The jump relations of the quadruple layer potential on a regular surface in three dimensions are derived. The jumps are shown to be proportional to the product of the density of the potential and the mean curvature of the underlying surface.

Jump Relations of the Quadruple Layer Potential on a Regular Surface in Three Dimensions

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1 Introduction

With the recent advances in fast algorithms (see, for example, [2]) for solving integral equations resulting from potential theory, there is a renewed interest in the classical potential theory. As is well known, the jump relations of the single and double layer potentials play an important role in the classical potential theory. In [3], the jump relations of the quadruple layer potential on a curve in two dimensions are derived; and it is shown that the jumps are proportional to the product of the density of the potential and the curvature of the curve. In this note, we derive the jump relations of the quadruple layer potential on a regular surface in three dimensions and show that the jumps of the quadruple layer potential are proportional to the product of the density and the mean curvature of the underlying surface. The result is summarized in Theorem 3.8.

2 Analytical Preliminaries

In this section, we collect some well known facts from classic analysis to be used in the remainder of the paper.

2.1 Notation

We will denote by S a sufficiently smooth (say, at least twice continuously differentiable) regular surface in $\mathbb{R}^3$. When S is an open surface with boundary, we assume that S is an open set, i.e., S does not contain its boundary C. For a point $x \in S$, we denote the unit normal vector to S at x by $N(x)$. For a vector $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ we will denote its length by $|x|$. Finally, for two vectors $x, y \in \mathbb{R}^3$, we denote their inner product by $\langle x, y \rangle$.

2.2 Single, Double, and Quadruple Layer Potentials

For any $x, t \in \mathbb{R}^3$ and $x \neq t$, we define the Green's function for the Laplace equation in $\mathbb{R}^3$ via the formula

$$G(x,t) = \frac{1}{|x-t|}. \tag{1}$$

Suppose now that S is a sufficiently smooth regular surface. For $t \in S$ we consider the directional derivatives of the function G with respect to t along the normal directions of S at t. It is easy to verify that

$$\frac{\partial G(x,t)}{\partial N(t)} = N(t) \cdot \nabla_t G(x,t) = \frac{\langle N(t), x-t \rangle}{|x-t|^3}. \tag{2}$$

$$\frac{\partial^2 G(x,t)}{\partial N(t)^2} = N(t) \cdot \nabla_t \nabla_t G(x,t) \cdot N(t) = \frac{3\langle N(t), x-t \rangle^2}{|x-t|^5} - \frac{1}{|x-t|^3}. \tag{3}$$

In the literature, (1), (2), (3) are often referred to as the single, double, and quadruple potentials respectively. Suppose further that $\sigma : S \to \mathbb{R}$ is a sufficiently smooth function. We will refer to the functions given by the formulae

$$\int_S G(x,t) \cdot \sigma(t) \cdot dt, \tag{4}$$

$$\int_S \frac{\partial G(x,t)}{\partial N(t)} \cdot \sigma(t) \cdot dt, \tag{5}$$

$$\int_S \frac{\partial^2 G(x,t)}{\partial N(t)^2} \cdot \sigma(t) \cdot dt, \tag{6}$$

as the single, double, and quadruple layer potentials, respectively.

2.3 Finite Part Integrals on a Regular Surface in $\mathbb{R}^3$

Finite part (also referred to as hypersingular) integrals on Euclidean spaces are well known and extensively used in mechanical engineering. Here we generalize the definition for the Euclidean spaces given by Samko (see Chapter 1, Section **5.3** in [4]) to define finite part integrals on a regular surface in $\mathbb{R}^3$.

Definition 2.1 *Suppose that S is a sufficiently smooth regular surface. Suppose further that the function f is integrable in $S - D_\epsilon(x)$ for all $\epsilon \in (0, \epsilon_0)$ for some $\epsilon_0 > 0$, where $D_\epsilon(x) = \{t \in S \,|\, |t-x| < \epsilon\}$. Then f is said to possess the* Hadamard property at x *if there exist constants a_k, b, and λ_k, possibly complex-valued, but $Re(\lambda_k) > 0$, $k = 1, 2, \ldots, N$, such that*

$$\int_{S-D_\epsilon(x)} f(t)dt = \sum_{k=1}^{N} a_k \cdot \epsilon^{-\lambda_k} + b \cdot \log\frac{1}{\epsilon} + I_0(\epsilon), \tag{7}$$

where $\lim_{\epsilon\to 0} I_0(\epsilon)$ *exists and is finite. In this case the* finite part *of the (divergent) integral* $\int_{S-D_\epsilon(x)} f(t)dt$ *is defined by the formula*

$$\text{f.p.} \int_S f(t)dt = \lim_{\epsilon\to 0} I_0(\epsilon). \tag{8}$$

Remark 2.1 When S is a flat surface embedded in $\mathbb{R}^2$, the above definition coincides with the conventional definition for finite part integrals in $\mathbb{R}^n$ given in [4].

2.4 Local Properties of a Regular Surface in $\mathbb{R}^3$

It is well known that locally any sufficiently smooth regular surface in $\mathbb{R}^3$ is the graph of a sufficiently smooth function. When the local coordinates are chosen along the principal directions, the surface admits a particularly simple parametrization. The following lemma summarizes this fact; it can be found in [1].

Lemma 2.1 *Suppose that* S *is a sufficiently smooth regular surface in* $\mathbb{R}^3$ *and that* x *is a point in* S*. Suppose further that* $x = (0,0,0)$ *and* $N(x) = (0,0,1)$*. Then there exists a neighborhood* $D_a(x) \in S$ *of* x *such that* $D_a(x)$ *admits a parametrization given by the formula*

$$D_a(x) = \{(u,v,g) \in S | g(u,v) = \frac{1}{2}(k_1 \cdot u^2 + k_2 \cdot v^2) + R(u,v)\}, \tag{9}$$

with k_1, k_2 *the principal curvatures of* S *at* x*, and the function* R *satisfying the conditions*

$$|R(u,v)| \le C \cdot (u^2+v^2)^{3/2}, \tag{10}$$

$$|R_u(u,v)|, |R_v(u,v)| \le C \cdot (u^2+v^2), \tag{11}$$

for some $C > 0$ *and all* $u^2+v^2 < a^2$*.*

3 Jump Relations of the Quadruple Layer Potential

In this section, we will derive the jump relations of the quadruple layer potential on a regular surface in three dimensions. In other words, for $x \in S$ let

$$u(x+\epsilon\cdot N(x)) = \int_S \frac{\partial^2 G(x+\epsilon\cdot N(x),t)}{\partial N(t)^2} \cdot \sigma(t)dt, \tag{12}$$

and we would like to compute the limits $\lim_{\epsilon\to 0+} u(x+\epsilon\cdot N(x))$ and $\lim_{\epsilon\to 0-} u(x+\epsilon\cdot N(x))$. We first consider the simple case when S is an open region in $\mathbb{R}^2$.

3.1 Planar Surface

In this case, we may assume that $x = (0,0,0)$, $N(x) = (0,0,1)$, $t = (u,v,0)$. Substituting these equations into (3), we obtain

$$\frac{\partial^2 G(x + \epsilon \cdot N(x), t)}{\partial N(t)^2} = \frac{3\epsilon^2}{(u^2+v^2+\epsilon^2)^{5/2}} - \frac{1}{(u^2+v^2+\epsilon^2)^{3/2}}, \tag{13}$$

and thus

$$\begin{aligned} u(x + \epsilon \cdot N(x)) &= \int_S \frac{\partial^2 G(x + \epsilon \cdot N(x), t)}{\partial N(t)^2} \cdot \sigma(t) \cdot dt \\ &= \int_S \left\{ \frac{3\epsilon^2}{(u^2+v^2+\epsilon^2)^{5/2}} - \frac{1}{(u^2+v^2+\epsilon^2)^{3/2}} \right\} \cdot \sigma(u,v) \cdot du \cdot dv. \end{aligned} \tag{14}$$

The following lemma states that the quadruple layer potential is continuous across S when S is flat.

Lemma 3.1 *Suppose that $\sigma : S \to \mathbb{R}$ is twice continuously differentiable. Then the function u defined by* (14) *satisfies the following relation:*

$$\begin{aligned} \lim_{\epsilon \to 0} u(x + \epsilon \cdot N(x)) &= \lim_{\epsilon \to 0} \int_S \left\{ \frac{3\epsilon^2}{(u^2+v^2+\epsilon^2)^{5/2}} - \frac{1}{(u^2+v^2+\epsilon^2)^{3/2}} \right\} \cdot \sigma(u,v) \cdot du \cdot dv \\ &= -\text{f.p.} \int_S \frac{1}{(u^2+v^2)^{3/2}} \cdot \sigma(u,v) \cdot du \cdot dv \\ &= -\lim_{\epsilon \to 0} \left\{ \int_{S - D_\epsilon} \frac{1}{(u^2+v^2)^{3/2}} \cdot \sigma(u,v) \cdot du \cdot dv - \frac{2\pi}{\epsilon} \cdot \sigma(0,0) \right\}, \end{aligned} \tag{15}$$

with $D_\epsilon = \{(u,v) | u^2 + v^2 < \epsilon^2\}$.

Proof. Obviously the problem is local, and we may assume that $S = D_R$ for some $R > 0$.

(a) If $\sigma(u,v) = 1$, then using polar coordinates we have

$$\begin{aligned} u(x + \epsilon \cdot N(x)) &= \int_{D_R} \left\{ \frac{3\epsilon^2}{(u^2+v^2+\epsilon^2)^{5/2}} - \frac{1}{(u^2+v^2+\epsilon^2)^{3/2}} \right\} \cdot 1 \cdot du \cdot dv \\ &= \frac{2\pi \cdot R^2}{(R^2+\epsilon^2)^{3/2}}, \end{aligned}$$

$$(16)$$

and thus

$$\lim_{\epsilon \to 0} u(x + \epsilon \cdot N(x)) = \frac{2\pi}{R}. \tag{17}$$

Also

$$\begin{aligned}
&-\text{f.p.} \int_S \frac{1}{(u^2+v^2)^{3/2}} \cdot 1 \cdot du \cdot dv \\
&= -\lim_{\epsilon \to 0} \left\{ \int_{D_R - D_\epsilon} \frac{1}{(u^2+v^2)^{3/2}} \cdot 1 \cdot du \cdot dv - \frac{2\pi}{\epsilon} \right\} \\
&= -\lim_{\epsilon \to 0} \left\{ \int_\epsilon^R \frac{1}{r^2} \cdot dr - \frac{2\pi}{\epsilon} \right\} \\
&= \frac{2\pi}{R}.
\end{aligned} \tag{18}$$

And the lemma clearly holds in this case.

(b) Similarly, the lemma holds for $\sigma = u$ and $\sigma = v$ since all related integrals are equal to zero by symmetry and $\sigma(0,0) = 0$.

(c) We now consider the general case $\sigma \in C^2(S)$. By Taylor's theorem we have

$$\sigma(u,v) = \sigma(0,0) + \sigma_u(0,0) \cdot u + \sigma_v(0,0) \cdot v + R_1(u,v), \tag{19}$$

where R_1 satisfies the condition that for all $u^2 + v^2 < \delta^2$

$$|R_1(u,v)| < M \cdot (u^2 + v^2), \tag{20}$$

with δ, M positive real numbers. By parts (a) and (b), we only need to prove that the lemma holds for $\sigma = R_1$. Since

$$\left| \left\{ \frac{3\epsilon^2}{(u^2+v^2+\epsilon^2)^{5/2}} - \frac{1}{(u^2+v^2+\epsilon^2)^{3/2}} \right\} \cdot R_1(u,v) \right| \le C \cdot \frac{1}{(u^2+v^2)^{1/2}} \tag{21}$$

for all $\epsilon < \epsilon_0$ and $u^2 + v^2 < \delta^2$, the integrands in (15) are only weakly singular (and hence absolutely integrable) even if $\epsilon = 0$. By Lebesgue's dominated convergence theorem, the order of limit and integration can be interchanged and the finite part integral is actually an ordinary (Lebesgue) integral. Hence, the lemma is proved.

□

Next, we will study the local properties of the quadruple potential near a regular surface.

3.2 Local Properties of the Quadruple Potential near a Regular surface

From now on, we use the parametrization given in Lemma 2.1, i.e., $x = (0,0,0)$, $N(x) = (0,0,1)$, and for $t \in D_a(x) \subset S$, it has the coordinates

$$\begin{aligned} t &= (u, v, g(u,v)) \\ &= (u, v, \frac{1}{2}(k_1 \cdot u^2 + k_2 \cdot v^2) + R(u,v)), \end{aligned} \tag{22}$$

with k_1, k_2 the principal curvatures at the point $x = (0,0,0)$, and the function R satisfying the conditions (10) and (11). We first compute the quadruple potential in the neighborhood of $D_a(x)$.

Lemma 3.2 *For* $t \in D_a(x) \subset S$,

$$\begin{aligned} \frac{\partial^2 G(x + \epsilon \cdot N(x), t)}{\partial N(t)^2} &= \frac{3\langle N(t), x + \epsilon \cdot N(x) - t\rangle^2}{|x + \epsilon \cdot N(x) - t|^5} - \frac{1}{|x + \epsilon \cdot N(x) - t|^3} \\ &= \frac{3 \cdot p_1^2(u,v,\epsilon)}{J^2(u,v) \cdot d_1^5(u,v,\epsilon)} - \frac{1}{d_1^3(u,v,\epsilon)}, \end{aligned} \tag{23}$$

where the functions p_1, J, d_1 *are defined by the formulae*

$$p_1(u,v,\epsilon) = \frac{1}{2}(k_1 \cdot u^2 + k_2 \cdot v^2) + R_u(u,v) \cdot u + R_v(u,v) \cdot v - R(u,v) + \epsilon, \tag{24}$$

$$J(u,v) = \sqrt{1 + k_1^2 \cdot u^2 + k_2^2 \cdot v^2 + 2k_1 \cdot u \cdot R_u + 2k_2 \cdot v \cdot R_v + R_u^2 + R_v^2}, \tag{25}$$

$$\begin{aligned} d_1(u,v,\epsilon) &= |x + \epsilon \cdot N(x) - t| \\ &= \sqrt{u^2 + v^2 + \left(\frac{1}{2}(k_1 \cdot u^2 + k_2 \cdot v^2) + R(u,v) - \epsilon\right)^2}. \end{aligned} \tag{26}$$

Proof. The first equality of (23) directly follows from the definition of the quadruple potential (3). And the second equality follows from direct computation and the details are as follows:

$$x + \epsilon \cdot N(x) - t = (-u, -v, -\frac{1}{2}(k_1 \cdot u^2 + k_2 \cdot v^2) - R(u,v) + \epsilon), \tag{27}$$

$$t_u = (1, 0, k_1 \cdot u + R_u), \quad t_v = (0, 1, k_2 \cdot v + R_v), \tag{28}$$

$$N(t) = \frac{t_u \times t_v}{|t_u \times t_v|} = \frac{(-k_1 \cdot u - R_u(u,v), -k_2 \cdot v - R_v(u,v), 1)}{J(u,v)}, \tag{29}$$

$$\langle N(t), x+\epsilon\cdot N(x)-t\rangle = \frac{p_1(u,v,\epsilon)}{J(u,v)}. \tag{30}$$

□

We now introduce the notation

$$d(u,v,\epsilon)=\sqrt{u^2+v^2+\epsilon^2}, \tag{31}$$

The following lemma provides the estimates for p_1, J, and d_1 to be used in the proof of Lemma 3.4.

Lemma 3.3 *there exist real positive numbers C_1, C_2, C, δ, ϵ_0 such that for all $t \in D_\delta(x)$ and $0 \le \epsilon < \epsilon_0$,*

$$|p_1(u,v,\epsilon)^2| \le C\cdot d(u,v,\epsilon)^2, \tag{32}$$

$$|p_1(u,v,\epsilon)^2-\epsilon^2-\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2)| \le C\cdot d(u,v,\epsilon)^4, \tag{33}$$

$$|J(u,v)-1| \le C\cdot d(u,v,\epsilon)^2, \tag{34}$$

$$C_1\cdot d(u,v,\epsilon) \le |d_1(u,v,\epsilon)| \le C_2\cdot d(u,v,\epsilon), \tag{35}$$

$$\left|\frac{1}{d_1(u,v,\epsilon)^3}-\frac{1}{d(u,v,\epsilon)^3}-\frac{3\cdot\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2)}{2\cdot d(u,v,\epsilon)^5}\right| \le C\cdot\frac{1}{d(u,v,\epsilon)}, \tag{36}$$

$$\left|\frac{1}{d_1(u,v,\epsilon)^5}-\frac{1}{d(u,v,\epsilon)^5}-\frac{5\cdot\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2)}{2\cdot d(u,v,\epsilon)^7}\right| \le C\cdot\frac{1}{d(u,v,\epsilon)^3}. \tag{37}$$

Proof. The estimates (32)–(35) directly follow from the definitions of p_1, J, d_1, d and the condition (10) satisfied by the function R. To prove (36), we first note that

$$|d(u,v,h)-d_1(u,v,h)| \le C\cdot d(u,v,\epsilon)^2, \tag{38}$$

$$\begin{aligned}&|d(u,v,h)^2-d_1(u,v,h)^2-\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2)|\\ =&\left|2\epsilon\cdot R(u,v)-\left(\frac{1}{2}(k_1\cdot u^2+k_2\cdot v^2)+R(u,v)\right)^2\right|\\ \le& C\cdot d(u,v,\epsilon)^4.\end{aligned} \tag{39}$$

For the sake of readability, we will sometimes denote $d_1(u,v,\epsilon)$, $d(u,v,\epsilon)$ simply by d_1, d respectively. We have

$$
\begin{aligned}
&\left|\frac{1}{d_1^3}-\frac{1}{d^3}-\frac{\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2)\cdot(d^2+d_1\cdot d+d_1^2)}{d^3\cdot d_1^3\cdot(d_1+d)}\right| \\
=&\left|\frac{(d^2-d_1^2+\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2))\cdot(d^2+d_1\cdot d+d_1^2)}{d^3\cdot d_1^3\cdot(d_1+d)}\right| \\
\leq& C\cdot\frac{1}{d(u,v,\epsilon)},
\end{aligned}
\tag{40}
$$

where the inequality follows from (35) and (39). Similarly

$$
\begin{aligned}
&\left|\frac{(d^2+d_1\cdot d+d_1^2)}{d_1^3\cdot(d_1+d)}-\frac{3}{2\cdot d^2}\right| \\
=&\left|\frac{(d-d_1)\cdot(3d_1^3+6d\cdot d_1^2+4d^2\cdot d_1+2d^3)}{2d^2\cdot d_1^3\cdot(d_1+d)}\right| \\
\leq& C\cdot\frac{1}{d(u,v,\epsilon)},
\end{aligned}
\tag{41}
$$

where the inequality follows from (35) and (38). Furthermore,

$$
\left|\frac{\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2)}{d(u,v,\epsilon)^3}\right|\leq C. \tag{42}
$$

Combining (40)–(42), we obtain (36). Finally, we note that (37) can be proved in an almost identical manner. □

The following lemma singles out most singular terms in the quadruple potential.

Lemma 3.4 *there exist real positive numbers C, δ, ϵ_0 such that for all $t \in D_\delta(x)$ and $0 \leq \epsilon < \epsilon_0$,*

$$
\begin{aligned}
&\left|\frac{\partial^2 G(x+\epsilon\cdot N(x),t)}{\partial N(t)^2}-\frac{3\epsilon^2}{d(u,v,\epsilon)^5}+\frac{1}{d(u,v,\epsilon)^3}\right. \\
&\left.-\frac{3\cdot\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2)}{2\cdot d(u,v,\epsilon)^5}-\frac{15\cdot\epsilon^3\cdot(k_1\cdot u^2+k_2\cdot v^2)}{2\cdot d(u,v,\epsilon)^7}\right| \\
\leq& C\cdot\frac{1}{d(u,v,\epsilon)}.
\end{aligned}
\tag{43}
$$

Proof. Combining (32)–(34), we have

$$
\left|\frac{p_1(u,v,\epsilon)^2}{J(u,v)^2\cdot d_1(u,v,\epsilon)^5}-\frac{\epsilon^2+\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2)}{d_1(u,v,\epsilon)^5}\right|\leq C\cdot\frac{1}{d(u,v,\epsilon)}. \tag{44}
$$

Combining (23) and (44), we obtain

$$\left|\frac{\partial^2 G(x+\epsilon\cdot N(x),t)}{\partial N(t)^2}-\frac{3(\epsilon^2+\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2))}{d_1(u,v,\epsilon)^5}+\frac{1}{d_1(u,v,\epsilon)^3}\right|\le C\cdot\frac{1}{d(u,v,\epsilon)}. \tag{45}$$

Finally, we obtain (43) by combining (36), (37), and (45). □

We now introduce the notation

$$r(x,t)=r(u,v)=\sqrt{u^2+v^2}. \tag{46}$$

The following corollary is obtained by letting $\epsilon=0$ in (43).

Corollary 3.5 *there exist real positive numbers C, δ such that for all $t\in D_\delta(x)$,*

$$\left|\frac{\partial^2 G(x,t)}{\partial N(t)^2}+\frac{1}{r(x,t)^3}\right|\le C\cdot\frac{1}{r(x,t)}. \tag{47}$$

3.3 General Regular Surface

We now consider the case of a general regular surface. Obviously since $1/r(x,t)$ in (47) is weakly singular (and thus absolutely integrable), we obtain the following lemma by Lebesgue's dominated convergence theorem and Corollary 3.5.

Lemma 3.6 *Suppose that $\sigma:S\to\mathbb{R}$ is twice continuously differentiable. Then*

$$\begin{aligned}\text{f.p.}\int_S\frac{\partial^2 G(x,t)}{\partial N(t)^2}\cdot\sigma(t)\cdot dt=&-\text{f.p.}\int_S\frac{1}{r^3(x,t)}\cdot\sigma(t)\cdot dt\\&+\int_S\left(\frac{\partial^2 G(x,t)}{\partial N(t)^2}+\frac{1}{r^3(x,t)}\right)\cdot\sigma(t)\cdot dt,\end{aligned} \tag{48}$$

where the last integral is interpreted in the sense of an ordinary (Lebesgue) integral.

The following lemma states that the last several terms in (43) are all "approximations to the identity" (or more precisely, a constant).

Lemma 3.7 *Suppose that $f\in L^p(D_a(x))$, $1\le p\le\infty$. Suppose further that x belongs to the Lebesgue set of f (see, for example, [5] Chapter I). Then*

$$\lim_{\epsilon\to0+}\int_{D_a(x)}\frac{\epsilon\cdot u^2}{d(u,v,\epsilon)^5}\cdot f(t)\cdot dt=\frac{2\pi}{3}\cdot f(x), \tag{49}$$

$$\lim_{\epsilon \to 0+} \int_{D_a(x)} \frac{\epsilon \cdot v^2}{d(u,v,\epsilon)^5} \cdot f(t) \cdot dt = \frac{2\pi}{3} \cdot f(x), \tag{50}$$

$$\lim_{\epsilon \to 0+} \int_{D_a(x)} \frac{\epsilon^3 \cdot u^2}{d(u,v,\epsilon)^7} \cdot f(t) \cdot dt = \frac{2\pi}{15} \cdot f(x), \tag{51}$$

$$\lim_{\epsilon \to 0+} \int_{D_a(x)} \frac{\epsilon^3 \cdot v^2}{d(u,v,\epsilon)^7} \cdot f(t) \cdot dt = \frac{2\pi}{15} \cdot f(x). \tag{52}$$

(x belongs to the Lebesgue set of f if $\lim_{\epsilon \to 0} \frac{1}{\epsilon^2} \int_{D_\epsilon(x)} |f(t) - f(x)| \cdot dt = 0$. In particular, x is a point of the Lebesgue set of f if f is continuous at x.)

Proof. The proofs of (49)–(52) are almost identical to each other and thus we will only prove (52). Representing dt in local coordinates, we have

$$\int_{D_a(x)} \frac{\epsilon^3 \cdot v^2}{d(u,v,\epsilon)^7} \cdot f(t) \cdot dt = \int_{u^2+v^2<a^2} \frac{\epsilon^3 \cdot v^2}{(u^2+v^2+h^2)^{7/2}} \cdot f(u,v) \cdot J(u,v) \cdot dudv, \tag{53}$$

where the function J is given in (25). Obviously, J is a bounded continuous function in $u^2 + v^2 \leq a^2$ and $J(0,0) = 1$. Hence, $f(0,0) \cdot J(0,0) = f(0,0) = f(x)$. By Theorem 1.25 of Chapter I in [5], we only need to prove that

$$\int_{\mathbb{R}^2} \frac{v^2}{(u^2+v^2+1)^{7/2}} \cdot du \cdot dv = \frac{2\pi}{15}. \tag{54}$$

But the above identity can easily be verified by evaluating the integral via polar coordinates. Hence, the lemma follows. □

We are now in a position to present the principal result of the paper.

Theorem 3.8 *Suppose that S is a sufficiently smooth regular surface in $\mathbb{R}^3$ and that $\sigma : S \to \mathbb{R}$ is twice continuously differentiable. Then the quadruple layer potential $u : \mathbb{R}^3 \to \mathbb{R}$ defined by the formula*

$$u(x) = \int_S \frac{\partial^2 G(x,t)}{\partial N(t)^2} \cdot \sigma(t) \cdot dt \tag{55}$$

satisfies the following jump relations at $x \in S$:

(a)

$$\lim_{\epsilon\to 0+} u(x+\epsilon\cdot N(x)) = \lim_{\epsilon\to 0+}\int_S \frac{\partial^2 G(x+\epsilon\cdot N(x),t)}{\partial N(t)^2}\cdot\sigma(t)\cdot dt$$
$$= 2\pi\cdot(k_1(x)+k_2(x))\cdot\sigma(x) + \text{f.p.}\int_S \frac{\partial^2 G(x,t)}{\partial N(t)^2}\cdot\sigma(t)\cdot dt; \tag{56}$$

(b)

$$\lim_{\epsilon\to 0-} u(x+\epsilon\cdot N(x)) = \lim_{\epsilon\to 0-}\int_S \frac{\partial^2 G(x+\epsilon\cdot N(x),t)}{\partial N(t)^2}\cdot\sigma(t)\cdot dt$$
$$= -2\pi\cdot(k_1(x)+k_2(x))\cdot\sigma(x) + \text{f.p.}\int_S \frac{\partial^2 G(x,t)}{\partial N(t)^2}\cdot\sigma(t)\cdot dt, \tag{57}$$

with $k_1(x)$, $k_2(x)$ the principal curvatures of S at x. (Note: $k_1(x)+k_2(x)$ is the trace of the second fundamental form of S at x, which changes sign if $N(x)$ changes sign (i.e., S changes its orientation).)

Proof. The proofs of (56) and (57) are almost identical, so we will only prove (56). By lemma 3.4, we have

$$\begin{aligned}
&\int_S \frac{\partial^2 G(x+\epsilon\cdot N(x),t)}{\partial N(t)^2}\cdot\sigma(t)\cdot dt = \int_S\left\{\frac{3\epsilon^2}{d(u,v,\epsilon)^5} - \frac{1}{d(u,v,\epsilon)^3}\right\}\cdot\sigma(t)\cdot dt\\
&+\int_S\left\{\frac{3\cdot\epsilon\cdot(k_1\cdot u^2+k_2\cdot v^2)}{2\cdot d(u,v,\epsilon)^5} + \frac{15\cdot\epsilon^3\cdot(k_1\cdot u^2+k_2\cdot v^2)}{2\cdot d(u,v,\epsilon)^7}\right\}\cdot\sigma(t)\cdot dt\\
&+\int_S R(u,v,\epsilon)\cdot\sigma(t)\cdot dt\\
&:= I_1+I_2+I_3,
\end{aligned} \tag{58}$$

where the function R satisfies the conditions

$$|R(u,v,\epsilon)| \le C\cdot\frac{1}{d(u,v,\epsilon)}, \tag{59}$$

$$R(u,v,0) = \frac{\partial^2 G(x,t)}{\partial N(t)^2} + \frac{1}{r^3(x,t)}. \tag{60}$$

We now analyze the above three items separately. First, by Lemma 3.1, we have

$$\lim_{\epsilon\to 0+} I_1 = -\text{f.p.}\int_S \frac{1}{r(x,t)^3}\cdot\sigma(t)\cdot dt. \tag{61}$$

Second, by Lemma 3.6, we have

$$\lim_{\epsilon\to 0+} I_2 = 2\pi\cdot(k_1(x)+k_2(x))\cdot\sigma(x). \tag{62}$$

Third, by Lebesgue's dominated convengence theorem, we have

$$\lim_{\epsilon\to 0+} I_3 = \int_S R(u,v,0)\cdot\sigma(t)\cdot dt = \int_S\left\{\frac{\partial^2 G(x,t)}{\partial N(t)^2}+\frac{1}{r^3(x,t)}\right\}\cdot\sigma(t)\cdot dt. \tag{63}$$

Finally, we obtain (56) by combining (48), (58), (61)–(63). □

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