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CONVEX HULLS OF f- AND β-VECTORS

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ABSTRACT. In this paper we describe the convex hulls of the sets of f- and β-vectors of different classes of simplicial complexes on n vertices. These include flag complexes, order posets, matroid complexes and general abstract simplicial complexes. As a result of this investigation, standard linear programming problems on these sets can be solved, including maximization of the Euler characteristics or of the sum of the Betti numbers.

1. INTRODUCTION

In this paper we investigate extremal questions concerning f- and β-vectors for different classes of abstract simplicial complexes. The classes treated here are:

1. flag (also knows as clique) complexes on n vertices;
2. abstract simplicial complexes on n vertices.

Consider for example a flag complex on n vertices. Its f-vector (or β-vector) is a point on the integer grid $\mathbb{Z}^n$ in $\mathbb{R}^n$. One could ask questions like: what is the maximum (or minimum) of the Euler characteristic on such complexes, what is the maximum of the sum of the Betti numbers or just of β_{11}? Extremal problems of this type have been treated thoroughly in the literature. For example Theorem 1.4 in [BK] answers the question: *what is the maximum of the Euler characteristic and the sum of the Betti numbers of an abstract simplicial complex.* Some other papers where similiar questions have been treated are [Koz],[Mar],[Re], [SYZ],[Z]. Relations between f- and β-vectors were studied, see for example [BK],[May]. Another direction of research has been to investigate convex hulls associated to Sperner families with different conditions imposed. Some of this work can be found in [DGH],[EE],[EFK1] [EFK2],[EFK3],[En],[KS]. This research is relevant here, because of the fact that β-vectors of simplicial complexes with at most $n+1$ vertices form exactly the same set as f-vectors of Sperner families on the set $\{1, 2, \ldots, n\}$.

One unifying approach to this kind of problems would be to consider them all as standard problems of the linear programming. Namely, given some set of points in $\mathbb{R}^n$, optimize (i.e. maximize or minimize) some linear function on this set.

Obviously finding the convex hull of this set of points would settle all questions of that type in one move. Because once we know the points in the corners of this convex hull, then in order to prove some linear inequality it suffices to check it for these extremal points only. For example we get a different proof for the result of Björner and Kalai cited above.

In sections 2,3 and 4 we will find the convex hulls of the sets of f- and β-vectors for the classes of complexes described above. The corresponding problems

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for matroid complexes and complexes associated to finite posets follow at once from our results.

At the end we give a conjecture, concerning convex hulls of f- and β-vectors for r-colorable complexes.

2. Basic notations and definitions

Let G be a simple graph. We denote by $E(G)$ the set of its edges and by $V(G)$ the set of its vertices. We say that $S \subseteq V(G)$ forms a **clique** if the corresponding induced subgraph is complete. Let $c_k(G)$ be the number of k-cliques of G, then the vector $(c_1, c_2, \ldots, c_n)$ is called the **clique vector** of G.

If G_1 and G_2 are two graphs with disjoint vertex sets then $G = G_1 \oplus G_2$ will denote the graph defined by

$$V(G) = V(G_1) \cup V(G_2)$$

and

$$E(G) = E(G_1) \cup E(G_2) \cup \{(x, y) | x \in V(G_1),\, y \in V(G_2)\}$$

We say that G is an **r-partite complete graph** (of type $(k_1, \ldots, k_r)$), if

$$G = A_1 \oplus A_2 \oplus \cdots \oplus A_r$$

where $E(A_i) = \emptyset$ and $|V(A_i)| = k_i$ for all $i = 1, \ldots, r$.

We say that a poset P is of **level type** $(p_1, \ldots, p_r)$ if $P \simeq \hat{0} \oplus p_1 1 \oplus \cdots \oplus p_r 1 \oplus \hat{1}$, where $\oplus$ denotes the ordered sum, and $k1$ denotes an antichain consisting of k elements. Sometimes we just say that a poset has level type. For any inquiries concerning posets we refer to Chapter 3 of [S].

Finally a few words about our terminology concerning algebraic topology. Let C be an abstract simplicial complex. Let f_k denote the number of faces of dimension k, and β_k denote the kth Betti number of C (in this paper we will consider reduced homology only), then we call the vector $(f_0, \ldots, f_{n-1})$ **f-vector** of C and the vector $(\beta_0, \ldots, \beta_{n-1})$ **β-vector** of C.

We say that a complex C is a **complete k-skeleton** if C has all possible faces up to cardinality k and no faces of cardinality $k+1$ and more.

Let x be some vertex of C, then $\mathrm{st}(x)$ and $\mathrm{lk}(x)$ are simplicial subcomplexes of C defined by

$$\mathrm{st}(x) = \{X \in C | X \cup \{x\} \in C\},\ \mathrm{lk}(x) = \{X \in C | x \notin X,\, X \cup \{x\} \in C\}$$

From now on, n will always denote the fixed number of vertices in our graph or complex.

3. The case of flag complexes

Definition 3.1. *Let us take an undirected graph G on n vertices. Define an abstract simplicial complex C associated to this graph in the following way: we take the set of vertices of the graph as the set on which we define our simplicial complex and we say that a collection of vertices forms a side if and only if the corresponding collection of vertices of G forms a complete subgraph (or a* **clique***). The abstract simplicial complexes obtained in this way are called* **flag (clique) complexes**. *We will denote such complexes by $C(G)$.*

An example of flag complexes is provided by the complexes associated to posets (having a finite poset P, we take its elements as vertices of a complex and the sets forming chains as sides).

Definition 3.2. *We call a graph G an* r**th Turán graph on** n **vertices** *or a* **Turán graph number** r *if G is a complete r-partite graph with sizes of the maximal independent sets as equal as possible. We will denote this graph by $T_r(n)$ or just T_r.*

The Turán graphs come up in different contexts all over extremal graph theory and are optimal in many senses (one can find a nice survey in [B]). In our case they turn out to determine the corners in the convex hulls of the sets of f-vectors and β-vectors of flag complexes.

Let us denote the f- and β-vectors of $T_r(n)$ by $F_r(n)$ (or just F_r) and $B_r(n)$ (or just B_r) respectively. Then

$$F_r(n)_{i-1} = \sum_{1\le j_1<\cdots<j_i\le r} k_{j_1}\dots k_{j_i}, \text{ where } k_i = \left\lceil \frac{n+i-1}{r} \right\rceil$$

and

$$B_r(n)_{r-1} = \prod_{i=1}^{r}(k_i - 1), \quad B_r(n)_i = 0,\ i \neq r-1$$

We will need an operation on graphs, which we call compression. Its poset version has previously been used in [Koz, Z]. In the context of graphs it appears in for example [MM]. Let x and y be vertices in a graph G with no edge in between and let $\{x_1,\dots,x_m\}$ be the set of vertices having an edge to x, and let $\{y_1,\dots,y_k\}$ be the set of vertices having an edge to y, then we define an (x,y)**-compression** (or an y **to** x **compression**) of G as a graph G^* given by:

$$E(G^*) = (E(G)\setminus\{(y,y_1),\dots,(y,y_k)\})\cup\{(y,x_1),\dots,(y,x_m)\}$$

Let G_x and G_y be the subgraphs of G induced by $\{x,x_1,\dots,x_m\}$, resp. $\{y,y_1,\dots,y_k\}$. We denote $C=C(G)$, $C^*=C(G^*)$, $C_x=C(G_x)$ and $C_y=C(G_y)$ then the (x,y)-compression changes the f-vector linearly, namely

$$f(C^*) = f(C) + f(C_x) - f(C_y) \tag{3.1}$$

So if l is a linear function on f-vectors then

$$l(f(C^*)) = l(f(C)) + l(f(C_x)) - l(f(C_y)) \tag{3.2}$$

If G^{**} is the (y,x)-compression of G and $C^{**}=C(G^{**})$ then either $l(f(C^*)) \ge l(f(C))$ or $l(f(C^{**})) \ge l(f(C))$, thus if $l(f(C))$ is maximal then $l(f(C)) = l(f(C^*)) = l(f(C^{**}))$ and we can compress however we want to, preserving the value of the function l. We use this observation in the proof of the next theorem.

Theorem 3.3. *The convex hull of f-vectors of flag complexes on n vertices is given by* $conv\{F_1, F_2, \dots, F_n\}$.

Proof: Choose a linear function l and a complex C where l achieves its maximal value. Pick some point x and let $\{y_1,\dots,y_m\}$ be the points with no edge to x. Then the argument above allows us to compress these points to x and thus obtain a graph G^*, such that

$$G^* = G_1 \oplus \{x, y_1, \dots, y_m\}$$

where $G_1 = G\setminus\{x,y_1,\dots,y_m\}$. Now we can continue compressing G_1 and so on. When the process terminates, we end up with a complete k-partite graph where l achieves its maximum.

We prove now that we actually can obtain a Turán graph. Assume that our k-partite graph is not a Turán graph, then we can find two maximal independent sets of sizes a and b such that $a-b \geq 2$. Let $p_1, \ldots, p_a$ and $q_1, \ldots, q_b$ be the elements of these independent sets. Form a new graph T by reconnecting the vertex p_a, we erase its old connections and connect it to $p_1, \ldots, p_b$ and all other vertices in the graph which are outside of the two considered independent sets. Then the clique vector of T is the same as that of G, since the links of p_a are the same in both graphs and $T \setminus \{p_a\}$ is equal to $G \setminus \{p_a\}$. Now we can shift p_a to q_1 and obtain a new k-partite graph with independent sets of more equal sizes and the value of l the same as that of G. Continuing in that way we eventually end up with a Turán graph. □

Theorem 3.4. *The convex hull of β-vectors of flag complexes on n vertices is given by $conv\{B_1, B_2, \ldots, B_n\}$.*

Proof: Let again l be a linear function (though this time in the space of β-vectors) and let C be a flag complex where l achieves its maximum, we denote the underlying graph by G. To apply the same kind of argument as before we need to know how our compression operation influences l. Observe that the simplex $conv\{B_1, B_2, \ldots, B_n\}$ is obtained by cutting the positive cone (i.e. the cone defined by $x_i \geq 0,\ i = 1, \ldots, n$) in $\mathbb{R}^n$ by a hyperplane with a positive normal vector. Hence to prove the statement of the theorem it would be enough to show that the B_i's maximize all linear functions with non-negative coefficients, i.e. we can assume that

$$l(\beta_0, \ldots, \beta_{n-1}) = \sum_{k=0}^{n-1} \alpha_k \beta_k, \text{ where } \alpha_k \geq 0.$$

Let x and y be vertices of G. We introduce the following notions: $G_1 = G \setminus \{y\}$, $C_1 = C \setminus \{y\}$ and let G^* (resp. C^*) be the result of the y to x compression of G (resp. C).

First let us note that C is a union of the two simplicial complexes C_1 and $\operatorname{st}(y)$. The intersection of these two simplicial complexes is obviously $\operatorname{lk}(y)$, hence we get a Mayer-Vietoris sequence

$$\ldots \longrightarrow H_k(C_1) \longrightarrow H_k(C) \longrightarrow H_{k-1}(\operatorname{lk}(y)) \longrightarrow \ldots \longrightarrow H_0(C_1) \longrightarrow H_0(C) \longrightarrow 0.$$

Note that we used the fact that $\operatorname{st}(y)$ is a cone and hence has trivial homology groups. The sequence above is an exact sequence, hence looking at its short subsequences of the type

$$\ldots \longrightarrow H_k(C_1) \longrightarrow H_k(C) \longrightarrow H_{k-1}(\operatorname{lk}(y)) \longrightarrow \ldots$$

we can conclude that

$$\beta_k(C) \leq \beta_k(C_1) + \beta_{k-1}(\operatorname{lk}(y)).$$

Summing it up with coefficients α_i (which are ≥ 0 !) we obtain

$$l(\beta(C)) \leq l(\beta(C_1)) + l'(\beta(\operatorname{lk}(y))) \tag{3.3}$$

where l' is obtained from l by shifting the arguments by one.

On the other hand we can apply the same argument to the complex C^*, which is a union of C_1 and a cone (namely $\operatorname{st}(x)$, where x is exchanged to y) with the intersection $\operatorname{lk}(x)$. The only difference will be that all the mappings

$$i_k\ :\ H_k(\operatorname{lk}(x)) \longrightarrow H_k(C_1)$$

are zero mappings, because $\mathrm{lk}(x)$ is mapped inside a cone in C_1. Hence instead of inequalities as above we get exact equalities for the Betti numbers of C^*, namely

$$l(\beta(C^*)) = l(\beta(C_1)) + l'(\beta(\mathrm{lk}(x))). \tag{3.4}$$

If $l'(\beta(\mathrm{lk}(x))) > l'(\beta(\mathrm{lk}(y)))$ then according to equations 3.3 and 3.4 we get

$$l(\beta(C^*)) > l(\beta(C))$$

which is a contradiction to our assumption of the optimality of C. If $l'(\beta(\mathrm{lk}(x))) < l'(\beta(\mathrm{lk}(y)))$, then an x to y compression increases the value of l, which again yields a contradiction. Hence $l'(\beta(\mathrm{lk}(x))) = l'(\beta(\mathrm{lk}(y)))$ and using equations 3.3 and 3.4 again we see that we are free to shift whichever way we want. So we are back in the situation of the previous proof.

Shifting the same way as above we end up with a complete k-partite graph, say of type $(t_1, \ldots, t_k)$. Then all the Betti numbers are equal to 0, except for the $k-1$th one which is equal to $\prod_{i=1}^{k}(t_i - 1)$. Hence the corresponding β-vector gives a point on the kth coordinate axis which is inside our simplex since the maximum of such a product is achieved when t_i's are as equal as possible, i.e. for Turán graphs. □

Corollary 3.5. *The convex hulls of f- and β-vectors of complexes on n vertices associated with posets are given by conv$\{F_1, F_2, \ldots, F_n\}$ and conv$\{B_1, B_2, \ldots, B_n\}$ respectively.*

Proof: The clique complex of a Turán graph is the same as the order complex of a level type poset. On the other hand if we have a poset P, we can associate to it a graph G, by taking the elements of P as vertices of G and connecting two vertices with an edge whenever the corresponding elements are comparable. Then the order complex of P translates obviously to the clique complex of G. We can conclude that the set of order complexes is a subset of the set of all flag complexes and it contains the complexes associated to Turán graphs, hence the convex hulls of f- and β-vectors must be the same. □

Note. Corollary 3.5 generalizes Theorem 4.4 in [Koz] and Theorem 2.5 in [Z].

4. The case of simplicial complexes

In this section we consider the same problem as above for the class of simplicial complexes on n vertices.

First we introduce some notation. Let S_k be the complete k-skeleton complex on n vertices (S_1 will denote a complex with no faces except for vertices). We denote the f- and β-vectors of S_k by $F_k(n)$ and $B_k(n)$ respectively. Then

$$F_k(n) = (\binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{k}, 0, \ldots, 0)$$

and

$$B_k(n)_{k-1} = \binom{n-1}{k}, \quad B_k(n)_i = 0, \ i \neq k-1$$

Let us define an operation on a complex which we call a **generalized compression**. Let C be a simplicial complex on n vertices and x and y two of its vertices. Let furthermore

$$\{X_1, \ldots, X_k\} = \{X \in C | x \in X, \, y \notin X\}$$
$$\{Y_1, \ldots, Y_m\} = \{Y \in C | x \notin Y, \, y \in Y\}$$

Then we define an **(x,y)-compression** of C (or a **y to x compression**) as the complex C^* given by:

$$C^* = (C \setminus \{Y_1, \dots, Y_m\}) \cup \{(X_i \setminus \{x\}) \cup \{y\} | i = 1, \dots, k\}$$

Let us see that C^* is again a simplicial complex. It is enough to verify that if $S \subseteq (X_1 \setminus \{x\}) \cup \{y\}$ then $S \in C^*$. We consider two cases. First, if $y \notin S$ then $S \subseteq X_1$, hence $S \in C$, but $S \notin \{Y_1, \dots, Y_m\}$, which gives $S \in C^*$. If on the contrary $y \in S$, let us denote $S' = (S \setminus \{y\}) \cup \{x\}$. Then $S' \subseteq X_1$, hence $S' \in C$, so $S' \in \{X_1, \dots, X_k\}$. Say $S' = X_i$, then $S = (X_i \setminus \{x\}) \cup \{y\}$ and so $S \in C^*$.

Observe that an (x,y)-compression is different from another similiar and frequently used operation on simplicial complexes called an (i,j)-shift (see Chapter 4 in [F] for a description). An (i,j)-shift preserves the f-vector, while our compression operation changes the f-vector linearly:

$$f(C^*) = f(C) + f(X_1, \dots, X_k) - f(Y_1, \dots, Y_m) \tag{4.1}$$

So if l is a linear function on f-vectors then

$$l(f(C^*)) = l(f(C)) + l(f(X_1, \dots, X_k)) - l(f(Y_1, \dots, Y_m)) \tag{4.2}$$

Theorem 4.1. *The convex hull of f-vectors of simplicial complexes on n vertices is given by $conv\{F_1, F_2, \dots, F_n\}$.*

Proof 1: Take l a linear function on f-vectors and let C be a simplicial complex where l achieves its maximum. In the same way as in the proof of Theorem 3.3, but using formula 4.2 instead of 3.2, we observe that we can perform (x,y)-compressions without changing the value of l. Though now we have a simpler situation, as we do not need to bother that x and y have no edge in between. So eventually we end up with a complex C^* where nothing can be compressed (just pick from the beginning some special x and compress everything to it). This means that for any two vertices of C, x and y, we have

$$\{X \setminus \{x\} | x \in X, y \notin X\} = \{Y \setminus \{y\} | x \notin Y, y \in Y\}.$$

It is a routine argument to see now, that this property implies that C^* is a complete k-skeleton, as we have that

$$x \in X, y \notin X, X \in C^* \Rightarrow (X \setminus \{x\}) \cup \{y\} \in C^*.$$

So we see that for any linear function, one of the points where it achieves the maximum is the f-vector of a k-skeleton. This proves the result. □

It is also possible to show this result using a different method involving some linear algebra. We give this proof in the next section.

Theorem 4.2. *The convex hull of β-vectors of simplicial complexes on n vertices is given by $conv\{B_1, B_2, \dots, B_n\}$.*

Proof: The proof of this theorem is very similiar to the one of 3.4. For that reason we will only show how to adopt the argument from above to this particular case. The only difference from Theorem 3.4 is that we use generalized compression instead of the usual one. Let again l be a linear function with nonnegative coefficients, C an optimal complex, x, y vertices of C, $C_1 = C \setminus \{y\}$ and C^* the result of the y to x compression. Then just as above we can write both C and C^* as unions of two simplicial complexes and using the Mayer-Vietoris exact sequences estimate the Betti numbers. The inequality 3.3 is valid also here. The only special thing about compression was that we actually obtained equality in 3.4. The argument was that in a compressed complex we have $\mathrm{lk}(x) = \mathrm{lk}(y)$ and hence the mapping induced by inclusion:

$$H_k(\mathrm{lk}(y)) \to H_k(C^* \setminus \{y\})$$

was trivial. But this is true even for the generalized compression. Though links of x and y are not equal any more, we still have that $\mathrm{lk}(y)$ is contained in $\mathrm{st}(x)$ and hence is mapped into a cone. So the mapping above is trivial even in this case and all the arguments used in the proof of Theorem 3.4 go through. □

Note. Observe that the f-vectors of simplicial complexes on n vertices are completely characterized by the Kruskal-Katona theorem, see [Kr],[Ka]. Also the characterization for the set of β-vectors can be obtained through the Theorem 1.3 in [BK] and the characterization of the f-vectors of Sperner families, independently discovered by [C] and [DGH].

Corollary 4.3. *The convex hulls of f- and β-vectors of matroid complexes on n vertices are given by $conv\{F_1, \ldots, F_n\}$ and $conv\{B_1, \ldots, B_n\}$ respectively.*

Proof: Matroid complexes are just a special case of simplicial complexes, on the other hand, the complete k-skeletons correspond to uniform matroids (see [W] for the definition), hence the result follows. □

Note. It would be interesting to define the corresponding convex hulls for more restricted classes of matroids, for example for the class of graphic matroids.

5. A DIRECT PROOF OF THEOREM 4.1

Here we give another method for finding the convex hull of a set of points. It provides another proof of Theorem 4.1.

Proof 2: We know that $conv\{F_1, \ldots, F_n\}$ is a simplex and that $F_k = (\binom{n}{1}, \ldots, \binom{n}{k}, 0, \ldots, 0)$. Let us compress our space along each coordinate by the binomial coefficients, i.e. we divide the kth coordinates by $\binom{n}{k}$. Let v_k denote the vector with first k coordinates equal to one and the rest equal to zero for $k = 0, \ldots, n$, $v_0 = 0$. Then we are left to prove that whenever we have an f-vector $(f_0, \ldots, f_{n-1})$ of a simplicial complex on n vertices, the point $g = (g_0, \ldots, g_{n-1})$, given by $g_i = f_i / \binom{n}{i+1}$, is contained in the simplex spanned by $v_0, \ldots, v_n$. Then the vector g is inside this simplex if and only if

$$g = \sum_{k=1}^{n} \alpha_k v_k, \text{ such that } \sum_{k=1}^{n} \alpha_k \leq 1 \text{ and } \alpha_k \geq 0,\ k = 1, \ldots, n$$

Let M be the matrix with vectors v_k as columns. M is an upper triangular matrix and if α is the vector $(\alpha_1, \ldots, \alpha_n)$, then we have a matrix identity:

$$g = M \cdot \alpha \text{ , or } \alpha = M^{-1} \cdot g.$$

But since

$$M^{-1} = \begin{pmatrix} 1 & -1 & 0 & \ldots & 0 \\ 0 & 1 & -1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 0 & 1 \end{pmatrix}$$

we get

$$\alpha = M^{-1} \cdot g = (g_0 - g_1, g_1 - g_2, \ldots, g_{n-1})$$

and the conditions on the vector α translate into the following inequalities:

$$1 \geq g_0 \geq g_1 \geq \cdots \geq g_{n-1}$$

Since it is obvious that $1 \geq g_0$ we only need to prove that $g_{k-1} \geq g_k$ for $k = 1, \dots, n-1$. It means that we have to prove that

$$f_{k-1}/\binom{n}{k} \geq f_k/\binom{n}{k+1}$$

or that

$$f_{k-1}\binom{n}{k+1} \geq f_k\binom{n}{k}$$

which after cancellation of common factors from factorials transforms into

$$(n-k)f_{k-1} \geq (k+1)f_k \tag{5.1}$$

This inequality is well known and can be found in, for example, Section 8 of [GK]. We give here a short simple argument.

Let C to be a simplicial complex with f-vector $(f_0, \dots, f_{n-1})$ and let us count pairs (A, B), where $A, B \in C$, $|A| = k+1$, $|B| = k$ and $B \subset A$. On one hand this number is equal to $(k+1)f_k$ since every face with $k+1$ elements contains exactly $k+1$ faces of cardinality k. On the other hand every face with k elements is contained in at most $n-k$ faces of cardinality $k+1$, hence the number of pairs that we count is at most $(n-k)f_{k-1}$. This yields the inequality 5.1. □

6. Open problems

We finish this paper with a conjecture concerning the class of r-colorable simplicial complexes.

Definition 6.1. *We call a simplicial complex r-***colorable** *if its 2-skeleton is r-colorable in the graph-theoretical sense.*

Conjecture 6.2. *The convex hull of f-vectors (resp. β-vectors) of r-colorable complexes on n vertices is equal to $conv\{F_1, \dots, F_n\}$ (resp. $conv\{B_1, \dots, B_n\}$), where F_i (B_i) denotes the f-vector (β-vector) of the complete i-skeleton of the flag complex associated to $T_r(n)$.*

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