It is well known that Jacobi polynomials $P_n^{(\alpha,\beta)}$ form complete orthogonal systems in the weighted space $L^2_{\alpha,\beta}[-1,1]$ whenever $\alpha, \beta > -1$. On the other hand, in certain physical applications, (for example, in the angular momentum calculations in quantum mechanics), there naturally occur polynomials $P_n^{(\alpha,-k)}$ with integer α and k. We show that for any $\alpha > -1$ and integer $k \geq 1$ the Jacobi polynomials $P_n^{(\alpha,-k)}$ $(n = k, k+1, \cdots)$ form complete orthogonal systems in $L^2_{\alpha,-k}[-1,1]$ with the weight $w_{\alpha,-k}(x) = (1-x)^\alpha \cdot (1+x)^{-k}$. In addition, for $\alpha \geq 0$ and $n \geq 1$ we obtain an upper bound on $[-1,1]$ for the function $((1-x)/2)^{\alpha/2} \cdot P_n^{(\alpha,-1)}(x)$, which is similar to the well known bound for the function $((1-x)/2)^{\alpha/2+1/4} \cdot P_n^{(\alpha,0)}(x)$.

On the Jacobi Polynomials $P_n^{(\alpha,-k)}$

Gregory Matviyenko
Research Report YALEU/DCS/RR–1076
June 13, 1995

This research was supported in part by the National Science Foundation under Grant DMS-901213595, and in part by the Office of Naval Research under Grant N00014–89–J–1527.
Approved for public release: distribution is unlimited.
Keywords: *Jacobi Polynomials* $P_n^{(\alpha,-k)}$, *Upper Bounds.*

1 Introduction

Jacobi polynomials $P_n^{(\alpha,\beta)}$ play an important role in pure and applied mathematics, numerical analysis, physics, and engineering. It is well known that all other classical polynomials orthogonal on $[-1,1]$ with appropriately chosen weights (Legendre, Chebyshev, Gegenbaurer), as well as certain combinations of elementary functions, are particular cases of Jacobi polynomials (see, for example, Askey [2]).

Normally one assumes that $\alpha, \beta > -1$, in part due to the well known facts (see, for example Chapts. 3 and 4 of Szegö [6], Lecture 2 of Askey [2]) summarized in Theorems 1.1 and 1.2 below. In Theorem 1.2 and elsewhere in the paper $L^2_{\alpha,\beta}[-1,1]$ denotes the weighted space with the weight function $w_{\alpha,\beta} : [-1,1] \to \mathbb{R}$ defined by the formula

$$w_{\alpha,\beta}(x) \stackrel{\text{def}}{=} (1-x)^{\alpha}(1+x)^{\beta}. \tag{1}$$

Theorem 1.1. *For any integer $n, m \geq 0$ and arbitrary real $\alpha, \beta \geq -1$,*

$$\int_{-1}^{1} w_{\alpha,\beta}(x) \cdot P_n^{(\alpha,\beta)}(x) \cdot P_m^{(\alpha,\beta)}(x)dx = \delta_{nm} h^n_{\alpha,\beta}, \tag{2}$$

where δ_{nm} is Kroneker's delta, and

$$h^n_{\alpha,\beta} \stackrel{\text{def}}{=} \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \cdot \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!}.\bullet \tag{3}$$

Theorem 1.2. *Let $\alpha, \beta > -1$ and suppose, that for an arbitrary function $f \in L^2_{\alpha,\beta}[-1,1]$ the coefficients f_n are defined by the formula*

$$f_n \stackrel{\text{def}}{=} \int_{-1}^{1} w_{\alpha,\beta}(x) P_n^{(\alpha,\beta)}(x) f(x)dx. \tag{4}$$

Then in $L^2_{\alpha,\beta}[-1,1]$,

$$f = \sum_{n=0}^{\infty} \frac{f_n}{h^n_{\alpha,\beta}} P_n^{(\alpha,\beta)}.\bullet \tag{5}$$

On the other hand, polynomials $P_n^{(\alpha,-k)}$ (for integer α and k) occur naturally in certain physical applications (see, for example, Chapts. 3 and 5 of Biedenharn and Louck [3] and references herein), which stimulates interest in this type of Jacobi polynomials.

In this paper we generalize Theorems 1.1 and 1.2 for the case of negative integer β. Namely we show that for all $k \geq 1$ and $\alpha > -1$ any sequence $\{P_n^{(\alpha,-k)}\}$ $(n = k, k+1, \cdots)$ forms an orthogonal complete system in $L^2_{\alpha,-k}[-1,1]$ (see Theorems 1.4 and 1.5 below). This property is a consequence of the observation that the polynomials $\{P_n^{(\alpha,-k)}\}$ for all $n \geq k$ have a zero of exactly k–th order at $x = -1$ (see Theorem 1.3 and Corollary 1.1 below), so that the relevant inner products involving $w_{\alpha,-k}$ and $P_n^{(\alpha,-k)}$ (i. e. analogues of integrals (2) and (4)) exist.

In addition, we obtain a uniform upper bound for the function $((1-x)/2)^{\alpha/2} \cdot P_n^{(\alpha,-1)}(x)$ on $[-1,1]$ (see Theorem 1.6 below). This bound is similar to the well known Szegö's bound for the function $((1-x)/2)^{\alpha/2+1/4} \cdot P_n^{(\alpha,0)}(x)$ (see formula (23) below). Upper bounds for Jacobi polynomials are of significant interest in certain applications (see, for example, Chap. 7 of Szegö [6], Nevai, Erdélyi, and Magnus [5], Elbert and Laforgia [4], and references herein). Note that most upper bounds for the polynomials $P_n^{(\alpha,\beta)}$ have been derived for the case $\alpha, \beta \geq -1/2$.

The plan and main results of the paper are as follows.

Section 2 contains relevant mathematical facts to be used in the remainder of the paper.

In Section 3 we establish a formula connecting the polynomials $P_n^{(\alpha,-k)}$ and $P_{n-k}^{(\alpha,k)}$, and prove the completeness of the system $\{P_n^{(\alpha,-k)}\}$. The main results of this section are Theorems 1.3, 1.4 and 1.5, and Corollary 1.1 below.

Theorem 1.3. *For any integer n and k such that $n \geq k$, and arbitrary real $\alpha > -1$,*

$$P_n^{(\alpha,-k)}(x) = \frac{\Gamma(n+\alpha+1)\Gamma(n-k+1)}{\Gamma(n-k+\alpha+1)\Gamma(n+1)} \cdot \left(\frac{1+x}{2}\right)^k \cdot P_{n-k}^{(\alpha,k)}(x). \bullet \qquad (6)$$

Corollary 1.1. *For all $\alpha > -1$ and $k \leq n$, $P_n^{(\alpha,-k)}$ has the zero of k–th order at $x = -1$. The remaining $n-k$ zeroes of $P_n^{(\alpha,-k)}$ are located in the interior of the interval* $[-1,1]$. $\bullet$

Remark 1.1. The formula (6) for integer α is known and widely used in the rotation group computations in quantum mechanics (see, for example, Chap. 3 of Biedenharn and Louck [3]) $\bullet$.

Remark 1.2. Combining (6) with the formula (21) below we can rewrite the relation

(6) in the form

$$P_n^{(\alpha,-k)}(x) = \left(\frac{1+x}{2}\right)^k \cdot \frac{P_n^{(\alpha,k)}(1)}{P_{n-k}^{(\alpha,k)}(1)} \cdot P_{n-k}^{(\alpha,k)}(x).\bullet \tag{7}$$

Theorem 1.4. *For any integer k, n, and m such that $n, m \geq k$, and arbitrary $\alpha > -1$,*

$$\int_{-1}^{1} w_{\alpha,-k}(x) \cdot P_n^{(\alpha,-k)}(x) \cdot P_m^{(\alpha,-k)}(x) = \delta_{nm} h_{\alpha,-k}^n, \tag{8}$$

where the function $w_{\alpha,\beta}$ is defined in (1), and

$$h_{\alpha,-k}^n = h_{\alpha,\beta}^n\Big|_{\beta=-k} = \frac{2^{\alpha-k+1}}{2n+\alpha-k+1} \cdot \frac{\Gamma(n+\alpha+1)(n-k)!}{\Gamma(n-k+\alpha+1)n!}.\bullet \tag{9}$$

Theorem 1.5. *Let n and k be integers such that $n \geq k$, and suppose that $\alpha > -1$. Suppose further, that for an arbitrary function $f \in L^2_{\alpha,-k}[-1,1]$ the coefficients f_n are defined by the formula*

$$f_n \stackrel{\text{def}}{=} \int_{-1}^{1} w_{\alpha,-k}(x) P_n^{(\alpha,-k)}(x) f(x) dx. \tag{10}$$

Then in $L^2_{\alpha,-k}[-1,1]$,

$$f = \sum_{n=k}^{\infty} \frac{f_n}{h_{\alpha,-k}^n} P_n^{(\alpha,-k)}.\bullet \tag{11}$$

In Section 4 we obtain an upper bound for the function $((1-x)/2)^{\alpha/2} \cdot P_n^{(\alpha,-1)}(x)$ on $[-1,1]$. The main result of this section is Theorem 1.6 below.

Theorem 1.6. *For all $\alpha \geq 0$ and integer $n \geq 1$,*

$$\max_{x\in[-1,1]} \left(\frac{1-x}{2}\right)^{\alpha/2} \left|P_n^{(\alpha,-1)}(x)\right| \leq 8\left(\frac{e}{3\pi}\right)^{1/2}.\bullet \tag{12}$$

Remark 1.3. Substituting $x = \cos\theta$ we can rewrite (12) in the form

$$\max_{\theta\in[0,\pi]} \left(\sin\frac{\theta}{2}\right)^{\alpha} \left|P_n^{(\alpha,-1)}(\cos\theta)\right| \leq 8\left(\frac{e}{3\pi}\right)^{1/2}.\bullet \tag{13}$$

2 Relevant Mathematical Facts

All formulae of this section that are given without a reference can be found in Chapts. 6 and 22 of Abramowitz and Stegun [1].

2.1 The Gamma Function

The gamma function Γ: $\mathbb{C} \to \mathbb{C}$ is an analytic function for all arguments $x \in \mathbb{C}$, save for the points $x = 0, -1, -2, \cdots$, where it has simple poles. The function $1/\Gamma$: $\mathbb{C} \to \mathbb{C}$ is an analytic function for all arguments $x \in \mathbb{C}$, and

$$1/\Gamma(-n) = 0 \quad \text{for all } n = 0, 1, \cdots. \tag{14}$$

For all $x > 0$ the function Γ can be written in the form

$$\Gamma(1 + x) = (2\pi)^{1/2} \cdot x^{x+1/2} \exp(-x + \vartheta/x), \tag{15}$$

where $0 < \vartheta < 1/12$.

On $[1, 2]$ the function Γ has the unique minimum

$$\gamma_0 \stackrel{\text{def}}{=} \min_{x \in [1,2]} \Gamma(x) = 0.8856031 \cdots. \tag{16}$$

2.2 Jacobi Polynomials

Jacobi polynomials $P_n^{(\alpha,\beta)}$ can be defined by the Rodrigues formula

$$P_n^{(\alpha,\beta)}(x) \stackrel{\text{def}}{=} \frac{(-1)^n}{2^n n!}(1-x)^{-\alpha}(1+x)^{-\beta}\frac{d^n}{dx^n}\left((1-x)^{\alpha+n}(1+x)^{\beta+n}\right), \tag{17}$$

and for $\alpha, \beta > -1$ they have the following explicit form:

$$P_n^{(\alpha,\beta)}(x) = \left(\frac{x-1}{2}\right)^n \times$$

$$\sum_{m=0}^{n} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha-m+1)\Gamma(m+\beta+1)m!(n-m)!}\left(\frac{x+1}{x-1}\right)^m. \tag{18}$$

Below we give certain relevant well known equalities for Jacobi polynomials.

$$(2n+\alpha+\beta)P_n^{(\alpha,\beta-1)}(x) = (n+\alpha+\beta)P_n^{(\alpha,\beta)}(x) + (n+\alpha)P_{n-1}^{(\alpha,\beta)}(x), \tag{19}$$

$$\frac{d}{dx}\left((1-x)^\alpha(1+x)^\beta P_n^{(\alpha,\beta)}(x)\right) = -2(n+1)(1-x)^{\alpha-1}(1+x)^{\beta-1}P_{n+1}^{(\alpha-1,\beta-1)}(x), \tag{20}$$

$$P_n^{(\alpha,\beta)}(1) = (-1)^n P_n^{(\beta,\alpha)}(-1) = \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)n!}. \tag{21}$$

Finally we cite three inequalities for the polynomials $P_n^{(\alpha,\beta)}$; they are valid for all integer $n \geq 0$, $\alpha, \beta \geq -1/2$, and $-1 \leq x \leq 1$.

$$\left|P_n^{(\alpha,\beta)}(x)\right| \leq \max\left[\left|P_n^{(\alpha,\beta)}(1)\right|, \left|P_n^{(\alpha,\beta)}(-1)\right|\right], \tag{22}$$

$$\left(\frac{1-x}{2}\right)^{\alpha/2+1/4}\left|P_n^{(\alpha,0)}(x)\right| \leq 1, \tag{23}$$

$$(1-x)^{\alpha/2+1/4}(1+x)^{\beta/2+1/4}\left|P_n^{(\alpha,\beta)}(x)\right| \leq$$

$$\left(\frac{2e}{\pi}\right)^{1/2}\left(2+\left(\alpha^2+\beta^2\right)^{1/2}\right)^{1/2}\left(h_{\alpha,\beta}^n\right)^{1/2}, \tag{24}$$

where $h_{\alpha,\beta}^n$ is defined in (3). The inequality (22) is standard (see, for example, Chap. 22 of Abramowitz and Stegun [1]), the inequality (23) can be found in Chap. 7 of Szegö [6], and the inequality (24) was recently obtained by Nevai, Erdélyi, and Magnus [5].

3 Orthogonality and Completeness of Systems of Polynomials $\{P_n^{(\alpha,-k)}\}$

In this section we prove Theorems 1.3, 1.4, and 1.5. Note that Corollary 1.1 immediately follows from (6) and Theorem 3.3.1 of Szegö [6].

Proof of Theorem 1.3. We begin with the observation that for any integer k and m such that $k \leq n$ and $m < k$,

$$\frac{\Gamma(n-k+1)}{\Gamma(m-k+1)(n-m)!} = 0, \tag{25}$$

which is an immediate consequence of (14).

Next, an inspection of the formula (18) shows that for any fixed $x \in \mathbb{C}$ and $\alpha > -1$, Jacobi polynomials are analytic functions of $\beta \in \mathbb{C}$. Therefore by the principle of analytic continuation we can substitute $\beta = -k$ into (18) which in combination with (25) yields

$$P_n^{(\alpha,-k)}(x) = \left(\frac{x-1}{2}\right)^n \times$$

$$\sum_{m=k}^{n} \frac{\Gamma(n+\alpha+1)\Gamma(n-k+1)}{\Gamma(n+\alpha-m+1)\Gamma(m-k+1)m!(n-m)!} \left(\frac{x+1}{x-1}\right)^m . \qquad (26)$$

Substituting $m = k + l$ into (26) we have

$$P_n^{(\alpha,-k)}(x) = \left(\frac{x-1}{2}\right)^{n-k} \left(\frac{x+1}{2}\right)^k \times$$

$$\sum_{l=0}^{n-k} \frac{\Gamma(n+\alpha+1)\cdot\Gamma(n-k+1)}{\Gamma(n-k+\alpha-l+1)\cdot\Gamma(l+k+1)\cdot l!\cdot(n-k-l)!} \left(\frac{x+1}{x-1}\right)^l . \qquad (27)$$

Now (6) is an immediate consequence of (18) and (27). •

Lemma 3.1 below can by easily proven by combining (3), (9), and (21); this result will be used in the proofs of Theorems 1.4 and 1.5.

Lemma 3.1 *For any $k \leq n$ and $\alpha > -1$,*

$$h_{\alpha,-k}^{n+k} = \left(\frac{P_{n+k}^{(\alpha,k)}(1)}{2^k P_n^{(\alpha,k)}(1)}\right)^2 h_{\alpha,k}^n . \bullet \qquad (28)$$

Proof of Theorem 1.4. Substituting (7) into the left hand side of (8) and using (2) we have

$$\int_{-1}^{1} w_{\alpha,-k}(x)\cdot P_n^{(\alpha,-k)}(x)\cdot P_m^{(\alpha,-k)}(x)dx =$$

$$\frac{1}{2^{2k}}\frac{P_n^{(\alpha,k)}(1)}{P_{n-k}^{(\alpha,k)}(1)}\cdot\frac{P_m^{(\alpha,k)}(1)}{P_{m-k}^{(\alpha,k)}(1)}\int_{-1}^{1} w_{\alpha,k}(x)\cdot P_{n-k}^{(\alpha,k)}(x)\cdot P_{m-k}^{(\alpha,k)}(x)dx =$$

$$\left(\frac{P_n^{(\alpha,k)}(1)}{2^k P_{n-k}^{(\alpha,k)}(1)}\right)^2 h_{\alpha,k}^{n-k}\delta_{nm}. \qquad (29)$$

Combining (28) and (29) we immediately obtain (8). •

Proof of Theorem 1.5. We begin with an observation that if $f \in L^2_{\alpha,-k}[-1,1]$ then $\tilde{f} \in L^2_{\alpha,k}[-1,1]$, where the function $\tilde{f}$ is defined by the formula

$$\tilde{f}(x) \stackrel{\text{def}}{=} \frac{f(x)}{(1+x)^k}. \tag{30}$$

By Theorem 1.2 we have in $L^2_{\alpha,k}[-1,1]$

$$\tilde{f} = \sum_{n=0}^{\infty} \frac{\tilde{f}_n}{h^n_{\alpha,k}} P_n^{(\alpha,k)}, \tag{31}$$

where

$$\tilde{f}_n \stackrel{\text{def}}{=} \int_{-1}^{1} w_{\alpha,k}(x) P_n^{(\alpha,k)}(x) \tilde{f}(x) dx. \tag{32}$$

Combining (32) with (7) and (30) we obtain

$$\tilde{f}_n = 2^k \frac{P_n^{(\alpha,k)}(1)}{P_{n+k}^{(\alpha,k)}(1)} \int_{-1}^{1} w_{\alpha,-k}(x) P_{n+k}^{(\alpha,-k)}(x) f(x) dx = 2^k \frac{P_n^{(\alpha,k)}(1)}{P_{n+k}^{(\alpha,k)}(1)} f_{n+k}. \tag{33}$$

Next, the substitution of (33) into (31) in combination with (7) and (30) yields

$$f = \sum_{n=0}^{\infty} \left(\frac{2^k P_n^{(\alpha,k)}(1)}{P_{n+k}^{(\alpha,k)}(1)} \right)^2 \frac{a_{n+k}}{h^n_{\alpha,k}} P_{n+k}^{(\alpha,-k)}, \tag{34}$$

and now (11) immediately follows from (28) and (34). •

4 An Upper Bound for Functions $((1-x)/2)^{\alpha/2} P_n^{(\alpha,-1)}(x)$

In this section we prove Theorem 1.6. It is carried out by means of considering five regions of parameters n, x, and α, obtaining an upper bound for each region separately, and finally choosing the largest such bound as a uniform upper bound for the function $((1-x)/2)^{\alpha/2} \left| P_n^{(\alpha,-1)}(x) \right|$ ($\alpha \geq 0$, $n \geq 1$) on $[-1,1]$. Throughout the proof of the theorem we will use the notation

$$x_0 \stackrel{\text{def}}{=} 1 - \left(\frac{2+\alpha}{2n+\alpha-1} \right)^2. \tag{35}$$

We begin with two preliminary results summarized in Lemmas 4.1 and 4.2. below. Their proofs are immediate consequences of (19) and (24), respectively.

Lemma 4.1. *For any $x \in \mathbb{C}$, $n \geq 1$ and $\alpha > -1$,*

$$\left|P_n^{(\alpha,-1)}(x)\right| \leq \frac{n+\alpha}{2n+\alpha}\left(\left|P_n^{(\alpha,0)}(x)\right| + \left|P_{n-1}^{(\alpha,0)}(x)\right|\right). \tag{36}$$

Lemma 4.2. *For any $0 \leq x \leq 1$ and $\alpha \geq -1/2$,*

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,0)}(x)\right| \leq 2\left(\frac{e}{\pi}\right)^{1/2}\left(\frac{2+\alpha}{2n+\alpha+1}\right)^{1/2}\frac{1}{(1-x)^{1/4}} \tag{37}$$

Proof of Theorem 1.6.

Region 1. We define this region by the inequalities

$$\alpha \geq 0, \tag{38}$$

$$n \geq 1, \tag{39}$$

$$-1 \leq x \leq 0. \tag{40}$$

From (23) and (40) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,0)}(x)\right| \leq \left(\frac{1-x}{2}\right)^{-1/4} \leq 2^{1/4}, \tag{41}$$

which in combination with (36) yields

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,-1)}(x)\right| \leq 2^{5/4} = 2.3784\cdots. \tag{42}$$

Region 2. We define this region by the inequalities

$$\alpha \geq 0, \tag{43}$$

$$n = 1, \tag{44}$$

$$0 < x \leq 1. \tag{45}$$

The formulae (6) and (18) yield

$$P_1^{(\alpha,-1)}(x) = (1+\alpha)\left(\frac{1+x}{2}\right) P_0^{(\alpha,1)}(x) = (1+\alpha)\left(\frac{1+x}{2}\right), \tag{46}$$

and now combining (46) with (43) and (45) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_1^{(\alpha,-1)}(x)\right| = (1+\alpha)\left(\frac{1+x}{2}\right)\left(\frac{1-x}{2}\right)^{\alpha/2} \le \max_{\alpha\ge 0}(1+\alpha)2^{-\alpha/2} = 1.5011\cdots. \tag{47}$$

Region 3. We define this region by the inequalities

$$\alpha \ge 0, \tag{48}$$

$$n \ge 2, \tag{49}$$

$$0 < x \le x_0, \tag{50}$$

where x_0 is defined in (35).

Combining (37) and (50) we can write

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,0)}(x)\right| \le 2\left(\frac{e}{\pi}\right)^{1/2}\left(\frac{2+\alpha}{2n+\alpha+1}\right)^{1/2}\frac{1}{(1-x_0)^{1/4}}, \tag{51}$$

and substituting (35) into (51) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,0)}(x)\right| \le 2\left(\frac{e}{\pi}\right)^{1/2}. \tag{52}$$

Similarly,

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_{n-1}^{(\alpha,0)}(x)\right| \le 2\left(\frac{e}{\pi}\right)^{1/2}. \tag{53}$$

Now substituting (52) and (53) into (36) we obtain

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,-1)}(x)\right| \le 4\left(\frac{e}{\pi}\right)^{1/2} = 3.7207\cdots. \tag{54}$$

Region 4. We define this region by the inequalities

$$0 \le \alpha \le 1, \tag{55}$$

$$n \ge 2, \tag{56}$$

$$x_0 < x \le 1. \tag{57}$$

Combining (21), (22), and (57) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2} \left|P_n^{(\alpha,0)}(x)\right| \le \left(\frac{1-x_0}{2}\right)^{\alpha/2} \frac{\Gamma(n+\alpha+1)}{\Gamma(\alpha+1)\Gamma(n+1)}. \tag{58}$$

Substituting (15) and (35) into (58) and using (16) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2} \left|P_n^{(\alpha,0)}(x)\right| \le \frac{1}{\gamma_0} \left(\frac{2+\alpha}{2^{1/2}}\right)^{\alpha} \left(\frac{\alpha+n}{2n+\alpha-1}\right)^{\alpha} \left(1+\frac{\alpha}{n}\right)^{n+1/2} \times$$

$$\exp(-\alpha)\exp(\vartheta_1/(\alpha+n) - \vartheta_2/n), \tag{59}$$

where $0 < \vartheta_1, \vartheta_2 < 1/12$.

Next, one can easily verify that for all $n \ge 2$ and $0 \le \alpha \le 1$,

$$\left(\frac{2+\alpha}{2^{1/2}}\right)^{\alpha} \le \frac{3}{2^{1/2}}, \tag{60}$$

$$\left(\frac{\alpha+n}{2n+\alpha-1}\right)^{\alpha} \le 1, \tag{61}$$

$$\left(1+\frac{\alpha}{n}\right)^{n} \le \exp(\alpha), \tag{62}$$

$$\left(1+\frac{\alpha}{n}\right)^{1/2} \le \left(\frac{3}{2}\right)^{1/2}, \tag{63}$$

$$\exp(\vartheta_1/(\alpha+n) - \vartheta_2/n) \le \exp(1/24), \tag{64}$$

and

$$\frac{n+\alpha}{2n+\alpha} \le \frac{3}{5}. \tag{65}$$

Now combining (36) with (59–65) we have

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,-1)}(x)\right| \le \frac{3^{5/2}}{10\gamma_0}\exp(1/24) = 1.8350\cdots. \tag{66}$$

Region 5. We define this region by the inequalities

$$\alpha > 1, \tag{67}$$

$$n \ge 2, \tag{68}$$

$$x_0 < x \le 1. \tag{69}$$

The formula (20) yields

$$(1-x)^\alpha(1+x)P_n^{(\alpha,1)}(x) = 2(n+1)\int_x^1 (1-t)^{\alpha-1}P_{n+1}^{(\alpha-1,0)}(t)dt, \tag{70}$$

while from (6) we have

$$P_n^{(\alpha,1)}(x) = \frac{2(n+1)}{(1+x)(n+\alpha+1)}P_{n+1}^{(\alpha,-1)}(x). \tag{71}$$

Combining (70) and (71) we obtain

$$(1-x)^\alpha P_n^{(\alpha,-1)}(x) = (n+\alpha)\int_x^1 (1-t)^{\alpha-1}P_n^{(\alpha-1,0)}(t)dt. \tag{72}$$

Next, combining (37) and (72) we have

$$\begin{aligned}
&\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,-1)}(x)\right| \le \\
&2^{-1/2}(1-x)^{-\alpha/2}(n+\alpha)\int_x^1 \left[\left(\frac{1-t}{2}\right)^{\alpha/2-1/2}\left|P_n^{(\alpha-1,0)}(t)\right|\right](1-t)^{\alpha/2-1/2}dt \le \\
&2^{1/2}(1-x)^{-\alpha/2}(n+\alpha)\left(\frac{e}{\pi}\right)^{1/2}\left(\frac{1+\alpha}{2n+\alpha}\right)^{1/2}\int_x^1 (1-t)^{\alpha/2-3/4}dt.
\end{aligned} \tag{73}$$

Integration in (73) in combination with (69) produces

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,-1)}(x)\right| \le 2^{1/2}\left(\frac{e}{\pi}\right)^{1/2}\frac{n+\alpha}{\alpha/2+1/4}\left(\frac{1+\alpha}{2n+\alpha}\right)^{1/2}(1-x)^{1/4} \le$$

$$2^{1/2}\left(\frac{e}{\pi}\right)^{1/2}\frac{n+\alpha}{\alpha/2+1/4}\left(\frac{1+\alpha}{2n+\alpha}\right)^{1/2}(1-x_0)^{1/4}, \tag{74}$$

which after the substitution of x_0 from (35) becomes

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,-1)}(x)\right| \le 2^{1/2}\left(\frac{e}{\pi}\right)^{1/2}\frac{n+\alpha}{\alpha/2+1/4}\left(\frac{1+\alpha}{2n+\alpha}\right)^{1/2}\left(\frac{2+\alpha}{2n+\alpha-1}\right)^{1/2}. \tag{75}$$

For all $n \ge 2$ and $\alpha > 1$ we have

$$\frac{(1+\alpha)^{1/2}(2+\alpha)^{1/2}}{\alpha/2+1/4} \le \frac{4}{3}6^{1/2}, \tag{76}$$

and

$$\frac{n+\alpha}{(2n+\alpha)^{1/2}(2n+\alpha-1)^{1/2}} \le 1. \tag{77}$$

Finally, substituting (76) and (77) into (75) we obtain

$$\left(\frac{1-x}{2}\right)^{\alpha/2}\left|P_n^{(\alpha,-1)}(x)\right| \le 8\left(\frac{e}{3\pi}\right)^{1/2} = 4.2963\cdots. \tag{78}$$

Now the conclusion of the theorem is a consequence of (42), (47), (54), (66), and (78). •

5 Conclusions and Generalizations

We have proven the orthogonality and completeness of the system $\{P_n^{(\alpha,-k)}\}$ and obtained an upper bound for the function $((1-x)/2)^{\alpha/2}P_n^{(\alpha,-1)}(x)$. These results can be extended in the following two directions.

First, it appears that there exist analogues of Theorems 1.3, 1.4, and 1.5 for Laguerre polynomials L_n^α with $\alpha = -1, -2, \cdots$.

Second, one can try to obtain an inequality sharper than (12). Our numerical experiments indicate that the constant $8(e/3\pi)^{1/2} = 4.29\cdots$ in (12) is not optimal.

This work is currently in progress and its results will be reported at a later date.

References

[1] M. Abramowitz and I. A. Stegun (Eds.), "Handbook of Mathematical Functions," National Bureau of Standards, Washington, DC, 1964.

[2] R. Askey, "Orthogonal Polynomials and Special Functions", SIAM, Philadelphia, PA, 1975

[3] L. C. Biedenharn and J. D. Louck, "Angular Momentum in Quantum Physics: Theory and Application", Addison–Wesley Publishing Co., Reading, MA, 1981

[4] A. Elbert and A. Laforgia, An Inequality for Legendre Polynomials, *J. Math. Phys.* **35** (1994), 13448–1360.

[5] P. Nevai, T. Erdélyi, and A. P. Magnus, Generalized Jacobi Weights, Christoffel Functions, and Jacobi Polynomials, *SIAM J. Math. Anal.* **25** (1994), 602–614.

[6] G. Szegö, "Orthogonal Polynomials", Amer. Math. Soc. Colloq. Publ. 23, American Mathematical Society, Providence, RI, 1939, forth ed., 1975.