Department of Computer Science
YALE UNIVERSITY
New Haven, Connecticut 06520

July 1972

ON THE RATE OF CONVERGENCE OF THE BERGMAN-VEKUA METHOD FOR THE NUMERICAL SOLUTION OF ELLIPTIC BOUNDARY VALUE PROBLEMS

by

Stanley C. Eisenstat

Research Report Number 72-2

This research was supported in part by the Office of Naval Research, N0014-67-A-0097-0016.

ACKNOWLEDGEMENTS

I would like to express my deepest gratitude to Professor Cleve Moler for suggesting this area of study and for his continual guidance and encouragement; to the other members of my reading committee, Professor John G. Herriot, Professor Stefan Bergman, and Professor M. M. Schiffer, for their careful reading and thoughtful suggestions; to Professor Gene Golub and Professor George Dantzig, for their encouragement over the years; and lastly, to the people of Serra House, for making it all more enjoyable.

This work was supported in part by the National Science Foundation and the Office of Naval Research.

On the Rate of Convergence of the Bergman-Vekua Method for the Numerical Solution of Elliptic Boundary Value Problems

Stanley C. Eisenstat

Abstract

Consider the elliptic partial differential equation with analytic coefficients

$$Lu \equiv \Delta u + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0$$

in a simply connected domain D of the complex plane. By a classic result of Picard, every solution u is analytic for $(x,y) \varepsilon D$ and, by a more recent result of Vekua, can be continued analytically to the solution of a complex formally hyperbolic equation on the product domain $D \times D^*$. By solving a Goursat problem for the hyperbolic equation, Bergman and Vekua have independently derived equivalent representations for solutions of $Lu = 0$ in terms of an integral operator V,

$$u = \text{Re}\, V[\phi], \quad \phi \text{ analytic,}$$

a generalization of the representation of harmonic functions as the real part of an analytic function. Thus the study of the solution u can be reduced to the study of the associated analytic function ϕ.

To that end, we shall prove that if the p^{th} derivatives of u are uniformly Hölder continuous with exponent γ, then the p^{th} derivative of ϕ is also uniformly Hölder continuous with exponent γ. Moreover, we shall show that the asymptotic expansions for u in the neighborhood of a corner obtained by Lehman give rise to analogous asymptotic expansions for ϕ.

The generalized harmonic polynomials $u_n = \text{Re}\, V[p_n]$, where p_n denotes a polynomial of degree n, comprise a distinguished class of solutions which is dense in the space of all solutions. Using a result

of Mergelyan on the degree of approximation of analytic functions by polynomials, we shall prove that the degree of approximation of the solution u by generalized harmonic polynomials is $\approx O(n^{-(p+\gamma)})$, where p,γ reflect the smoothness of u.

The Bergman-Vekua method of particular solutions as developed for use on high speed digital computers by Bergman and Herriot and Schryer approximates the solution of the boundary value problem

$$Lu = 0 \text{ in } D; \quad u = f \text{ on } \partial D$$

by the generalized harmonic polynomials which in some sense best approximates the boundary data f along the boundary. We shall prove that the asymptotic rate of convergence of this method is $\approx O(n^{-(p+\gamma)})$, where p,γ reflect the smoothness of both the boundary and the boundary data. For domains with piecewise smooth boundary, we shall show that we can treat corners by introducing certain singular particular solutions to the approximation as suggested by the form of the asymptotic expansion for ϕ.

The method of particular solutions has been applied to the membrane eigenvalue problem

$$\Delta u + \lambda u = 0 \text{ in } D; \quad u = 0 \text{ on } \partial D$$

by Fox, Henrici, and Moler. We shall develop a similar method implicit in the work of Bergman which defines approximate eigenvalues as local solutions of a minimum problem, and prove that the asymptotic rate of convergence is the same as in the boundary value problem.

§1. Introduction

Consider the elliptic partial differential equation with analytic coefficients

$$Lu \equiv \Delta u + a(x,y)\frac{\partial u}{\partial x} + b(x,y)\frac{\partial u}{\partial y} + c(x,y)u = 0$$

in a simply connected domain D of the complex plane. By a classic result of Picard, every solution u is analytic for $(x,y) \in D$ and, by a more recent result of Vekua [16], can be continued analytically to the solution of a complex formally hyperbolic equation on the product domain $D \times D^*$. By solving a Goursat problem for the hyperbolic equation, Bergman [2] and Vekua [16] have independently derived equivalent representations for solutions of $Lu = 0$ in terms of an integral operator V,

$$u = \mathrm{Re}\, V[\phi] \quad , \quad \phi \text{ analytic},$$

a generalization of the representation of harmonic functions as the real part of an analytic function. Thus the study of the solution u can be reduced to the study of the associated analytic function ϕ.

To that end, we shall prove that if the p^{th} derivatives of u are uniformly Hölder continuous with exponent γ, then the p^{th} derivative of ϕ is also uniformly Hölder continuous with exponent γ. Moreover, we shall show that the asymptotic expansions for u in the neighborhood of a corner obtained by Lehman [9] give rise to analogous asymptotic expansions for ϕ.

The generalized harmonic polynomials $u_n = \mathrm{Re}\, V[p_n]$, where p_n denotes a polynomial of degree n, comprise a distinguished class of solutions which is dense in the space of all solutions. Using a result of Mergelyan [11] on the degree of approximation of analytic functions by polynomials, we shall prove that the degree of approximation of the solution u by generalized harmonic polynomials is $\approx O(n^{-(p+\gamma)})$, where p,γ reflect the smoothness of u.

The Bergman-Vekua method of particular solutions as developed for use on high speed digital computers by Bergman and Herriot [4,5] and Schryer [14] approximates the solution of the boundary value problem

$$Lu = 0 \text{ in } D; \quad u = f \text{ on } \partial D$$

by the generalized harmonic polynomial which in some sense best approximates the boundary data f along the boundary. We shall prove that the asymptotic rate of convergence of this method is $\approx O(n^{-(p+\gamma)})$, where p,γ reflect the smoothness of both the boundary and the boundary data. For domains with piecewise smooth boundary, we shall show that we can treat corners by introducing certain singular particular solutions to the approximation as suggested by the form of the asymptotic expansion for ϕ.

The method of particular solutions has been applied to the membrane eigenvalue problem

$$\Delta u + \lambda u = 0 \text{ in } D; \quad u = 0 \text{ on } \partial D$$

by Fox, Henrici, and Moler [7]. We shall develop a similar method implicit in the work of Bergman [3] which defines approximate eigenvalues as local solutions of a minimum problem, and prove that the asymptotic rate of convergence is the same as in the boundary value problem.

§2. Notation

Let D denote a bounded, simply connected domain in the complex plane with piecewise smooth boundary ∂D. Let the boundary be parametrized with respect to arc-length by

$$x = x(s), \quad y = y(s), \quad 0 \leq s \leq \ell_D$$

with x(s), y(s) periodic and piecewise differentiable. The tangent to the curve exists whenever x(s), y(s) are continuously differentiable; a point $z_q = (x(s_q), y(s_q))$ at which the tangent fails to exist is said to be (the <u>vertex of</u>) <u>a corner</u>. Let Γ_1 and Γ_2 denote the adjacent boundary arcs and let $\frac{\pi}{\alpha_q}$ be the interior <u>angle</u> between the tangents to Γ_1 and Γ_2 at z_q. The domain D is said to be of <u>class $\mathcal{R}$</u> if $0 < \frac{\pi}{\alpha_q} < 2\pi$ at each corner z_q (interior and exterior cusps are excluded); D is said to be of <u>class $\mathcal{R}(\mu)$</u> $(0 < \mu \leq 1)$ if, in addition, $\pi\mu \leq 2\pi - \frac{\pi}{\alpha_q}$ (the exterior angle at each corner is at least $\pi\mu$). Clearly a domain with no corners is of class $\mathcal{R}(1)$.

A function g(z)(*) defined on a closed subset of the complex plane is said to be Hölder continuous with exponent γ $(0 < \gamma \leq 1)$ if there exists a constant K such that

$$|g(z_1) - g(z_2)| \leq K|z_1 - z_2|^{\gamma}, \quad \forall\, z_1, z_2 \in S.$$

We introduce the function spaces

$C(D \cup \partial D)$	space of functions continuous in $D \cup \partial D$
$A(D)$	space of functions analytic in D

(*) For convenience, we shall frequently write g(z) for a function g(x,y).

$C^{p,\gamma}(D\cup\partial D)$ space of functions whose p^{th} derivatives are Hölder continuous with exponent γ in $D\cup\partial D$

$C^{p,\gamma}(D\cup\partial D - \{z_1,\ldots,z_q\})$ space of functions whose p^{th} derivatives are Hölder continuous with exponent γ in every compact subset of $D\cup\partial D - \{z_1,\ldots,z_q\}$

If the functions x(s), y(s) in the boundary parametrization have p^{th} derivatives with respect to arclength which are Hölder continuous with exponent γ, then we write $\partial D \in C^{p,\gamma}$. If $\partial D \in C^{p,\gamma}$ or $\partial D \in C^{p,\gamma}$ except at corners $z_1, \ldots, z_q$, then the spaces $C^{p,\gamma}(\partial D)$ and $C^{p,\gamma}(\partial D - \{z_1,\ldots,z_q\})$ are defined in the obvious manner. We state without proof the obvious

<u>Lemma</u>: If $g \in C^{p,\gamma}$, then $g \in C^{p',\gamma'}$ for $p' + \gamma' \leq p + \gamma$.

A corner determined by the vertex z_q and interior angle $\frac{\pi}{\alpha_q}$ is said to be an <u>analytic corner</u> (z_q,α_q) if the adjacent (closed) boundary arcs are segments of analytic curves. If g(z) is defined on ∂D and piecewise analytic near z_q, i.e.,

$$g(x,y) = \psi_1(x,y) \text{ on } \Gamma_1, \quad g(x,y) = \psi_2(x,y) \text{ on } \Gamma_2$$

where the functions ψ_j are analytic functions of x,y near z_q, then we write $g \in PA(z_q)$. The space $PA(z_1,\ldots,z_q)$ is similarly defined.

Finally, we introduce the norms

$$\|g\|_{D\cup\partial D} = \max_{z\in D\cup\partial D} |g(z)| \qquad \|g\|_{\partial D} = \max_{z\in\partial D} |g(z)|$$

$$\|g\|_{2,D} = \{\int_D g^2\, dxdy\}^{1/2} \qquad \|g\|_{2,\partial D} = \{\int_{\partial D} g^2 ds\}^{1/2}$$

§3. An Integral Representation

By a classic result of analysis, every harmonic function, or equivalently every solution of Laplace's equation $\Delta u = 0$, can be represented as the real part of an analytic function. For solutions of the more general equation

$$Lu \equiv \Delta u + a(x,y) \frac{\partial u}{\partial x} + b(x,y) \frac{\partial u}{\partial y} + c(x,y)u = 0,$$

an analogous representation has been derived independently by Berman [2] and by Vekua [16].

Let D be a simply connected domain of class $\mathcal{R}$ and let u satisfy $Lu = 0$ in D. For the moment, we shall assume that the coefficients a,b,c are analytic. Then by a classic result of Picard, the solution u is analytic in D.

Treating x and y as independent complex variables, we introduce the variables $z = x + iy$, $z^* = x - iy$, which are conjugate complex if and only if x and y are real, and the corresponding differential operators

$$\frac{\partial}{\partial z} \equiv \frac{1}{2} \{\frac{\partial}{\partial x} - i \frac{\partial}{\partial y}\}, \qquad \frac{\partial}{\partial z^*} \equiv \frac{1}{2} \{\frac{\partial}{\partial x} + i \frac{\partial}{\partial y}\} \equiv \frac{\partial}{\partial \bar{z}} .$$

Let $\mathcal{D}$ be a simply connected domain with $D \cup \partial D \subset \mathcal{D}$ and set $\mathcal{D}^* = \{z^* | \overline{z^*} \in \mathcal{D}\}$, the conjugate domain to $\mathcal{D}$. Then the functions

$$A(z,z^*) = \frac{1}{4} \left\{ a(\frac{z+z^*}{2}, \frac{z-z^*}{2i}) + i\, b(\frac{z+z^*}{2}, \frac{z-z^*}{2i}) \right\}$$

$$B(z,z^*) = \frac{1}{4} \left\{ a(\frac{z+z^*}{2}, \frac{z-z^*}{2i}) - i\, b(\frac{z+z^*}{2}, \frac{z-z^*}{2i}) \right\}$$

$$C(z,z^*) = \frac{1}{4}\, c(\frac{z+z^*}{2}, \frac{z-z^*}{2i})$$

are analytic for $(z,z^*) \in \mathcal{D} \times \mathcal{D}^*$. We may now relax our assumptions on the coefficients $a(x,y)$, $b(x,y)$, $c(x,y)$ and assume only that

$A(z,z^*)$, $B(z,z^*)$, $C(z,z^*)$ are analytic for $(z,z^*) \in \mathcal{D} \times \mathcal{D}^*$.

Theorem 3.1: (Vekua [16]) The solution $u(x,y)$ can be continued analytically to a solution

$$U(z,z^*) = u\left(\frac{z+z^*}{2}, \frac{z-z^*}{2i}\right), \quad (z,z^*) \in D \times D^*$$

of the equation

$$\mathcal{L}U \equiv \frac{\partial^2 U}{\partial z \partial z^*} + A(z,z^*) \frac{\partial U}{\partial z} + B(z,z^*) \frac{\partial U}{\partial z^*} + C(z,z^*)U = 0.$$

By solving a Goursat problem for the complex, formally hyperbolic equation $\mathcal{L}U = 0$, Bergman [2] and Vekua [16] have derived equivalent integral representations for the solution u in terms of an analytic function.

With our assumptions on the coefficients, the complex Riemann function $G(t,t^*,z,z^*)$ for the operator $\mathcal{L}$ (Henrici [8]) is defined and analytic for $t,z \in \mathcal{D}$, $t^*,z^* \in \mathcal{D}^*$. For any function ϕ analytic in D and $z_0 \in D$, we define the integral operator

$$I[\phi;z_0](z,z^*) \equiv \frac{1}{2}\left\{G(z,\bar{z}_0,z,z^*)\phi(z) + \int_{z_0}^{z} \phi(t)H(t,\bar{z}_0,z,z^*)\,dt\right.$$

$$\left. + G(z_0,z^*,z,z^*)\phi^*(z^*) + \int_{\bar{z}_0}^{z^*} \phi^*(t^*)H^*(z_0,t^*,z,z^*)\,dt^*\right\}$$

where $\phi^*(z^*) = \overline{\phi(\bar{z}^*)}$ and

$$H(t,t^*,z,z^*) = B(t,t^*)G(t,t^*,z,z^*) - \frac{\partial G}{\partial t}(t,t^*,z,z^*)$$

$$H^*(t,t^*,z,z^*) = A(t,t^*)G(t,t^*,z,z^*) - \frac{\partial G}{\partial t^*}(t,t^*,z,z^*).$$

When $z^* = \bar{z}$, this simplifies to

$$I[\phi;z_0](z,\overline{z}) = \mathrm{Re}\{G(z,\overline{z}_0,z,\overline{z})\phi(z) + \int_{z_0}^{z} \phi(t)H(t,\overline{z}_0,z,\overline{z})\,dt\}$$

$$\equiv \mathrm{Re}\{V[\phi;z_0](z,\overline{z})\}.$$

Theorem 3.2: (Vekua[16]) Fix $z_0 \in D$. Then there exists a unique function ϕ analytic in D with $\phi(z_0)$ real such that

$$u(x,y) = \mathrm{Re}\, V[\phi;z_0](z,\overline{z}), \quad z = x + iy \in D$$

$$U(z,z^*) = I[\phi;z_0](z,z^*), \qquad (z,z^*) \in D \times D^*.$$

Moreover,

$$\phi(z) = 2U(z,\overline{z}_0) - U(z_0,\overline{z}_0)G(z_0,\overline{z}_0,z,\overline{z}_0).$$

For Laplace's equation, the Riemann function is $G(t,t^*,z,z^*) = 1$ and the integral representation reduces to $u = \mathrm{Re}\{\phi\}$, a classic result for harmonic functions.

The Vekua integral representation is of limited value in calculating a particular solution u_k given an analytic function ϕ_k since the Riemann function is known for only a few special equations. However, Bergman [2] has derived an equivalent integral representation which can be constructed directly from the coefficients A,B,C. For the case of polynomial coefficients, the method has been implemented by Bergman and Herriot [4,5] and by Schryer [14].

Finally, we note that if $\phi \in C(D \cup \partial D)$, then

$$\|\mathrm{Re}\ V[\phi;z_0]\|_{D\cup\partial D} \leq \|G\|\ \|\phi\|_{D\cup\partial D} + \int_{z_0}^{z} \|H\|\ \|\phi\|_{D\cup\partial D}\,d|t| \leq K_V\|\phi\|_{D\cup\partial D}$$

since the domain is bounded and the functions G, H are bounded (indeed analytic) in the appropriate domain.

§4. The Continuity of u vs. ϕ

The integral representation of Bergman and Vekua associates an analytic function with every solution of $Lu = 0$ in D. In this section, we shall relate the smoothness of the associated analytic function to the smoothness of the solution.

Theorem 4.1: Let D be a simply connected domain of class $\mathcal{R}$ and fix $z_0 \varepsilon D$. Let u satisfy $Lu = 0$ in D and let ϕ be the associated analytic function. If $u \varepsilon C^{p,\gamma}(D\cup\partial D)$, then $\phi \varepsilon C^{p,\gamma}(D\cup\partial D)$ $(p + \gamma > 0)$.

The proof of the Theorem depends on three Lemmas. Let

$$r(z) = \min_{t\varepsilon\partial D} |t-z|,$$

the distance from $z \varepsilon D$ to the boundary.

Lemma 4.1: Let D be a simply connected domain of class $\mathcal{R}$ and let u satisfy $Lu = 0$ in D. If $u \varepsilon C^{0,\gamma}(D\cup\partial D)$, then there exists a constant K independent of z such that

$$|U_z(z,\bar{z})| = |\frac{1}{2} u_x(x,y) - \frac{i}{2} u_y(x,y)| \leq Kr(z)^{\gamma-1} \qquad (0 < \gamma \leq 1).$$

Proof:

Fix $z \varepsilon D$ and let $R = \frac{1}{2} r(z)$. The function $v(\zeta) \equiv u(z+R\zeta) - u(z)$ defined on the unit disk $D_0 = \{\zeta \,|\, |\zeta| < 1\}$ satisfies

$$\Delta_\zeta v(\zeta) = -R^2\{au_x(z+R\zeta) + bu_y(z+R\zeta) + cu(z+R\zeta)\} \equiv f(\zeta), \quad \zeta \ \varepsilon \ D_0.$$

Since $u \varepsilon C^{0,\gamma}(D\cup\partial D)$,

$$|v(\zeta)| = |u(z + R\zeta) - u(z)| \leq K_1|R\zeta|^\gamma = K_1R^\gamma, \qquad \zeta \ \varepsilon \ \partial D_0.$$

From the Schauder Interior Estimates,

$$r(t)\{|u_x(t)| + |u_y(t)|\} \leq K_2, \quad t \ \varepsilon \ D$$

whence

$$|u_x(z+R\zeta)|,\ |u_y(z+R\zeta)| \leq \frac{K_2}{r(z+R\zeta)} \leq \frac{K_2}{R}, \quad \zeta \in D_0$$

since $r(z+R\zeta) \geq R$ for $|\zeta| \leq 1$. Therefore

$$|f(\zeta)| \leq R^2\{\|a\|_{D\cup\partial D}\frac{K_2}{R} + \|b\|_{D\cup\partial D}\frac{K_2}{R} + \|c\|_{D\cup\partial D}\|u\|_{D\cup\partial D}\}$$

$$\leq K_3R + K_4R^2 \leq K_5R^\gamma, \quad \zeta \in D_0$$

since $R \leq$ diameter (D). Let $\Gamma(\zeta,\tau)$ be the Green's function for Laplace's equation $\Delta u = 0$ in D_0:

$$\Gamma(\zeta,\tau) = -\frac{1}{2\pi}\log\left|\frac{\zeta-\tau}{1-\zeta\bar{\tau}}\right| .$$

Then

$$v(\zeta) = \int_{|\tau|=1}\frac{\partial\Gamma}{\partial n_\tau}(\zeta,\tau)v(\tau)ds_\tau - \int_{|\tau|\leq 1}\Gamma(\zeta,\tau)\Delta_\tau v(\tau)dA_\tau .$$

Differentiating,

$$\frac{\partial v}{\partial\zeta}(\zeta) = \int_{|\tau|=1}\frac{\partial^2\Gamma}{\partial\zeta\partial n_\tau}(\zeta,\tau)v(\tau)ds_\tau - \int_{|\tau|\leq 1}\frac{\partial\Gamma}{\partial\zeta}(\zeta,\tau)f(\tau)dA_\tau$$

and

$$\left|\frac{\partial v}{\partial\zeta}(0)\right| \leq \int_{|\tau|=1}\left|\frac{\partial^2\Gamma}{\partial\zeta\partial n_\tau}(0,\tau)\right||v(\tau)|ds_\tau + \int_{|\tau|\leq 1}\left|\frac{\partial\Gamma}{\partial\zeta}(0,\tau)\right||f(\tau)|dA_\tau$$

$$\leq K_1R^\gamma\int_{|\tau|=1}\left|\frac{\partial^2\Gamma}{\partial\zeta\partial n_\tau}(0,\tau)\right|ds_\tau + K_5R^\gamma\int_{|\tau|\leq 1}\left|\frac{\partial\Gamma}{\partial\zeta}(0,\tau)\right|dA_\tau$$

$$= K_1R^\gamma + \frac{1}{3}K_5R^\gamma = K_6R^\gamma .$$

But then

$$\left|\frac{\partial U}{\partial z}(z,\bar{z})\right| = R^{-1}\left|\frac{\partial v}{\partial\zeta}(0)\right| \leq K_6R^{\gamma-1} = 2^{1-\gamma}K_6r(z)^{\gamma-1}.$$

Q.E.D.

Lemma 4.2: Let $z_0 \in D$ and let

$$\chi(z) = \int_{z_0}^{z} M(z,t,\overline{t})dt + N(z,t,\overline{t})d\overline{t}$$

where the integral is path independent and

$$|M(z,t,\overline{t})| \leq Kr(t)^{\gamma-1}, \quad |N(z,t,\overline{t})| \leq Kr(t)^{\gamma-1} \qquad (0 < \gamma \leq 1)$$

uniformly for $z \in D \cup \partial D$. Then $\chi(z)$ is uniformly bounded for $z \in D \cup \partial D$ and the bound depends only on K and the domain D.

Proof:

Let $D_\delta = \{z \in D | r(z) > \delta\}$. We shall prove that $\chi(z)$ is uniformly bounded in D_δ for $\delta > 0$ sufficiently small and in a neighborhood of every boundary point z_B. The result then follows from the compactness of $D \cup \partial D$.

Let

$$\psi(z,\zeta) = \int_{z_0}^{z} M(\zeta,t,\overline{t})\, dt + N(\zeta,t,\overline{t})\, d\overline{t}, \quad z,\zeta \in D \cup \partial D.$$

For $\delta > 0$ sufficiently small, the domain D_δ is simply connected and $z_0 \in D_\delta$. Thus for $z \in D_\delta$, $\zeta \in D \cup \partial D$,

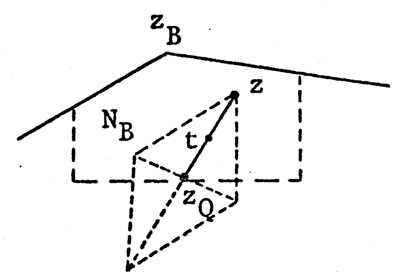
$$|\psi(z,\zeta)| \leq \int_{z_0}^{z} \{|M(\zeta,t,\overline{t})| + |N(\zeta,t,\overline{t})|\}ds_t$$

$$\leq 2K \int_{z_0}^{z} r(t)^{\gamma-1}ds_t \leq 2K\delta^{\gamma-1} \int_{z_0}^{z} ds_t \leq K_\delta$$

since the path from z_0 to z can be chosen to lie in $D_\delta \cup \partial D_\delta$ and the length of the shortest such path is uniformly bounded. Since $\chi(z) = \psi(z,z)$,

$$|\chi(z)| \leq K_\delta, \quad z \in D_\delta.$$

Let $z_B \in \partial D$. Since the boundary is piecewise smooth and contains neither interior nor exterior cusps, without loss of generality we can assume that there exists a sufficiently small neighborhood N_B of z_B with the property that there exists a point $z_Q \in N_B \cap D$ and a constant η_B such that for every $z \in N_B$, the diamond



$\tilde{z} = 2z_Q - z$

$$\{t \in D \,|\, |\sin \arg \frac{t-z}{z_Q-z}| < \eta_B,$$

$$|\sin \arg \frac{t-\tilde{z}}{z_Q-\tilde{z}}| < \eta_B\},$$

where $\tilde{z} = 2z_Q - z$, is contained in D. Therefore

$$r(t) \geq \eta_B \, |t-z|, \quad t \in \overrightarrow{z_Q z}$$

for every $z \in N_B$. Note that

$$\psi(z,\zeta) = \psi(z_Q,\zeta) + \int_{z_Q}^{z} M(\zeta,t,\bar{t})dt + N(\zeta,t,\bar{t})d\bar{t}.$$

For $\delta > 0$ sufficiently small, $z_Q \in D_\delta$ whence

$$|\psi(z_Q,\zeta)| \leq K_\delta, \quad \zeta \in D \cup \partial D.$$

Thus for $z \in N_B$, $\zeta \in D \cup \partial D$,

$$|\psi(z,\zeta)| \leq |\psi(z_Q,z)| + 2K \int_{\zeta_B}^{z} r(t)^{\gamma-1} ds_t$$

$$\leq K_\delta + 2K\eta_B^{\gamma-1} \int_{\zeta_B}^{z} |t-z|^{\gamma-1} ds_t$$

$$= K_\delta + \frac{2K}{\gamma} \eta_B^{\gamma-1} |z - \zeta_B|^\gamma \leq K_B.$$

taking $\overrightarrow{z_Q z}$ as the path from z_Q to z. Since $\chi(z) = \psi(z,z)$,

$$|\chi(z)| \leq K_B, \quad z \in N_B.$$

Q.E.D.

Indeed, the result is still valid when $z_0 \in \partial D$. For fixing $\tilde{z}_0 \in D$,

$$\chi(z) = -\tilde{\psi}(\tilde{z}_0, z) + \tilde{\psi}(z, z)$$

where

$$\tilde{\psi}(z,\zeta) = \int_{\tilde{z}_0}^{z} M(z,t,\overline{t})dt + N(z,t,\overline{t})d\overline{t}, \quad z,\zeta \in D\cup\partial D.$$

The argument used to prove the Lemma shows that $\tilde{\psi}(z,\zeta)$ is uniformly bounded for $z,\zeta \in D\cup\partial D$. Thus $\chi(z)$ is uniformly bounded for $z \in D\cup\partial D$.

<u>Lemma 4.3</u>: Let $z_0 \in D\cup\partial D$ and let

$$\chi(z) = \int_{z_0}^{z} M(z,t,\overline{t})dt + N(z,t,\overline{t})d\overline{t}$$

where the integral is path independent and

$$|M(z,t,\overline{t})| \leq Kr(t)^{\gamma-1}, \quad |N(z,t,\overline{t})| \leq Kr(t)^{\gamma-1} \quad (0 < \gamma \leq 1)$$

uniformly for $z \in D\cup\partial D$. Moreover, letting

$$\tilde{M}(z_1,z_2,t,\overline{t}) = \frac{M(z_1,t,\overline{t}) - M(z_2,t,\overline{t})}{z_1 - z_2} \qquad z_1,z_2 \in D\cup\partial D$$

$$\tilde{N}(z_1,z_2,t,\overline{t}) = \frac{N(z_1,t,\overline{t}) - N(z_2,t,\overline{t})}{z_1 - z_2}, \qquad z_1 \neq z_2$$

assume that

$$|\tilde{M}(z_1,z_2,t,\overline{t})| \leq Kr(t)^{\gamma-1}, \quad |\tilde{N}(z_1,z_2,t,\overline{t})| \leq Kr(t)^{\gamma-1}$$

uniformly for $z_1,z_2 \in D\cup\partial D$, $z_1 \neq z_2$. Then $\chi \in C^{0,\gamma}(D\cup\partial D)$.

<u>Proof</u>:

We must prove that the ratio

$$\psi(z_1,z_2) = \frac{\chi(z_1) - \chi(z_2)}{(z_1 - z_2)^{\gamma}}$$

is uniformly bounded for $z_1, z_2 \in D \cup \partial D$, $z_1 \neq z_2$. But

$$\psi(z_1,z_2) = (z_1-z_2)^{1-\gamma}\int_{z_0}^{z_1} \tilde{M}(z_1,z_2,t,\bar{t})dt + \tilde{N}(z_1,z_2,t,\bar{t})d\bar{t}$$

$$+ \int_{z_1}^{z_2} \frac{M(z_2,t,\bar{t})}{(z_1-z_2)^\gamma} dt + \frac{N(z_2,t,\bar{t})}{(z_1-z_2)^\gamma} d\bar{t}$$

$$= \psi_1(z_1,z_2) + \psi_2(z_1,z_2)$$

By Lemma 4.2, $\psi_1(z_1,z_2)$ is uniformly bounded for $z_1,z_2 \in D\cup\partial D$, $z_1 \neq z_2$. Moreover, if $|z_1 - z_2| \geq \varepsilon > 0$, then $\psi_2(z_1,z_2)$ is uniformly bounded again by Lemma 4.2 (the bound will of course depend on ε). We shall prove that for every $z_Q \in D\cup\partial D$, there exists a neighborhood N_Q of z_Q such that $\psi_2(z_1,z_2)$ is uniformly bounded for $z_1,z_2 \in N_Q$. By the compactness of $D\cup\partial D$, there exist a finite number of such neighborhoods with the property that for some $\varepsilon > 0$ sufficiently small, if $z_1,z_2 \in D\cup\partial D$ and $|z_1 - z_2| < \varepsilon$, then z_1,z_2 belong to at least one such neighborhood and $\psi_2(z_1,z_2)$ is bounded by the corresponding bound. Thus $\psi_2(z_1,z_2)$ will be uniformly bounded.

If $z_Q \in D$, then take $N_Q = \{z \in D \,\big|\, |z - z_Q| < \frac{1}{2} r(z_Q)\}$. Then

$$r(z) \geq r(z_Q) - |z - z_Q| = \frac{1}{2} r(z_Q), \quad z \in N_Q.$$

Thus, for $z_1,z_2 \in N_Q$,

$$|\psi_2(z_1,z_2)| \leq |z_1 - z_2|^{-\gamma} \int_{z_1}^{z_2} \{|M(z_2,t,\bar{t})| + |N(z_2,t,\bar{t})|\}ds_t$$

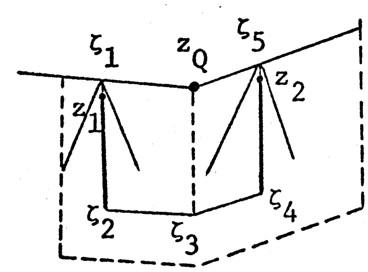
$$\leq |z_1 - z_2|^{-\gamma} \int_{z_1}^{z_2} 2Kr(t)^{\gamma-1}ds_t$$

$$\leq |z_1 - z_2|^{-\gamma}2^{2-\gamma}Kr(z_Q)^{\gamma-1} \int_{z_1}^{z_2} ds_t$$

$$= |z_1 - z_2|^{1-\gamma}\, 2^{2-\gamma} K^r (z_Q)^{\gamma-1} \le 2^{2-\gamma} K$$

taking the direct path from z_1 to z_2 and noting that $|z_1 - z_2| \le r(z_Q)$.

Let $z_Q \in \partial D$. Since the boundary is piecewise smooth and contains neither interior nor exterior cusps, without loss of generality we can assume that there exists a sufficiently small neighborhood N_Q of z_Q with the property that there exists a constant η_Q such that for any $z_1, z_2 \in N_Q$ with $h = |z_1 - z_2| > 0$,



$$r(t) \ge \eta_Q\, |t - \zeta_1|\ , \quad t \in \overrightarrow{\zeta_1\zeta_2}$$

$$r(t) \ge \eta_Q h\ , \quad t \in \overrightarrow{\zeta_2\zeta_3\zeta_4}$$

$$r(t) \ge \eta_Q |t - \zeta_5|\ , \quad t \in \overrightarrow{\zeta_4\zeta_5}$$

where $\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5$ are taken such that $|\zeta_1 - \zeta_2| = |\zeta_5 - \zeta_4| = h$.

Then

$$|\psi_2(z_1 z_2)| \le |z_1 - z_2|^{-\gamma} \left\{\int_{z_1}^{\zeta_2} + \int_{\zeta_2}^{\zeta_4} + \int_{\zeta_4}^{\zeta_5}\right\} \{|M(z_2,t,\bar{t})| + |N(z_2,t,\bar{t})|\} ds_t$$

$$\le h^{-\gamma}\, 2K \left\{\int_{z_1}^{\zeta_2} + \int_{\zeta_2}^{\zeta_4} + \int_{\zeta_4}^{\zeta_5}\right\} r(t)^{\gamma-1} ds_t$$

$$\le 2Kh^{-\gamma}\eta_Q^{\gamma-1} \left\{\int_{z_1}^{\zeta_2} |t - \zeta_1|^{\gamma-1} ds_t + \int_{\zeta_2}^{\zeta_4} h^{\gamma-1} ds_t + \int_{\zeta_4}^{z_2} |t - \zeta_5|^{\gamma-1} ds_t\right\}$$

$$\le 2Kh^{-\gamma}\eta_Q^{\gamma-1} \left\{\int_0^{|\zeta_2-\zeta_1|} s^{\gamma-1} ds + h^{\gamma-1}\{|\zeta_4 - \zeta_3| + |\zeta_3 - \zeta_2|\}\right.$$

$$+ \int_0^{|\zeta_5-\zeta_4|} s^{\gamma-1} ds$$

$$= 2Kh^{-\gamma}\eta_Q^{\gamma-1}\{\frac{1}{\gamma} h^\gamma + 2h^\gamma + \frac{1}{\gamma} h^\gamma\}$$

$$= 4K(\frac{1}{\gamma} + 1)\eta_Q^{\gamma-1}.$$

Q.E.D.

Proof of Theorem 4.1:

The extended solution $U(z,z^*)$ satisfies (Lewy [10])

$$U(z,z^*) = U(\overline{z^*},z^*)G(\overline{z^*},z^*,z,z^*)$$

$$+ \int_{\overline{z^*}}^{z} U(t,\overline{t})[G_{t^*}(t,\overline{t},z,z^*) - A(t,\overline{t})G(t,\overline{t},z,z^*)]d\overline{t}$$

$$+ [B(t,\overline{t})U(t,\overline{t}) + U_z(t,\overline{t})]G(t,\overline{t},z,z^*)dt$$

where the integral is path-independent and G is the Riemann function. From Theorem 3.2, the associated analytic function $\phi(z)$ satisfies

$$\phi(z) = 2U(z,\overline{z}_0) - U(z_0,\overline{z}_0)G(z_0,\overline{z}_0,z,\overline{z}_0)$$

so that

$$\phi(z) = U(z_0,\overline{z}_0)G(z_0,\overline{z}_0,z,\overline{z}_0)$$

$$+ 2\int_{z_0}^{z} U(t,\overline{t})[G_{t^*}(t,\overline{t},z,\overline{z}_0) - A(t,\overline{t})G(t,\overline{t},z,\overline{z}_0)]d\overline{t}$$

$$+ [B(t,\overline{t})U(t,\overline{t}) + U_z(t,\overline{t})]G(t,\overline{t},z,\overline{z}_0)dt$$

Differentiating,

$$\frac{d^p\phi}{dz^p}(z) = U(z_0,\overline{z}_0)\frac{\partial^p G}{\partial z^p}(z_0,\overline{z}_0,z,\overline{z}_0)$$

$$+ 2\sum_{k=0}^{p-1}\frac{d^{p-k-1}}{dz^{p-k-1}}[U(z,\overline{z})B(z,\overline{z})\frac{\partial^k G}{\partial z^k}(z,\overline{z},z,\overline{z}_0)$$

$$+ \frac{\partial U}{\partial z}(z,\bar{z}) \frac{\partial^k G}{\partial z^k}(z,\bar{z},z,\bar{z}_0)]$$

$$+ 2 \int_{z_0}^{z} U(t,\bar{t})[\frac{\partial^p G_{t^*}}{\partial z^p}(t,\bar{t},z,\bar{z}_0) - A(t,\bar{t})\frac{\partial^p G}{\partial z^p}(t,\bar{t},z,\bar{z}_0)]d\bar{t}$$

$$+ [B(t,\bar{t})U(t,\bar{t}) + U_z(t,\bar{t})] \frac{\partial^p G}{\partial z^p}(t,\bar{t},z,\bar{z}_0)dt$$

$$= C_p(z) + \sum_{j=0}^{p} C_{pj}(z,\bar{z}) \frac{\partial^{p-j} U}{\partial^{p-j} z}(z,\bar{z}) + R_p(z,\bar{z})$$

where the coefficients

$$C_p(z) = \frac{\partial^p G}{\partial z^p}(z_0,\bar{z}_0,z,\bar{z}_0)$$

$$C_{pj}(z,\bar{z}) = 2\sum_{k=0}^{j}\left\{\binom{p-k-1}{j-k-1}\frac{d^{j-k-1}}{dz^{j-k-1}}\left[B(z,\bar{z})\frac{\partial^k G}{\partial z^k}(z,\bar{z},z,\bar{z}_0)\right]\right.$$

$$\left. + \binom{p-k-1}{j-k}\frac{d^{j-k}}{dz^{j-k}}\left[\frac{\partial^k G}{\partial z^k}(z,\bar{z},z,\bar{z}_0)\right]\right\}$$

are analytic for $(z,z^*) \in \mathcal{D} \times \mathcal{D}^*$ and the integral

$$R_p(z,\bar{z}) = 2\int_{z_0}^{z} U(t,\bar{t})\left[\frac{\partial^p G_{t^*}}{\partial z^p}(t,\bar{t},z,\bar{z}_0) - A(t,\bar{t})\frac{\partial^p G}{\partial z^p}(t,\bar{t},z,\bar{z}_0)\right]d\bar{t}$$

$$+ [B(t,\bar{t})U(t,\bar{t}) + U_z(t,\bar{t})]\frac{\partial^p G}{\partial z^p}(t,\bar{t},z,\bar{z}_0)\, dt$$

$$= \int_{z_0}^{z} N(z,t,\bar{t})d\bar{t} + M(z,t,\bar{t})dt$$

is path independent. Moreover,

$$\frac{\partial^k U}{\partial z^k}(z,\bar{z}) = [\frac{1}{2}(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y})]^k u(x,y) = 2^{-k}\sum_{\ell=0}^{k}\binom{k}{\ell} i^{k-\ell}\frac{\partial^k u}{\partial x^\ell \partial y^{k-\ell}} .$$

Since $u \in C^{p,\gamma}(D \cup \partial D)$, a fortiori $u \in C^{k,\gamma}(D\cup\partial D)$ $(0 \le k \le p)$ and

$\frac{\partial^k U}{\partial z^k}(z,\bar{z}) \in C^{0,\gamma}(D \cup \partial D)$. From Lemma 4.1,

$$\left|\frac{\partial U}{\partial z}(z,\bar{z})\right| \leq Kr(z)^{\gamma-1}.$$

Thus the functions $M(z,t,\bar{t})$, $N(z,t,\bar{t})$ satisfy the conditions of Lemma 4.3, and $R_p(z,\bar{z}) \in C^{0,\gamma}(D \cup \partial D)$. Therefore $\frac{d^p\phi}{dz^p} \in C^{0,\gamma}(D \cup \partial D)$ or $\phi \in C^{p,\gamma}(D \cup \partial D)$. Q.E.D.

Theorem 4.1 requires that u be more than merely continuous and that ∂D have neither interior nor exterior cusps. The necessity of these restrictions is shown by the following examples.

Example (Privalov [13]):

If u is merely continuous, then ϕ need not be continuous. For let $D = \{z \,|\, |z| < 1\}$, the unit disk, and let the sequence $\{a_k\}_{k=1}^{\infty}$ satisfy(*)

$$a_1 > a_2 > a_3 > \ldots ; \quad \lim_{k\to\infty} a_k = 0; \quad \sum_{k=1}^{\infty} \frac{a_k}{k} = +\infty .$$

Define

$$u(r,\theta) = \sum_{k=1}^{\infty} \frac{a_k r_k \sin k\theta}{k}, \qquad re^{i\theta} \in D \cup \partial D$$

$$\phi(z) = \frac{1}{i} \sum_{k=1}^{\infty} \frac{a_k z^k}{k}, \qquad z \in D$$

Then $\Delta u = 0$ in D, ϕ is analytic in D, and

$$u(r,\theta) = \mathrm{Re}\, \phi(z) = \mathrm{Re}\, V[\phi;0](z,\bar{z}), \quad z = re^{i\theta} \in D.$$

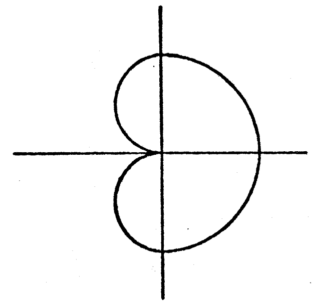
However, $\phi \notin C(D \cup \partial D)$ even though $u \in C(D \cup \partial D)$ since the series for

(*) E.g., $a_k = \frac{1}{1 + \log k}$

the conjugate harmonic function

$$v(r,\theta) = \mathrm{Im}\,\phi(z) = \sum_{k=1}^{\infty} \frac{a_k r^k \cos k\theta}{k}$$

diverges at $z = 1$.



Example:

If u is Hölder continuous but ∂D contains an interior (reentrant) cusp, then ϕ need not be Hölder continuous with the same exponent. For let

$$D = \{re^{i\theta} \,\big|\, 0 < r < 1;\ -\pi < \theta < \pi;\ |\sin\theta| > r \text{ if } |\theta| > \tfrac{\pi}{2}\}$$

and define

$$u(r,\theta) = r^{\gamma}\cos\gamma\theta \qquad z = re^{i\theta} \in D \quad (0 < \gamma < 1).$$

$$\phi(z) = z^{\gamma} \equiv r^{\gamma}(\cos\gamma\theta + i\sin\gamma\theta)$$

Then $\Delta u = 0$ in D, ϕ is analytic in D, and

$$u(r,\theta) = \mathrm{Re}\,\phi(z) = \mathrm{Re}\,V[\phi;0](z,\bar{z}), \quad z = re^{i\theta} \in D.$$

However,

(A) $\qquad \phi \notin C^{0,\gamma}(D \cup \partial D)$

even though

(B) $\qquad u \in C^{0,\gamma}(D \cup \partial D)$

Proof of (A):

For $0 < r < 2^{1-\frac{1}{\gamma}}$, let $\theta = \theta(r)$ satisfy

$$\sin\theta = 2^{\frac{1}{\gamma}-1} r, \quad \frac{\pi}{2} < \theta < \pi$$

and define $z = z(r) = re^{i\theta}$. Then $z, \bar{z} \in D$ but

$$\frac{|\phi(z) - \phi(\bar{z})|}{|z - \bar{z}|^\gamma} = \frac{|2ir^\gamma \sin\gamma\theta|}{|2ir\sin\theta|^\gamma} = \frac{2^{1-\gamma}\sin\gamma\theta}{(\sin\theta)^\gamma} = \frac{\sin\gamma\theta}{r^\gamma}$$

$$\geq \frac{1}{r^\gamma} \inf_{\frac{\pi}{2}<\theta<\pi} \{\sin\gamma\theta\} = \frac{1}{r^\gamma} \min\{\sin\frac{\gamma\pi}{2}, \sin\gamma\pi\} \to +\infty$$

as $r \to 0$.

Q.E.D.

Proof of (B):

Let $r_1e^{i\theta_1}$, $r_2e^{i\theta_2} \in D \cup \partial D$ where without loss of generality $r_1 \geq r_2$. Then

$$|u(r_1,\theta_1)-u(r_2,\theta_2)| \leq |r_1^\gamma \cos\gamma\theta_1 - r_2^\gamma \cos\gamma\theta_1| + |r_2^\gamma \cos\gamma\theta_1 - r_2^\gamma\cos\gamma\theta_2|$$

$$= |r_1^\gamma - r_2^\gamma||\cos\gamma\theta_1| + r_2^\gamma|\cos\gamma\theta_1 - \cos\gamma\theta_2|$$

$$\leq |r_1^\gamma - r_2^\gamma| + 2r_2^\gamma\left|\sin\frac{\gamma(\theta_1-\theta_2)}{2}\right|\left|\sin\frac{\gamma(\theta_1+\theta_2)}{2}\right|$$

$$\leq |r_1 - r_2|^\gamma + \gamma r_2^\gamma \min(|\theta_1-\theta_2|, |\theta_1+\theta_2|)$$

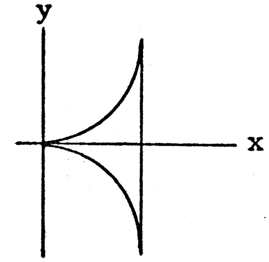
$$\leq |r_1-r_2|^\gamma + \frac{\gamma}{\pi} r_2^\gamma |e^{i\theta_1} - e^{i\theta_2}|$$

$$\leq |r_1e^{i\theta_1} - r_2e^{i\theta_1}|^\gamma + \frac{2^{1-\gamma}\gamma}{\pi}|r_2e^{i\theta_1} - r_2e^{i\theta_2}|^\gamma$$

$$\leq (1 + \frac{2^{1-\gamma}\gamma}{\pi})|r_1e^{i\theta_1} - r_2e^{i\theta_2}|^\gamma.$$

Q.E.D.

<u>Example</u>: (Babuska [1])



If u is Hölder continuous but ∂D contains an exterior cusp, then ϕ <u>need not even be continuous</u>. For let

$$D = \{(x,y)\,|\,0 < x < \frac{1}{4},\ |y| < \frac{1}{4},\ \rho(x,y) \equiv \frac{|y|^{\frac{1-\gamma}{2}}}{x^2+y^2} < 1\} \qquad (0 < \gamma < 1)$$

and define

$$u(x,y) = \frac{y}{x^2+y^2}, \quad \phi(z) = \frac{i}{z}, \quad z = x + iy \in D.$$

Then $\Delta u = 0$ in D, ϕ is analytic in D, and

$$u(x,y) = \operatorname{Re} \phi(z) = \operatorname{Re} V[\phi;0](z,\bar{z}), \quad z = x + iy \in D.$$

However, ϕ is not continuous in $D \cup \partial D$ (ϕ has a pole at $z = 0$) even though

$$\text{(C)} \qquad u \in C^{0,\gamma}(D \cup \partial D).$$

<u>Proof of (C)</u>:

Let (x_1,y_1), $(x_2,y_2) \in D \cup \partial D$ where without loss of generality $x_1 \leq x_2$. Then letting $\lambda = \frac{1-\gamma}{2}$,

$$|u(x_1,y_1) - u(x_2,y_2)| \leq \left|\frac{y_1}{x_1^2+y_1^2} - \frac{y_1}{x_2^2 y_1^2}\right| + \left|\frac{y_1}{x_2^2 y_1^2} - \frac{y_2}{x_2^2 y_2^2}\right|$$

$$= \frac{|y_1|}{(x_1^2+y_1^2)(x_2^2+y_1^2)} \left|x_1^2 - x_2^2\right| + \frac{|x_2^2 - y_1 y_2|}{(x_2^2+y_1^2)(x_2^2+y_2^2)} |y_1 - y_2|$$

$$\leq \frac{|y_1|\,|x_1+x_2|}{(x_1^2+y_1^2)(x_2^2+y_1^2)} |x_1 - x_2| + \frac{x_2^2}{(x_2^2+y_1^2)(x_2^2+y_2^2)} |y_1 - y_2|$$

$$+ \frac{|y_1|\,|y_2|}{(x_2^2+y_1^2)(x_2^2+y_2^2)} |y_1 - y_2|$$

$$= \frac{|y_1|^\lambda}{x_1^2 + y_1^2} \cdot \frac{|y_1|^\lambda}{x_2^2 + y_1^2} \cdot |y_1|^\gamma |x_1 + x_2| |x_1 - x_2|^{1-\gamma} \cdot |x_1 - x_2|^\gamma$$

$$+ \frac{x_2^2}{x_2^2 + \min(|y_1|,|y_2|)^2} \cdot \frac{\max(|y_1|,|y_2|)^\lambda}{x_2^2 + \max(|y_1|,|y_2|)^2}$$

$$\cdot \frac{|y_1 - y_2|^{2\lambda}}{\max(|y_1|^\lambda,|y_2|^\lambda)} \cdot |y_1 - y_2|^\gamma$$

$$+ \frac{|y_1|^\lambda}{x_2^2 + y_1^2} \cdot \frac{|y_2|^\lambda}{x_2^2 + y_2^2} \cdot |y_1 y_2|^{1-\lambda} |y_1 - y_2|^{1-\gamma} \cdot |y_1 - y_2|^\gamma$$

$$\leq |x_1 - x_2|^\gamma + |y_1 - y_2|^\gamma + |y_1 - y_2|^\gamma, \quad \text{since } (x_2,y_1) \in D \cup \partial D.$$

Q.E.D.

For computational purposes, it is desirable to allow the origin point z_0 to lie on ∂D as well as in D. The following result extends Theorem 3.2 in this regard and relates the smoothness of Re $V[\phi;z_0]$ to the smoothness of ϕ.

Theorem 4.2: Let D be a simply connected domain of class $\mathcal{R}$ and fix $z_0 \in D\cup\partial D$.

(i) Let $\phi \in C^{p,\gamma}(D\cup\partial D)$ be analytic in D and define

$$u(x,y) \equiv \text{Re } V[\phi;z_0](z,\bar{z}), \quad z = x + iy \in D\cup\partial D.$$

Then $u \in C^{p,\gamma}(D\cup\partial D)$ and satisfies $Lu = 0$ in D.

(ii) If $u \in C^{p,\gamma}(D\cup\partial D)$ satisfies $Lu = 0$ in D, then there exists a unique analytic function $\phi \in C^{p,\gamma}(D\cup\partial D)$ with $\phi(z_0)$ real such that

$$u(x,y) = \text{Re } V[\phi;z_0](z,\bar{z}), \quad z = x + iy \in D\cup\partial D$$

$$U(z,z^*) = I[\phi;z_0](z,z^*), \quad (z,z^*) \in (D\cup\partial D) \times (D^*\cup\partial D^*).$$

Moreover,

$$\phi(z) = 2U(z,\overline{z}_0) - U(z_0,\overline{z}_0)G(z_0,\overline{z}_0,z,\overline{z}_0).$$

Proof:

(i) Let $\phi \in C^{p,\gamma}(D \cup \partial D)$ be analytic in D. A long but straightforward formal calculation shows that $u \equiv \text{Re } V[\phi;z_0]$ satisfies $Lu = 0$ in D. On differentiating, the p^{th} derivatives of u are seen to depend only on $\frac{d^k\phi}{dz^k}$ $(0 \leq k \leq p)$. But $\phi \in C^{p,\gamma}(D \cup \partial D)$ whence $\frac{d^k\phi}{dz^k} \in C^{0,\gamma}(D \cup \partial D)$. Therefore $u \in C^{p,\gamma}(D \cup \partial D)$.

(ii) Let $u \in C^{p,\gamma}(D \cup \partial D)$ and fix $\tilde{z}_0 \in D$. By Vekua's Theorem 3.2, there exists a unique analytic function $\tilde{\phi} \in C^{p,\gamma}(D \cup \partial D)$ with $\tilde{\phi}(\tilde{z}_0)$ real such that $U = I[\tilde{\phi};\tilde{z}_0]$. As in part (i), $U \in C^{p,\gamma}$ as is

$$\psi(z,z_0) = 2U(z,\overline{z}_0) - U(z_0,\overline{z}_0)G(z_0,\overline{z}_0,z,\overline{z}_0)$$

for each $z_0 \in D \cup \partial D$. Moreover, since $U(z,z^*)$ is analytic for $z \in D$ given any $z^* \in D \cup \partial D$, we have $\psi(\cdot,z_0)$ is analytic in D for $z_0 \in D \cup \partial D$. But $u = \text{Re } V[\psi(\cdot,z_0);z_0]$ for every $z_0 \in D$ by Vekua's Theorem and therefore for every $z_0 \in D \cup \partial D$ using the continuity of Re V and $U(z,z^*)$. The same argument shows that $U = I[\psi(\cdot,z_0);z_0]$. Now take $\phi(z) = \psi(z,z_0)$.

Q.E.D.

§5. An Asymptotic Expansion

Lehman [9] has obtained asymptotic expansions in the neighborhood of an analytic corner for solutions of $Lu = 0$ which are analytic on each of the adjacent boundary arcs. In this section we shall give a similar asymptotic expansion for the associated analytic function.

Let D be a simply connected domain of class $\mathcal{R}$ with analytic corner (z_q, α_q), and let $\alpha \equiv \alpha_q$. The asymptotic expansions considered are in terms of

$$z - z_q, \quad (z - z_q)^{\alpha}$$

when α is irrational, and in terms of

$$z - z_q, \ (z - z_q)^{\alpha}, \ (z - z_q)^{\ell} \log (z - z_q)$$

when α is rational, $\alpha = \frac{\ell}{m}$ in lowest terms; for simplicity, we shall state and prove results only for the case α irrational.

A function $g(z)$ defined in D is said to have an asymptotic expansion of order η if

$$g(z) = P_{\eta}(z,\bar{z};z_q,\alpha) + O(|z - z_q|^{\eta})$$

as $z \to z_q$ in D, where

$$P_{\eta}(z,\bar{z};z_q,\alpha) = \sum_{\substack{j+k\alpha+\ell+m\alpha<\eta \\ j,k,\ell,m\geq 0}} C_{jk\ell m}(z-z_q)^{j+k\alpha}(\overline{z}-\overline{z}_q)^{\ell+m\alpha}.$$

The asymptotic expansion is said to be p-times differentiable if

$$\frac{\partial^k g}{\partial^j z \partial^{k-j} \bar{z}}(z) = \frac{\partial^k P_{\eta}}{\partial^j z \partial^{k-j} \bar{z}}(z,\bar{z};z_q,\alpha) + O(|z-z_q|^{\eta-k}), \quad 0 \leq j \leq k \leq p.$$

A function has at most one asymptotic expansion of any order (Lehman [9]). Moreover, if $g(z)$ has a p-times differentiable asymptotic expansion of order η, then $g(z)$ has a p-times differentiable asymptotic expansion of order η' for any $\eta' \leq \eta$ (simply neglect higher order terms).

Theorem 5.1: (Lehman [9]) Let $u \in C(D \cup \partial D) \cap \mathcal{PA}(z_q)$ satisfy $Lu = 0$ in D. Then for any $p \geq 0$, u has a p-times differentiable asymptotic expansion of order p.

Theorem 5.2: Let $u \in C^{0,\beta}(D \cup \partial D) \cap \mathcal{PA}(z_q)$ $(0 < \beta \leq 1)$ satisfy $Lu = 0$ in D. Fix $z^{(q)} \in D \cup \partial D$ and let ϕ_q be the corresponding associated analytic function: $u = \operatorname{Re} V[\phi_q; z^{(q)}]$. Then for any $p \geq 0$, ϕ_q has a p-times differentiable asymptotic expansion of order p:

$$\phi_q(z) = \sum_{\substack{j+k\alpha<p \\ j,k\geq 0}} c_{qjk}(z - z_q)^{j+k\alpha} + O(|z - z_q|^p).$$

Proof:

Let $Q_\nu(z,\bar{z};z_q,\alpha)$ denote a generic asymptotic expansion of order ν

$$Q_\nu(z,\bar{z};z_q,\alpha) = \sum_{\substack{j+k\alpha+\ell+m\alpha<\nu \\ j,k,\ell,m\geq 0}} \gamma_{jk\ell m}(z-z_q)^{j+k\alpha}(\bar{z}-\bar{z}_q)^{\ell+m\alpha}$$

where the coefficients may vary from occurrence to occurrence. Let $\tilde{Q}_\nu(z,\bar{z};z_q,\alpha_q)$ denote a generic asymptotic expansion of the same form except that the coefficients $\gamma_{j0\ell m}$ vanish.

The following properties of these generic asymptotic expansions are immediate (remember that α is irrational so that $j + k\alpha$ is non-integral for $k > 0$):

$$Q_\nu = O(1), \quad Q_\nu\tilde{Q}_\nu \text{ is of the form } \tilde{Q}_\nu + O(|z - z_q|^\nu)$$

$\frac{\partial^\nu Q_\nu}{\partial z^\nu}$ is of the form $(z-z_q)^{-\nu}\tilde{Q}_\nu$, $(z-z_q)^{-\nu}\tilde{Q}_\nu$ is of the form $\frac{\partial^\nu Q_\nu}{\partial z^\nu}$.

As in the proof of Theorem 4.1,

$$\frac{d^p\phi_q}{dz^p}(z) = C_p(z,\bar{z}) + \sum_{\nu=0}^{p} C_{p-\nu,p}(z,\bar{z}) \frac{\partial^\nu U}{\partial z^\nu}(z,\bar{z}) + R_p(z,\bar{z}).$$

Since the coefficients $C_{p-\nu,p}$ and C_p are analytic, the Taylor series yields the following differentiable asymptotic expansions:

$$C_{p-\nu,p}(z,\bar{z}) = \sum_{\substack{j+\ell<\nu \\ j,\ell\geq 0}} \frac{1}{j!}\frac{1}{\ell!} \frac{\partial^{j+\ell} C_{p-\nu,p}}{\partial z^j \partial \bar{z}^\ell}(z_q,\bar{z}_q)(z-z_q)^j(\overline{z-z_q})^\ell + O(|z-z_q|^\nu)$$

$$= Q_\nu(z,\bar{z};z_q,\alpha) + O(|z-z_q|^\nu)$$

$$C_p(z,\bar{z}) = O(1)$$

From Lehman's Theorem, $U(z,\bar{z})$ has a $\nu+1$ times differentiable asymptotic expansion of order ν for $0 \leq \nu \leq p$:

$$U(z,\bar{z}) = P_\nu(z,\bar{z};z_q,\alpha) + O(|z-z_q|^\nu)$$

$$\frac{\partial^\nu U}{\partial z^\nu}(z,\bar{z}) = \frac{\partial^\nu P_\nu}{\partial z^\nu}(z,\bar{z};z_q,\alpha) + O(1)$$

$$= (z-z_q)^{-\nu}\tilde{Q}_\nu(z,\bar{z};z_q,\alpha) + O(1),$$

the expansion for $\frac{\partial^\nu U}{\partial z^\nu}$ being again differentiable. Since $R_p \in C^{0,\beta}(D\cup\partial D)$, R_p has a differentiable asymptotic expansion

$$R_p(z,\bar{z}) = 0 + O(1)$$

for

$$\frac{\partial R_p}{\partial \bar{z}}(z,\bar{z}) = 2U(z,\bar{z})\left\{\frac{\partial^p G_{t*}}{\partial z^p}(z,\bar{z},z,\bar{z}) - A(z,\bar{z})\frac{\partial^p G}{\partial z^p}(z,\bar{z},z,\bar{z})\right\}$$

so that $\frac{\partial R_p}{\partial \bar{z}}$ is continuous whence bounded in $D\cup\partial D$, and

$$\frac{\partial R_p}{\partial \bar{z}}(z,\bar{z}) = O(|z-z_q|^{-1}).$$

Combining these results,

$$\frac{d^p\phi_q}{dz^p}(z) = C_p(z,\bar{z}) + \sum_{\nu=0}^{p} C_{p-\nu,p}(z,\bar{z})\frac{\partial^\nu U}{\partial z^\nu}(z,\bar{z}) + R_p(z,\bar{z})$$

$$= O(1) + \sum_{\nu=0}^{p}\{Q_\nu + O(|z-z_q|^\nu)\}\{(z-z_q)^{-\nu}\tilde{Q}_\nu + O(1)\} + O(1)$$

$$= \sum_{\nu=0}^{p}\{(z-z_q)^{-\nu}\tilde{Q}_\nu + O(1)\} + O(1)$$

$$= (z-z_q)^{-p}\left\{\sum_{\nu=0}^{p}(z-z_q)^{p-\nu}\tilde{Q}_\nu\right\} + O(1)$$

$$= (z-z_q)^{-p}\hat{Q}_p(z,\bar{z};z_q,\alpha) + O(1)$$

an asymptotic expansion differentiable with respect to $\bar{z}$. But ϕ_q is analytic in D whence

$$0 = \frac{\partial}{\partial \bar{z}}\frac{d^p\phi_q}{dz^p}(z) = (z-z_q)^{-p}\frac{\partial \hat{Q}_p}{\partial \bar{z}}(z,\bar{z};z_q,\alpha) + O(|z-z_q|^{-1})$$

and

$$\frac{\partial \hat{Q}_p}{\partial \bar{z}}(z,\bar{z};z_q,\alpha) = O(|z-z_q|^{p-1}).$$

From the uniqueness of asymptotic expansions, $\frac{\partial \hat{Q}_p}{\partial \bar{z}} = 0$ so that $\hat{Q}_p$ is analytic in D and does not depend on $\bar{z}$:

$$\hat{Q}_p = \sum_{\substack{0<j+k\alpha<p \\ j\geq 0,k>0}} \hat{\gamma}_{jk}(z-z_q)^{j+k\alpha}.$$

Let

$$\hat{P}(z;z_q,\alpha) = \sum_{\substack{j+k\alpha<p \\ j\geq 0,k>0}} \left\{ \frac{\hat{\gamma}_{jk}}{\prod_{\ell=0}^{p-1} (j+k\alpha-\ell)} \right\} (z-z_q)^{j+k\alpha}.$$

Then

$$\frac{d^p\hat{P}}{dz^p} = (z-z_q)^{-p}\hat{Q}_p = \frac{d^p\phi_q}{dz^p} + O(1).$$

Integrating the preceding equation from z_q to z,

$$\frac{d^{p-1}\phi_q}{dz^{p-1}}(z) = \frac{d^{p-1}\hat{P}}{dz^{p-1}}(z) + \left[\frac{d^{p-1}(\phi_q-\hat{P})}{dz^{p-1}}\right]_{z=z_q} + O(|z-z_q|^1).$$

By induction,

$$\phi_q(z) = \hat{P}(z;z_q,\alpha) + \sum_{\ell=0}^{p-1} \frac{1}{\ell!} \left[\frac{d^\ell(\phi_q-\hat{P})}{dz^\ell}\right]_{z=z_q} (z-z_q)^\ell + O(|z-z_q|^p)$$

$$= \sum_{\substack{j+k\alpha<p \\ j,k\geq 0}} C_{jk}(z-z_q)^{j+k\alpha} + O(1)$$

Q.E.D.

§6. An Approximation Problem

The generalized harmonic polynomials $u_n = \text{Re } V[p_n;z_0]$, where p_n denotes a polynomial of degree n, comprise a special class of solutions of $Lu = 0$. In this section, we shall study the problem of approximating arbitrary solutions by generalized harmonic polynomials.

Let $u \in C(D \cup \partial D)$ satisfy $Lu = 0$ in D and let ϕ be the associated analytic function. Then the generalized harmonic polynomial $u_n = \text{Re } V[p_n;z_0]$ satisfies

$$\|u - u_n\|_{D\cup\partial D} = \|\text{Re } V[\phi;z_0] - \text{Re } V[p_n;z_0]\|_{D\cup\partial D}$$

$$= \|\text{Re } V[\phi - p_n;z_0]\|_{D\cup\partial D} \leq K_V\|\phi - p_n\|_{D\cup\partial D}.$$

Thus the approximation problem can be reduced to the problem of approximating an analytic function by polynomials, a problem studied by Walsh [17], Sewell [15], and Mergelyan [11].

<u>Theorem 6.1</u> (Mergelyan): Let D be a simply connected domain of class $R(\mu)$ $(0 < \mu \leq 1)$. If $\phi \in C^{p,\gamma}(D\cup\partial D)$ is analytic in D, then for any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ independent of n such that

$$E_n(\phi) \equiv \min_{p_n \in \mathbb{P}_n} \|\phi - p_n\|_{D\cup\partial D} \leq \frac{C(\varepsilon)}{n^{\mu(p+\gamma)-\varepsilon}} ,$$

where $\mathbb{P}_n$ denotes the space of polynomials of degree n.

As an immediate consequence of this result, we have

<u>Theorem 6.2</u>: Let D be a simply connected domain of class $R(\mu)$ $(0 < \mu \leq 1)$. If $u \in C^{p,\gamma}(D\cup\partial D)$ satisfies $Lu = 0$ in D, then for

any $\varepsilon > 0$ there exists a constant $K(\varepsilon)$ independent of n such that

$$E_n(u) \equiv \min_{a_j^{(n)}, b_j^{(n)}} \| u - u_n[a_j^{(n)}, b_j^{(n)}] \|_{D \cup \partial D} \leq \frac{K(\varepsilon)}{n^{\mu(p+\gamma)-\varepsilon}}$$

where

$$u_n[a_j^{(n)}, b_j^{(n)}] = \sum_{j=0}^{n} \{a_j^{(n)} \operatorname{Re} V[z^j; z_0] + b_j^{(n)} \operatorname{Im} V[z^j; z_0]\}$$

Proof:

By Theorem 4.2, if $u \in C^{p,\gamma}(D \cup \partial D)$, then $\phi \in C^{p,\gamma}(D \cup \partial D)$ and, by Mergelyan's Theorem, for any $\varepsilon > 0$ there exists a constant $C(\varepsilon)$ independent of n such that

$$\min_{p_n \in \mathbb{P}_n} \| \phi - p_n \|_{D \cup \partial D} \leq \frac{C(\varepsilon)}{n^{\mu(p+\gamma)-\varepsilon}}$$

whence

$$\min_{p_n \in \mathbb{P}_n} \| u - \operatorname{Re} V[p_n; z_0] \|_{D \cup \partial D} \leq \min_{p_n \in \mathbb{P}_n} K_V \| \phi - p_n \|_{D \cup \partial D} \leq \frac{K_V C(\varepsilon)}{n^{\mu(p+\gamma)-\varepsilon}} .$$

But writing $p_n(z) = \sum_{j=0}^{n} (a_j^{(n)} - i b_j^{(n)}) z^j$,

$$\operatorname{Re} V[p_n; z_0] = \sum_{j=0}^{n} \{a_j^{(n)} \operatorname{Re} V[z^j; z_0] - b_j^{(n)} \operatorname{Re} iV[z^j; z_0]\}$$

$$= u_n[a_j^{(n)}, b_j^{(n)}].$$

Q.E.D.

In domains with piecewise smooth boundary, the degree of generalized harmonic polynomial approximation may be limited by a lack of smoothness in the solution near the corners. However, if the corners are analytic corners and the solution is analytic on each of the adjacent boundary arcs, then the asymptotic expansions of §5 suggest the use of the singular particular solutions

$$\begin{Bmatrix} \mathrm{Re} \\ \mathrm{Im} \end{Bmatrix} V[(z - z_q)^{j+k\alpha_q}; z^{(q)}]$$

which have the correct singularities near the corners. (In practice, $z^{(q)} = z_q$ or $z^{(q)} = z_0$ but we shall not make any such assumptions on $z^{(q)}$.)

Theorem 6.3: Let D be a simply connected domain of class $\mathcal{R}(\mu)$ $(0 < \mu \leq 1)$ with analytic corners $(z_1, \alpha_1), \ldots, (z_q, \alpha_q)$ and let $u \in C^{p,\gamma}(D \cup \partial D - \{z_1,\ldots,z_q\}) \cap \mathcal{PA}(z_1,\ldots,z_q)$ $(p + \gamma > 0)$ satisfy $Lu = 0$ in D. Fix $z_0, z^{(1)}, \ldots, z^{(q)} \in D \cup \partial D$. Then for any $\varepsilon > 0$ there exists a constant $\tilde{K}(\varepsilon)$ independent of n such that

$$\tilde{E}_n(u) \equiv \min_{a_j^{(n)}, b_j^{(n)}, a_{\ell jk}^{(n)}, b_{\ell jk}^{(n)}} \| u - \tilde{u}_n[a_j^{(n)}, b_j^{(n)}, a_{\ell jk}^{(n)}, b_{\ell jk}^{(n)}] \|_{D\cup\partial D}$$

$$\leq \frac{\tilde{K}(\varepsilon)}{n^{\mu(p+\gamma)-\varepsilon}} .$$

where

$$\tilde{u}_n[a_j^{(n)}, b_j^{(n)}, a_{\ell jk}^{(n)}, b_{\ell jk}^{(n)}] = \sum_{j=0}^{n} \{a_j^{(n)} \mathrm{Re}\, V[z^j, z_0] + b_j^{(n)} \mathrm{Im}\, V[z^j; z_0]\}$$

$$+ \sum_{\ell=1}^{q} \sum_{\substack{j+k\alpha_\ell < p+\gamma \\ j,k \geq 0}} \{a_{\ell jk}^{(n)} \mathrm{Re}\, V[(z-z_\ell)^{j+k\alpha_\ell}; z^{(\ell)}]$$

$$+ b_{\ell jk}^{(n)} \mathrm{Im}\, V[(z-z_\ell)^{j+k\alpha_\ell}; z^{(\ell)}]\}.$$

Proof:

We first prove that $u \in C^{0,\beta}(D\cup\partial D)$ with $\beta = \min(\gamma, \alpha_1, \ldots, \alpha_q)$ so that we may apply Theorem 5.2 (note that $0 < \beta \leq 1$). By assumption, $u \in C^{p,\gamma}(D\cup\partial D - \{z_1,\ldots,z_q\})$ whence $u \in C^{0,\beta}(D\cup\partial D - \{z_1,\ldots,z_q\})$. From Lehman's Theorem, u has a differentiable asymptotic expansion of order β near each analytic corner z_ℓ:

$$u(z) = u(z_\ell) + O(|z - z_\ell|^\beta).$$

Differentiating,

$$\frac{\partial u}{\partial z} = O(|z - z_\ell|^{\beta-1}), \quad \frac{\partial u}{\partial \bar{z}} = O(|z - z_\ell|^{\beta-1}).$$

For $\delta_\ell > 0$ sufficiently small, the domain $D_\ell = \{z \in D \,\big|\, |z - z_\ell| < \delta_\ell\}$ is a simply connected domain of class R and

$$\left|\frac{\partial u}{\partial z}(z)\right| \leq K_\ell |z - z_\ell|^{\beta-1} \leq K_\ell r_\ell(z)^{\beta-1}$$

$$\forall\, z \in D_\ell$$

$$\left|\frac{\partial u}{\partial \bar{z}}(z)\right| \leq K_\ell |z - z_\ell|^{\beta-1} \leq K_\ell r_\ell(z)^{\beta-1}$$

where $r_\ell(z) = \min\limits_{t \in \partial D_\ell} |t - z|$. Since

$$u(z) = u(z_\ell) + \int_{z_q}^{z} u_z(t)dt + u_{\bar{z}}(t)d\bar{t},$$

we have $u \in C^{0,\beta}(D_\ell \cup \partial D_\ell)$ by Lemma 4.3. Therefore $u \in C^{0,\beta}(D \cup \partial D)$.

Let $v_q = u$ and let ϕ_q be the associated analytic function corresponding to the origin point $z^{(q)}$:

$$v_q = \text{Re } V[\phi_q; z^{(q)}].$$

By assumption, $v_q \in C^{p,\gamma}(D \cup \partial D - \{z_1, \ldots, z_q\})$ whence $\phi_q \in C^{p,\gamma}(D \cup \partial D - \{z_1, \ldots, z_q\})$. By Theorem 5.2, ϕ_q has a (p+1)-times differentiable asymptotic expansion of order $p + \gamma$:

$$\phi_q(z) = \sum_{\substack{j+k\alpha_q < p+\gamma \\ j,k \geq 0}} c_{qjk}(z-z_q)^{j+k\alpha_q} + O(|z-z_q|^{p+\gamma})$$

$$\equiv P_q(z) + O(|z - z_q|^{p+\gamma}).$$

Let $\tilde{v}_q = \operatorname{Re} V[P_q; z^{(q)}]$ and set $v_{q-1} = v_q - \tilde{v}_q$. Since P_q is analytic in $D \cup \partial D - \{z_q\}$, we have $P_q \in C^{p,\gamma}(D\cup\partial D - \{z_1,\ldots,z_q\})$ and $\tilde{v}_q \in PA(z_1,\ldots,z_{q-1})$. Thus $\phi_q' - P_q \in C^{p,\gamma}(D\cup\partial D - \{z_1,\ldots,z_q\})$ and $v_{q-1} \in PA(z_1,\ldots,z_{q-1})$. Since

$$\frac{d}{dz}\left\{\frac{d^p}{dz^p}(\phi_q - P_q)\right\}(z) = O(|z - z_q|^{\gamma-1}),$$

we have $\frac{d^p}{dz^p}(\phi_q - P_q) \in C^{0,\gamma}(D_q \cup \partial D_q)$ for $\delta_q > 0$ sufficiently small by the argument used above. Therefore $\frac{d^p}{dz^p}(\phi_q - P_q) \in C^{0,\gamma}(D\cup\partial D - \{z_1,\ldots,z_{q-1}\})$ or $\phi_q - P_q \in C^{p,\gamma}(D\cup\partial D - \{z_1,\ldots,z_{q-1}\})$ whence $v_{q-1} \in C^{p,\gamma}(D\cup\partial D - \{z_1,\ldots,z_{q-1}\})$.

By induction, there exist expansions

$$P_\ell(z) = \sum_{\substack{j+k\alpha_\ell < p+\gamma \\ j,k\geq 0}} c_{\ell jk}(z - z_\ell)^{j+k\alpha_\ell}$$

such that

$$v_0 \equiv u - \sum_{\ell=1}^{q} \operatorname{Re} V[P_\ell; z^{(\ell)}] \in C^{p,\gamma}(D\cup\partial D).$$

Letting $c_{\ell jk} = a_{\ell jk} - ib_{\ell jk}$,

$$v_0 = u - \sum_{\ell=1}^{q} \sum_{\substack{j+k\alpha_\ell < p+\gamma \\ j,k\geq 0}} \{a_{\ell jk} \operatorname{Re} V[(z - z_\ell)^{j+k\alpha_\ell};\ z^{(\ell)}]$$

$$+ b_{\ell jk} \operatorname{Im} V[(z - z_\ell)^{j+k\alpha_\ell};\ z^{(\ell)}]\}.$$

Therefore, applying Theorem 6.2 to v_0,

$$\tilde{E}_n(u) = \min_{a_j^{(n)}, b_j^{(n)}, a_{\ell jk}^{(n)}, b_{\ell jk}^{(n)}} \|u - \tilde{u}_n[a_j^{(n)}, b_j^{(n)}, a_{\ell jk}^{(n)}, b_{\ell jk}^{(n)}]\|_{D \cup \partial D}$$

$$\leq \min_{a_j^{(n)}, b_j^{(n)}} \|u - \tilde{u}_n[a_j^{(n)}, b_j^{(n)}, a_{\ell jk}, b_{\ell jk}]\|_{D \cup \partial D}$$

$$= \min_{a_j^{(n)}, b_j^{(n)}} \|v_0 - u_n[a_j^{(n)}, b_j^{(n)}]\|_{D \cup \partial D} \leq \frac{K(\varepsilon)}{n^{\mu(p+\gamma)-\varepsilon}} .$$

Q.E.D.

§7. The Boundary Value Problem

The method of particular solutions applied to the boundary value problem

$$Lu = \Delta u + au_x + bu_y + cu = 0 \quad \text{in } D, \qquad u = f \quad \text{on } \partial D \qquad \text{(BVP)}$$

approximates the solution by a linear combination of particular solutions fitted to the boundary data. In this section, we shall relate the smoothness of the boundary and of the boundary data to the smoothness of the solution, and, applying results on the approximation problem, we shall establish the asymptotic rate of convergence of the method.

Let D be a simply connected domain of class $\mathcal{R}$ and let $u \in C(D \cup \partial D)$ be a solution of the boundary value problem (BVP).

Theorem 7.1: If $\partial D \in C^{\max(p,1),\gamma}$ and $f \in C^{p,\gamma}(\partial D)$, then $u \in C^{p,\gamma}(D \cup \partial D)$ $(0 < \gamma < 1)$.

Vekua [16] proves the case $p = 0$ using the theory of singular integral equations, and the case $p = 1$ using the result for $p = 0$; the case $p \geq 2$ is an immediate consequence of the Schauder Boundary Estimates.

Combining this result with our results on the approximation problem, we have

Theorem 7.2: If $\partial D \in C^{\max(p,1),\gamma}$ and $f \in C^{p,\gamma}(\partial D)$, then for any $\varepsilon > 0$ there exists a constant $K(\varepsilon)$ independent of n such that

$$E_n(u) \equiv \min_{a_j^{(n)}, b_j^{(n)}} \|u - u_n\|_{D \cup \partial D} \leq \frac{K(\varepsilon)}{n^{p+\gamma-\varepsilon}}$$

where

$$u_n = \sum_{j=0}^{n} \{a_j^{(n)} \text{Re } V[z^j;z_0] + b_j^{(n)} \text{Im } V[z^j;z_0]\}.$$

Proof:

By Theorem 7.1, $u \in C^{p,\gamma}(D \cup \partial D)$. Since the boundary is smooth, the domain D is of class $R(1)$ and the result follows from Theorem 6.2.

Q.E.D.

The method of particular solutions for the boundary value problem (BVP) approximates the solution u by the linear combination of particular solutions u_n fitted to the boundary data f (Schryer [14]). If $c(x,y) \leq 0$ in D, then by the maximum principle,

$$\|u - u_n\|_{D\cup\partial D} = \|u - u_n\|_{\partial D} = \|f - u_n\|_{\partial D}$$

so that

$$E_n(f) \equiv \min_{a_j^{(n)},b_j^{(n)}} \|f - u_n\|_{\partial D} = \min_{a_j^{(n)},b_j^{(n)}} \|u - u_n\|_{D\cup\partial D} \leq \frac{K(\varepsilon)}{n^{p+\gamma-\varepsilon}}$$

under the assumptions of Theorem 7.2. Therefore, if the coefficients $a_j^{(n)}$, $b_j^{(n)}$ are chosen to minimize $\|f - u_n\|_{\partial D}$, then the rate of convergence for the method of particular solutions is at least $O(n^{-(p+\gamma)+\varepsilon})$.

For domains with only piecewise smooth boundary, these results do not apply. However, if the corners are analytic corners and the boundary data is analytic on each of the adjacent boundary arcs, then we may treat the corners by introducing the singular particular solutions

$$\begin{Bmatrix} \text{Re} \\ \text{Im} \end{Bmatrix} V[(z - z_\ell)^{j+k\alpha_\ell};\ z^{(\ell)}].$$

Theorem 7.3: Let D be a simply connected domain of class $R(\mu)$ $(0 < \mu \leq 1)$ with $\partial D \in C^{\max(p,1),\gamma}$ except at analytic corners (z_1,α_1), $\ldots,(z_q,\alpha_q)$ and let $f \in C^{p,\gamma}(\partial D - \{z_1,\ldots,z_q\}) \cap PA(z_1,\ldots,z_q)$. Fix z_0, $z^{(1)}$, $\ldots$, $z^{(q)} \in D \cup \partial D$ and let

$$\tilde{u}_n = \sum_{j=0}^{n} \{a_j^{(n)} \operatorname{Re} V[z^j;z_0] + b_j^{(n)} \operatorname{Im} V[z^j;z_0]\}$$

$$+ \sum_{\ell=1}^{q} \sum_{\substack{j+k\alpha_\ell<p+\gamma \\ j,k\geq 0}} \{a_{\ell jk}^{(n)} \operatorname{Re} V[(z-z_\ell)^{j+k\alpha_\ell};z^{(\ell)}]$$

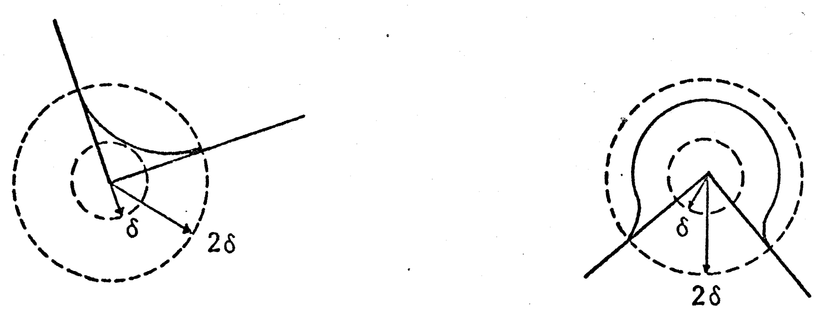
$$+ b_{\ell jk}^{(n)} \operatorname{Im} V[(z-z_\ell)^{j+k\alpha_\ell};z^{(\ell)}]\}.$$

Then for any $\varepsilon > 0$ there exists a constant $\tilde{K}(\varepsilon)$ independent of n such that

$$\tilde{E}_n(u) = \min_{a_j^{(n)},b_j^{(n)},a_{\ell jk}^{(n)},b_{\ell jk}^{(n)}} \|u - \tilde{u}_n\|_{D\cup\partial D} \leq \frac{\tilde{K}(\varepsilon)}{n^{\mu(p+\gamma)-\varepsilon}} .$$

Proof:

For any $\delta > 0$ sufficiently small, we can smoothly round each analytic corner (z_ℓ,α_ℓ) of D (as shown in the figure) to form a simply connected sub-domain D_δ with $\partial D_\delta \in C^{\max(p,1),\gamma}$:



From Lehman's Theorem, u has a (p+1)-times differentiable asymptotic expansion of order 0:

$$u(z) = u(z_\ell) + O(|z - z_\ell|^0).$$

Differentiating,

$$\frac{\partial^{j+k} u}{\partial^j x \, \partial^k y}(z) = O(|z - z_\ell|^{-p-1}), \quad j+k = p+1, \quad j, k \geq 0$$

so that there exists a constant K_ℓ such that

$$\left|\frac{\partial^{j+k} u}{\partial^j x \, \partial^k y}(x,y)\right| \leq \frac{K_\ell}{|z - z_\ell|^{p+1}} \leq K_\ell \delta^{-p-1}, \quad \delta \leq |z - z_\ell| \leq 3\delta$$

for $\delta > 0$ sufficiently small. From this we conclude that $u \in C^{p,\gamma}(\{z \in \partial D_\delta \,\big|\, |z - z_\ell| \leq 3\delta\})$ for each analytic corner (z_ℓ, α_ℓ).

Let f_δ denote the restriction of u to ∂D_δ. By assumption, $u = f$ on ∂D and $u \in C^{p,\gamma}(\partial D - \{z_1,\ldots,z_q\})$. Thus $f_\delta \in C^{p,\gamma}(\partial D_\delta)$. But u is a solution of the boundary value problem

$$Lv = 0 \quad \text{in } D_\delta; \qquad u = f_\delta \quad \text{on } \partial D_\delta$$

whence $u \in C^{p,\gamma}(D_\delta)$ by Theorem 7.1. Since every compact subset of $D \cup \partial D - \{z_1,\ldots,z_q\}$ is contained in D_δ for $\delta > 0$ sufficiently small, we have $u \in C^{p,\gamma}(D \cup \partial D - \{z_1,\ldots,z_q\})$ by definition and the result follows immediately from Theorem 6.3.

Q.E.D.

The rate of convergence of the analogous method of particular solutions is $O(n^{-\mu(p+\gamma)+\varepsilon})$. The analysis is the same as for the case of domains with smooth boundary.

§8. The Eigenvalue Problem

The method of particular solutions has been applied to the membrane eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } D; \qquad u = 0 \quad \text{on } \partial D \qquad \text{(EVP)}$$

with excellent results (Fox, Henrici, and Moler [7]). In this section we shall study a method implicit in the work of Bergman [3] for which the degree of approximation is at least $O(n^{-\mu(p+\gamma)+\varepsilon})$.

If λ^* and u^* are an approximate eigenvalue and eigenfunction, approximate in the sense that they satisfy the differential equation

$$\Delta u^* + \lambda^* u^* = 0 \quad \text{in } D$$

but not the boundary conditions, then a bound on the relative error in λ^* is given by the following Theorem:

<u>Theorem 8.1</u> (Moler and Payne [12]): Let w satisfy

$$\Delta w = 0 \quad \text{in } D, \qquad w = u^* \quad \text{on } \partial D.$$

If

$$\varepsilon = \frac{\|w\|_{2,D}}{\|u^*\|_{2,D}} < 1,$$

then there exists an eigenvalue $\hat{\lambda}$ such that

$$|\lambda^* - \hat{\lambda}| \leq \varepsilon |\hat{\lambda}|.$$

To apply this bound, in practice, the calculation of $\|w\|_{2,D}$ would involve the solution of a boundary value problem. However, using the following Theorem, we can give a somewhat weaker bound depending directly on the approximate eigenvector u^*.

<u>Theorem 8.2</u> (Bramble and Payne [6]): There exists an explicitly computable, domain-dependent constant K_D such that

$$\|w\|_{2,D} \leq K_D \|w\|_{2,\partial D}$$

for every w satisfying $\Delta w = 0$ in D.

Corollary: If

$$\varepsilon = \frac{K_D \|u^*\|_{2,\partial D}}{\|u^*\|_{2,D}} < 1,$$

then there exists an eigenvalue $\hat{\lambda}$ such that

$$|\lambda^* - \hat{\lambda}| \leq \varepsilon |\hat{\lambda}|.$$

Proof:

If w satisfies $\Delta w = 0$ in D, $w = u^*$ on ∂D, then

$$\frac{\|w\|_{2,D}}{\|u^*\|_{2,D}} \leq \frac{K_D \|w\|_{2,\partial D}}{\|u^*\|_{2,D}} = \frac{K_D \|u^*\|_{2,\partial D}}{\|u^*\|_{2,D}} < 1$$

and the result follows from Theorem 8.1.

Q.E.D.

The Corollary suggests a method for finding approximate eigenvalues. Given λ, let the corresponding approximate eigenfunction be

$$u_{n,\lambda} = \sum_{j=0}^{n} \{a_j^{(n,\lambda)} \mathrm{Re}\, V_\lambda[z^j;z_0] + b_j^{(n,\lambda)} \mathrm{Im}\, V_\lambda[z^j;z_0]\}$$

where V_λ is the integral operator associated with the equation

$$L_\lambda u = \Delta u + \lambda u = 0 \text{ in } D,$$

and the coefficients $a_j^{(n,\lambda)}$, $b_j^{(n,\lambda)}$ are chosen to minimize(*) the ratio

---

(*) The minimization of the ratio of two positive definite quadratic forms in the coefficients also arises in the Rayleigh-Ritz variational approximation to the eigenvalue problem. However, there the minimum ratio is an approximate eigenvalue rather than a measure of accuracy.

$$\frac{K_D \|u_{n,\lambda}\|_{2,\partial D}}{\|u_{n,\lambda}\|_{2,D}} = K_D \sqrt{\frac{\int_{\partial D} u_{n,\lambda}^2 ds}{\int_D u_{n,\lambda}^2 dxdy}}$$

Then the minimum value

$$\varepsilon_n(\lambda) = \min_{a_j^{(n,\lambda)},\, b_j^{(n,\lambda)}} \frac{K_D \|u_{n,\lambda}\|_{2,\partial D}}{\|u_{n,\lambda}\|_{2,D}}$$

is a continuous function of λ; and, if $\varepsilon_n(\lambda) < 1$, then there exists an eigenvalue $\hat{\lambda}$ such that

$$|\lambda - \hat{\lambda}| \leq \varepsilon_n(\lambda)|\hat{\lambda}|.$$

Moreover, if $\varepsilon_n(\lambda)$ is "small", then λ is "close" to $\hat{\lambda}$. Thus we are lead to the search for local minima of $\varepsilon_n(\lambda)$ as a method for finding approximate eigenvalues.

Theorem 8.3: Let $\partial D \in C^{\max(p,1),\gamma}$. Then there exist local minima λ_n^* for which

$$\varepsilon_n(\lambda_n^*) \leq \frac{K'(\varepsilon)}{n^{p+\gamma-\varepsilon}}$$

for any $\varepsilon > 0$, where the constant $K'(\varepsilon)$ is independent of n. Thus the degree of approximation is at least $O(n^{-(p+\gamma)+\varepsilon})$.

Proof:

We shall show that if $\hat{\lambda}$ is an eigenvalue, then $\varepsilon_n(\hat{\lambda}) \leq \frac{K'(\varepsilon)}{n^{p+\gamma-\varepsilon}}$. The result follows from the continuity of $\varepsilon_n(\lambda)$.

Let $u \in C(D \cup \partial D)$ be the corresponding normalized eigenvector

$$\Delta u + \hat{\lambda} u = 0 \text{ in } D, \quad \|u\|_{2,D} = 1.$$

By Theorem 7.2, for any $\varepsilon > 0$ there exists a constant $K(\varepsilon)$ independent of n such that

$$\|u - u_n\|_{D\cup\partial D} \le \frac{K(\varepsilon)}{n^{p+\gamma-\varepsilon}}$$

where u_n is the generalized harmonic polynomial of best approximation to u. Letting A_D denote the area of D and ℓ_D the length of ∂D,

$$\|u_n\|_{2,D} \ge \|u\|_{2,D} - \|u - u_n\|_{2,D} \ge 1 - \sqrt{A_D}\,\|u - u_n\|_{D\cup\partial D} \ge 1 - \frac{\sqrt{A_D}K(\varepsilon)}{n^{p+\gamma-\varepsilon}}$$

and

$$\|u_n\|_{2,\partial D} \le \sqrt{\ell_D}\,\|u_n\|_{\partial D} = \sqrt{\ell_D}\,\|u - u_n\|_{\partial D} \le \frac{\sqrt{\ell_D}\,K(\varepsilon)}{n^{p+\gamma-\varepsilon}}$$

so that

$$\varepsilon_n(\hat{\lambda}) \le \frac{K_D\|u_n\|_{2,\partial D}}{\|u_n\|_{2,D}} \le \frac{K_D\dfrac{\sqrt{\ell_D}\,K(\varepsilon)}{n^{p+\gamma-\varepsilon}}}{1 - \dfrac{\sqrt{A_D}\,K(\varepsilon)}{n^{p+\gamma-\varepsilon}}} \le \frac{K'(\varepsilon)}{n^{p+\gamma-\varepsilon}}$$

for n sufficiently large.

Q.E.D.

For domains with analytic corners, the inclusion of singular particular solutions gives an analogous method with degree of convergence at least $O(n^{-\mu(p+\gamma)+\varepsilon})$:

<u>Theorem 8.4</u>: Let D be a simply connected domain of class $\mathcal{R}(\mu)$ $(0 < \mu \le 1)$ with $\partial D \in C^{\max(p,1),\gamma}$ except at analytic corners $(z_1,\alpha_1), \ldots, (z_q,\alpha_q)$. Fix $z_0, z^{(1)}, \ldots, z^{(q)} \in D\cup\partial D$ and let

$$\tilde{u}_{n,\lambda} = \sum_{\ell=1}^{q} \sum_{\substack{j+k\alpha_\ell < p+\gamma \\ j,k \geq 0}} \{a_{\ell jk}^{(n,\lambda)} \operatorname{Re} V_\lambda[(z - z_\ell)^{j+k\alpha_\ell}; z^{(\ell)}]$$

$$+ b_{\ell jk}^{(n,\lambda)} \operatorname{Im} V_\lambda[(z - z_\ell)^{j+k\alpha_\ell}; z^{(\ell)}]\}$$

$$+ \sum_{j=0}^{n} \{a_j^{(n,\lambda)} \operatorname{Re} V_\lambda[z^j; z_0] + b_j^{(n,\lambda)} \operatorname{Im} V_\lambda[z^j; z_0]\}.$$

Set

$$\varepsilon_n(\lambda) = \min_{a_{\ell jk}^{(n,\lambda)}, b_{\ell jk}^{(n,\lambda)}, a_j^{(n,\lambda)}, b_j^{(n,\lambda)}} \frac{K_D \|\tilde{u}_{n,\lambda}\|_{2,\partial D}}{\|\tilde{u}_{n,\lambda}\|_{2,D}} .$$

Then there exist local minima $\tilde{\lambda}_n^*$ for which

$$\tilde{\varepsilon}_n(\tilde{\lambda}_n^*) \leq \frac{\tilde{K}'(\varepsilon)}{n^{\mu(p+\gamma)-\varepsilon}}$$

for any $\varepsilon > 0$, where the constant $\tilde{K}'(\varepsilon)$ is independent of n.

Proof:

The proof is exactly the same as the proof of the preceding Theorem except that Theorem 7.3 is used in place of Theorem 7.2.

Q.E.D.

Bibliography

[1] Babuska, Ivo. Private Communication.

[2] Bergman, Stefan. Integral Operators in the Theory of Linear Partial Differential Equations. Berlin, Springer, 1961.

[3] Bergman, Stefan. "Über ein Verfahren zur Konstruktion der Näherunglösungen der Gleichung $\Delta u + \tau^2 u = 0$." Prikladnaia Matematika i Mekhanika. 3 (1936), 97-107.

[4] Bergman, Stefan, and John G. Herriot. "Application of the Method of the Kernel Function for Solving Boundary Value Problems." Numerische Mathematik. 3 (1961), 209-225.

[5] Bergman, Stefan, and John G. Herriot. "Numerical Solution of Boundary Value Problems by the Method of Integral Operators." Numerische Mathematik. 7 (1965), 42-65.

[6] Bramble, J. H., and L. E. Payne. "Bounds for Solutions of Second Order Elliptic Partial Differential Equations." Contributions to Differential Equations I. (1963), 95-127.

[7] Fox, L., P. Henrici, and C. Moler. "Approximations and Bounds for Eigenvalues of Elliptic Operators." SIAM Journal on Numerical Analysis. 4 (1967), 89-102.

[8] Henrici, P. "A Survey of I. N. Vekua's Theory of Elliptic Partial Differential Equations with Analytic Coefficients." ZAMP. 8(1957), 169-203.

[9] Lehman, R. Sherman. "Developments at an Analytic Corner of Solutions of Elliptic Partial Differential Equations." Journal of Mathematics and Mechanics. 8 (1959), 727-760.

[10] Lewy, Hans. "On the Reflection Laws of Second Order Differential Equations in Two Independent Variables." Bulletin of the AMS. 65 (1959), 37-58.

[11] Mergelyan, S. N. "Uniform Approximations to Functions of a Complex Variable." AMS Translations, Series I, No. 101. Translated from Uspehi Matematiceskih Nauk (N.S.). 7, no. 2 (48), 31-122 (1952).

[12] Moler, C. B., and L. E. Payne. "Bounds for Eigenvalues and Eigenvectors of Symmetric Operators." SIAM Journal on Numerical Analysis. 5 (1968), 64-70.

[13] Privaloff, M. I. "Sur les Fonctions Conjugèes." Bulletin de la Société Mathématique de France. 44 (1916), 100-103.

[14] Schryer, N. L. Symbolic Computer Solution of Elliptic Boundary Value Problems. Thesis, University of Michigan, 1969. (a shortened version, "Constructive Approximation of Solutions to Linear Elliptic Boundary Value Problems" to appear in SIAM Journal on Numerical Analysis).

[15] Sewell, W. E. Degree of Approximation by Polynomials in the Complex Domain. Princeton, Princeton University Press, 1942.

[16] Vekua, I. N. New Methods for Solving Elliptic Equations. North-Holland Publishing Company, Amsterdam, 1967.

[17] Walsh, J. L. Interpolation and Approximation by Rational Functions in the Complex Domain. AMS Colloquium Publication Volume 20. American Mathematical Society, Providence, Rhode Island, 1935.