -0.8566330131781384D+00 | -0.2294529593974142D+01 |

| k= 2 | | |
|---|---|---|
| 1 | 0.2519163888422707D+01 | 0.5366129541098682D+01 |
| 2 | -0.4395885396191253D+01 | -0.1218320712530640D+02 |
| 3 | 0.3280452346850634D+01 | 0.1052950605531380D+02 |
| 4 | -0.9037308390820877D+00 | -0.3212428471106080D+01 |

| k= 3 | | |
|---|---|---|
| 1 | 0.3589497168479460D+01 | 0.7821414573355062D+01 |
| 2 | -0.1028194998745706D+02 | -0.2920918483691252D+02 |
| 3 | 0.1524412419817439D+02 | 0.5018377890599299D+02 |
| 4 | -0.1245093824201635D+02 | -0.4539338065890004D+02 |
| 5 | 0.5347037290133003D+01 | 0.2107727158454216D+02 |
| 6 | -0.9477704273134525D+00 | -0.3979899568077642D+01 |

| k= 4 | | |
|---|---|---|
| 1 | 0.4602911286529744D+01 | 0.1020296324267832D+02 |
| 2 | -0.1835599861488110D+02 | -0.5312486862606378D+02 |
| 3 | 0.4073518974328119D+02 | 0.1366563786096513D+03 |
| 4 | -0.5530489773783668D+02 | -0.2054841186741071D+03 |
| 5 | 0.4736043505664168D+02 | 0.1902586450921973D+03 |
| 6 | -0.2509295082909035D+02 | -0.1073843305583267D+03 |
| 7 | 0.7541454492179474D+01 | 0.3403442687592432D+02 |
| 8 | -0.9861433968239469D+00 | -0.4659095961953657D+01 |

| k= 5 | | |
|---|---|---|
| 1 | 0.5576409349747767D+01 | 0.1253203265379597D+02 |
| 2 | -0.2852065698585619D+02 | -0.8377173491458909D+02 |
| 3 | 0.8432379551508595D+02 | 0.2871359566879038D+03 |
| 4 | -0.1601173988450523D+03 | -0.6038601656792564D+03 |
| 5 | 0.2054417067588763D+03 | 0.8377166700470196D+03 |
| 6 | -0.1811847384277427D+03 | -0.7870189584977965D+03 |
| 7 | 0.1087543813080456D+03 | 0.4981713229210595D+03 |
| 8 | -0.4259681068876286D+02 | -0.2042685138412166D+03 |
| 9 | 0.9843244561772901D+01 | 0.4914154195659545D+02 |
| 10 | -0.1019932546114451D+01 | -0.5278151333515754D+01 |

## 7.4 Quadrature Weights $\rho_j^k$ for Singular Functions

$$\int_{-b}^{b} f(x)dx \approx T^n_{\rho^p\beta^m}(f)$$
$$= T^n_{R\beta^m}(f) + T^n_{L\beta^m}(f) + h\sum_{j=0}^{l} \rho_j(\phi(x_j) + \phi(x_{-j}))$$

Note: $\rho_0$ is given for $h = 0.01$, For any other $h$, the following formula is used to calculate $\rho_0$.

$$\rho_0 = (-.918938533204672741 7803 + 0.5log(h)) - \sum_{j=1}^{p} \rho_j \quad (137)$$

| m = 3 | m = 5 | m = 7 |
|---|---|---|
| -0.3221523626198730D+01 | -0.3191075169140337D+01 | -0.3181467102013171D+01 |
| | -0.3044845705839327D-01 | -0.4325921322794724D-01 |
| | | 0.3202689042388491D-02 |

| m = 9 | m = 11 | m = 13 |
|---|---|---|
| -0.3176811195217937D+01 | -0.3174071153542312D+01 | -0.3172268092036274D+01 |
| -0.5024307342079858D-01 | -0.5462714010179898D-01 | -0.5763224261186158D-01 |
| 0.5996233119529027D-02 | 0.8188266460029230D-02 | 0.9905467894350714D-02 |
| -0.4655906795234226D-03 | -0.1091885919666338D-02 | -0.1735836457536894D-02 |
| | 0.7828690501786438D-04 | 0.2213870245446548D-03 |
| | | -0.1431001195267904D-04 |

| m = 15 | m = 17 | m = 19 |
|---|---|---|
| -0.3170992165916170D+01 | -0.3170041916020681D+01 | -0.3169306861514305D+01 |
| -0.5981954453204053D-01 | -0.6148248184914681D-01 | -0.6278924541603596D-01 |
| 0.1127253159446256D-01 | 0.1238115647253341D-01 | 0.1329589096935583D-01 |
| -0.2343420324253269D-02 | -0.2897732763288696D-02 | -0.3396678852464559D-02 |
| 0.4036621845595672D-03 | 0.6052303442088134D-03 | 0.8131245480320894D-03 |
| -0.4745095013720858D-04 | -0.9784299004952013D-04 | -0.1618104373797589D-03 |
| 0.2761744848710794D-05 | 0.1051436637368180D-04 | 0.2422167651587582D-04 |
| | -0.5537586803550720D-06 | -0.2381400032647607D-05 |
| | | 0.1142275845182835D-06 |

| m = 21 | m = 23 | m = 25 |
|---|---|---|
| -0.3168721384920306D+01 | -0.3168244070581230D+01 | -0.3167847493678468D+01 |
| -0.6384310328523506D-01 | -0.6471094753809971D-01 | -0.6543800519316396D-01 |
| 0.1406233305604607D-01 | 0.1471321624569456D-01 | 0.1527249136497475D-01 |
| -0.3843770069700536D-02 | -0.4244313571022683D-02 | -0.4603847576274231D-02 |
| 0.1019474340602541D-02 | 0.1219746091263614D-02 | 0.1411497560731106D-02 |
| -0.2355067918692057D-03 | -0.3156154921336350D-03 | -0.3995067600256630D-03 |
| 0.4387403771306165D-04 | 0.6890800654569580D-04 | 0.9851668933111745D-04 |
| -0.6066217757119951D-05 | -0.1195656336479857D-04 | -0.2018119747186014D-04 |
| 0.5477355521032650D-06 | 0.1529459820049702D-05 | 0.3260961737325821D-05 |
| -0.2408377597694342D-07 | -0.1274231726028842D-06 | -0.3871484601943021D-06 |
| | 0.5166969831297037D-08 | 0.2990271150667017D-07 |
| | | -0.1124351894335142D-08 |

| m = 27 | m = 29 | m = 31 |
|---|---|---|
| -0.3167512774719844D+01 | -0.3167226492889164D+01 | -0.3166978846772530D+01 |
| -0.6605594788600917D-01 | -0.6658761414298577D-01 | -0.6704988689403577D-01 |
| 0.1575801776649599D-01 | 0.1618335077207727D-01 | 0.1655894738230538D-01 |
| -0.4927531843955059D-02 | -0.5219948285292188D-02 | -0.5485075304276741D-02 |
| 0.1593569961301572D-02 | 0.1765579632676354D-02 | 0.1927601699833581D-02 |
| -0.4851878897058822D-03 | -0.5711927253932730D-03 | -0.6564674975812873D-03 |
| 0.1318371286512027D-03 | 0.1680496910458935D-03 | 0.2064233385304999D-03 |
| -0.3070344146767653D-04 | -0.4337783830581833D-04 | -0.5799637068090648D-04 |
| 0.5891522736279919D-05 | 0.9512778975749005D-05 | 0.1416413018600433D-04 |
| -0.8882076980903206D-06 | -0.1711220479787840D-05 | -0.2924616447680532D-05 |
| 0.9822897121976360D-07 | 0.2413616289062888D-06 | 0.4941524555505996D-06 |
| -0.7065765782430224D-08 | -0.2495734799324587D-07 | -0.6540388025633560D-07 |
| 0.2475589120039617D-09 | 0.1678885488869214D-08 | 0.6345793057687260D-08 |
| | -0.5505102218712507D-10 | -0.4007478791366100D-09 |
| | | 0.1234631631962446D-10 |

| m = 33 | m = 35 | m = 37 |
|---|---|---|
| -0.3166762511619708D+01 | -0.3166571903740287D+01 | -0.3166402692325675D+01 |
| -0.6745551530557688D-01 | -0.6781430660801560D-01 | -0.6813392816895060D-01 |
| 0.1689299430945688D-01 | 0.1719198706148916D-01 | 0.1746114206017127D-01 |
| -0.5726331418330602D-02 | -0.5946641867196491D-02 | -0.6148508116208072D-02 |
| 0.2079973982393914D-02 | 0.2223175774156742D-02 | 0.2357753273497796D-02 |
| -0.7402722529894703D-03 | -0.8221018482825148D-03 | -0.9016249160749561D-03 |
| 0.2463303649153491D-03 | 0.2872451625618713D-03 | 0.3287354588014059D-03 |
| -0.7432197238379932D-04 | -0.9211101483880897D-04 | -0.1111274006152623D-03 |
| 0.1984260034353227D-04 | 0.2651349126416089D-04 | 0.3412004557474223D-04 |
| -0.4580836910292848D-05 | -0.6715522004894009D-05 | -0.9348560035479858D-05 |
| 0.8916453665775555D-06 | 0.1466368276662483D-05 | 0.2246527693132364D-05 |
| -0.1418448246845963D-06 | -0.2695610269256914D-06 | -0.4646008810431617D-06 |
| 0.1767037741742959D-07 | 0.4047684210333943D-07 | 0.8082991536902293D-07 |
| -0.1614096203394717D-08 | -0.4759815470416764D-08 | -0.1148532768136401D-07 |
| 0.9602551109604562D-10 | 0.4105974377982503D-09 | 0.1278405465017250D-08 |
| -0.2789306492547372D-11 | -0.2308426950559283D-10 | -0.1044412720573741D-09 |
| | 0.6342175941576707D-12 | 0.5564945021538352D-11 |
| | | -0.1450213949229612D-12 |

| m = 39 | m = 41 | m = 43 |
|---|---|---|
| -0.3166251466775081D+01 | -0.3166115504658825D+01 | -0.3166115504658825D+01 |
| -0.6842046079112800D-01 | -0.6867878881201404D-01 | -0.6867878881201404D-01 |
| 0.1770469478902206D-01 | 0.1792611880692437D-01 | 0.1792611880692437D-01 |
| -0.6334072100094389D-02 | -0.6505172477564357D-02 | -0.6505172477564357D-02 |
| 0.2484274171602103D-02 | 0.2603300521146428D-02 | 0.2603300521146428D-02 |
| -0.9786376366601862D-03 | -0.1053029105125390D-02 | -0.1053029105125390D-02 |
| 0.3704506824517389D-03 | 0.4121099047922528D-03 | 0.4121099047922528D-03 |
| -0.1311507079674222D-03 | -0.1519803191376791D-03 | -0.1519803191376791D-03 |
| 0.4259144483911754D-04 | 0.5184904980367620D-04 | 0.5184904980367620D-04 |
| -0.1248611531858182D-04 | -0.1612303155465844D-04 | -0.1612303155465844D-04 |
| 0.3255027605557997D-05 | 0.4509136652480968D-05 | 0.4509136652480968D-05 |
| -0.7428077534364396D-06 | -0.1119040467513331D-05 | -0.1119040467513331D-05 |
| 0.1457448522607878D-06 | 0.2428371655709533D-06 | 0.2428371655709533D-06 |
| -0.2404950901525398D-07 | -0.4528845255185268D-07 | -0.4528845255185268D-07 |
| 0.3241558798437558D-08 | 0.7103184896000958D-08 | 0.7103184896000958D-08 |
| -0.3423992518658962D-09 | -0.9102854426840433D-09 | -0.9102854426840433D-09 |
| 0.2656123735758442D-10 | 0.9146251630822981D-10 | 0.9146251630822981D-10 |
| -0.1344809528411308D-11 | -0.6753249440965090D-11 | -0.6753249440965090D-11 |
| 0.3332744815245408D-13 | 0.3256755515337396D-12 | 0.3256755515337396D-12 |
| | -0.7693371141612778D-14 | -0.7693371141612778D-14 |

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