BICUBIC INTERPOLATION OVER RIGHT TRIANGLES

by

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In this note we improve the error bound recently given by C. A. Hall in [2] for an interpolation scheme due to G. Birkhoff. This interpolation scheme is defined over right triangles using bicubic polynomials.

Consider the right triangle, \( \Delta \), with vertices \((0,0), (h,0), \) and \((0,k)\) and the set \( P \equiv \{ p(x,y) \} \) there exist real constants \( a_{ij}, 0 \leq i + j \leq 3, \) such that \( p(x,y) = \sum_{0 \leq i+j \leq 3} a_{ij} x^i y^j \) for all \((x,y) \in \Delta\) of all real bicubic polynomials on \( \Delta \). We define an interpolation mapping \( M \) from \( C^2(\Delta) \) to \( P \) by

1. \((D_x^i D_y^j Mf)(0,0) = \frac{\partial^{i+j}}{\partial x^i \partial y^j} Mf(0,0) = D_x^i D_y^j f(0,0), 0 \leq i, j \leq 1,\)

2. \((D_x^i D_y^j Mf)(0,k) = D_x^i D_y^j f(0,k), 0 \leq i + j \leq 1,\)

and

3. \((D_x^i D_y^j Mf)(h,0) = D_x^i D_y^j f(h,0), 0 \leq i + j \leq 1,\)

for all \( f \in C^2(\Delta).\)

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Following [2] we have that the interpolation mapping $M$ is well-defined.

**Theorem 1.** The interpolation mapping $M$ is well-defined on $C^2(\Delta)$, i.e., $Mf$ exists and is unique for all $f \in C^2(\Delta)$.

Before stating and proving our main result (Theorem 2) on an error bound, we consider two preliminary results.

**Lemma 1.** Let $f \in C^4[0,h]$ and $c(x)$ be the unique cubic polynomial such that if $e(x) \equiv c(x) - f(x)$ then $e(0) = e(h) = De(0) = De(h) = 0$. Then

\[
(4) \quad \|e\|_{L^\infty[0,h]} \leq \frac{1}{384} h^4 \|D^4 f\|_{L^\infty[0,h]},
\]

\[
(5) \quad \|De\|_{L^\infty[0,h]} \leq \frac{\sqrt{3}}{216} h^3 \|D^4 f\|_{L^\infty[0,h]},
\]

and

\[
(6) \quad \|De\|_{L^1[0,h]} \leq \frac{1}{72} h^4 \|D^4 f\|_{L^\infty[0,h]}.
\]

**Proof.** For a proof of (4) see any standard reference on interpolation theory and for a proof of (5) see [1]. The proof of (6) is as follows.

By Rolle's Theorem and the interpolation conditions, there exists a point $\xi \in (0,h)$ such that $De(\xi) = 0$. Define a new function

\[
F(z) \equiv De(z) - az(h - z)(z - \xi)
\]

for all $z \in [0,h]$, where $a$ is a real constant to be chosen.
Given a fixed $x \in (0, h)$ such that $x \neq \xi$, choose $\alpha$ such that

$$F(x) = 0, \quad \text{i.e., } \alpha \equiv \frac{D(x)}{[x(h-x)(x-\xi)].}$$

Then

$$F(0) = F(h) = F(\xi) = F(x) = 0$$

and by Rolle's Theorem there exists

$\theta \in [0, h]$ such that $D^3F(\theta) = 0$.

Computing $D^3F$, we find that $\alpha = \frac{1}{6} D^4f(\theta)$ and hence

$$D(x) = \frac{1}{6} x (h-x)(x-\xi) D^4f(\theta).$$

Thus,

$$\|D\|_{L^1[0,h]} \leq \frac{1}{6} \|D^4f\|_{L^\infty[0,h]} \max_{\xi \in [0,h]} \left( \int_0^\xi x(h-x)(\xi-x)dx \right. + \int_\xi^x x(h-x)(x-\xi)dx \bigg) \left. \right) \leq \frac{1}{6} \|D^4f\|_{L^\infty[0,h]} \max_{\xi \in [0,h]} \phi(\xi).$$

Since $D^2\phi(\xi) > 0$ for all $\xi \in (0, h)$, the maximum of $\phi(\xi)$ occurs for $\xi = 0$ and/or $\xi = h$. Thus $\phi(\xi) \leq \frac{1}{12} h^2$ and (6) follows immediately.

Q.E.D.

**Lemma 2.** Let $f \in C^3[0,h]$ and $q(x)$ be the unique quadratic polynomial such that if $e(x) \equiv q(x) - f(x)$ then $e(0) = e(h) = D(e)(0) = 0$.

Then

$$\|D\|_{L^\infty[0,h]} \leq \frac{1}{2} h^2 \|D^3f\|_{L^\infty[0,h]}$$
and

\[(8) \quad \|D^1_0 e\|_{L^1[0,h]} \leq \frac{1}{6} h^3 \|D^3 f\|_{L^\infty[0,h]} .\]

**Proof.**  By Rolle's Theorem and the interpolation conditions, there exists a point \(\xi \in (0,h)\) such that \(D_0 e(\xi) = 0\). Define a new function

\[F(z) \equiv D_0 e(z) - az(z - \xi)\] for all \(z \in [0,h]\), where \(a\) is a real constant to be chosen.

Given a fixed \(x \in (0,h)\) such that \(x \neq \xi\), choose \(a\) such that \(F(x) = 0\), i.e., \(a \equiv D_0 e(x)/[(x(\xi - x)]\). Then \(F(0) = F(\xi) = F(x) = 0\) and by Rolle's Theorem there exists \(\theta \in [0,h]\) such that \(D^2_0 f(\theta) = 0\).

Computing \(D^2_0 f\), we find that \(a = \frac{1}{2} D^3 f(\theta)\) and hence

\[D_0 e(x) = \frac{1}{2} x(x - \xi) D^3 f(\theta) .\] Thus

\[\|D^1_0 e\|_{L^\infty[0,h]} \leq \frac{1}{2} \|D^3 f\|_{L^\infty[0,h]} \max_{x \in [0,h]} \max_{\xi \in [0,h]} x |x - \xi|\]

\[\leq \frac{1}{2} h^2 \|D^3 f\|_{L^\infty[0,h]} ,\]

which proves (7).

Moreover,
\[
\|D_{e}\|_{L^1[0,h]} \leq \frac{1}{2} \|D^3 f\|_{L^\infty[0,h]} \left[ \int_0^\xi x(\xi - x)dx + \int_{\xi}^{h} x(x - \xi)dx \right]
\]

\[
\leq \frac{1}{2} \|D^3 f\|_{L^\infty[0,h]} \max_{\xi \in [0,h]} \left[ \frac{1}{3} \xi^3 + \frac{1}{3} h^3 - \frac{1}{2} h^2 \xi \right]
\]

\[
\leq \frac{h^3}{6} \|D^3 f\|_{L^\infty[0,h]},
\]

which proves (8). Q.E.D.

We proceed now to our main result, which improves Theorem 7 of [2].

Throughout the remainder of this note, we will let \(\|\cdot\|\) denote \(\|\cdot\|_{L^\infty(\Delta)}\).

**Theorem 2.** If \(f \in C^4(\Delta)\), then \(e(x,y) \equiv Mf(x,y) - f(x,y)\) satisfies

\[(9) \quad \|D_{xy} e\| \leq \frac{1}{2} h^2 \|D_x^4 D_y f\| + \frac{1}{2} k^2 \|D_x^3 D_y^2 f\| + \h k \|D_x^2 D_y^2 f\|,
\]

\[(10) \quad \|D_x e\| \leq \frac{8}{81} h^3 \|D_x^4 f\| + \frac{1}{2} h^2 k \|D_x^3 D_y f\| + \frac{1}{6} k^3 \|D_x^3 D_y f\|
\]

\[\quad \quad \quad \quad + \frac{1}{2} h k^2 \|D_x^2 D_y^2 f\|,
\]

\[(11) \quad \|D_y e\| \leq \frac{8}{81} k^3 \|D_y^4 f\| + \frac{1}{2} k^2 h \|D_x^3 D_y f\| + \frac{1}{6} h^3 \|D_x^3 D_y f\|
\]

\[\quad \quad \quad \quad + \frac{1}{2} h k^2 \|D_x^2 D_y^2 f\|,
\]

and

\[(12) \quad \|e\| \leq \frac{1}{2} \left( \frac{1}{384} + \frac{1}{72} \right) (h^4 \|D_x^4 f\| + k^4 \|D_y^4 f\|) + \frac{1}{6} h^3 k \|D_x^3 D_y f\|
\]

\[\quad \quad \quad \quad + \frac{1}{6} k^3 h \|D_y^3 D_x f\| + \frac{1}{4} h^2 k^2 \|D_x^2 D_y^2 f\|.
\]
Proof. If \((c,d)\) is any point in \(\Delta\),

\[
\Delta_{xy} D_x D_y e(c,d) = D_x D_y e(c,d) - D_x D_y e(c,0) - D_x D_y e(0,d) + D_x D_y e(0,0)
\]

\[
= \int_0^c \int_0^d D_x^2 D_y^2 e(x,y) dy \, dx
\]

\[
= \int_0^c \int_0^d D_x^2 D_y^2 f(x,y) dy \, dx
\]

and hence

\[
(13) \quad |\Delta_{xy} D_x D_y e(c,d)| \leq c d \|D_x^2 D_y^2 f\|.
\]

Using (13) and (7) we obtain

\[
(14) \quad |D_x D_y e(c,d)| \leq |D_x D_y e(c,0)| + |D_x D_y e(0,d)| + |D_x D_y e(0,0)|
\]

\[
+ c d \|D_x^2 D_y^2 f\|
\]

\[
\leq \frac{1}{2} h^2 \|D_x^3 D_y f\| + \frac{1}{2} k^2 \|D_y^3 D_x f\| + h k \|D_x^2 D_y^2 f\|,
\]

which proves (9).

Moreover,

\[
(15) \quad |D_x e(c,d)| \leq |D_x e(c,0)| + \int_0^d |D_y D_x e(c,y)| dy.
\]

Using (5), (14), and (8) to bound the right-hand side of (15),

we have
\[ |D_x e(c,d)| \leq \frac{8}{81} h^3 \|D_x^3 f\| + \int_0^d |D_x D_y e(c,0)| dy + \int_0^d |D_x D_y e(0,y)| dy \]

\[ + \int_0^d h y \|D_x^2 D_y^2 f\| dy \]

\[ \leq \frac{8}{81} h^3 \|D_x^4 f\| + \frac{1}{2} h^2 k \|D_x^3 D_y f\| + \frac{1}{6} k^3 \|D_y^3 D_x f\| \]

\[ + \frac{1}{2} h k^2 \|D_x^2 D_y^2 f\| , \]

which proves (10). Inequality (11) follows by symmetry.

Finally,

\[ |e(c,d)| \leq |e(0,d)| + \int_0^c |D_x e(x,d)| dx \]

\[ \leq |e(0,d)| + \int_0^c |D_x e(x,c,0)| dx + \int_0^c \int_0^d |D_y D_x e(x,y)| dy dx. \]

From (14) and (16) we have

\[ |e(c,d)| \leq |e(0,d)| + \int_0^c |D_x e(x,c,0)| dx \]

\[ + \int_0^c \int_0^d [ D_x D_y e(x,0) + D_x D_y e(0,y) + xy\|D_x^2 D_y^2 f\| ] dy dx. \]

Using (4), (6), and (8) to bound the right-hand side of (17),

we obtain

\[ |e(c,d)| \leq \frac{1}{384} k^4 \|D_y^4 f\| + \frac{1}{72} h^4 \|D_x^4 f\|_\infty + \frac{1}{6} h^3 k \|D_x^3 D_y f\| \]

\[ + \frac{1}{6} k^3 h \|D_y^3 D_x f\| + \frac{1}{4} h^2 k^2 \|D_x^2 D_y^2 f\| . \]
By symmetry, we also have

\[ |e(c,d)| \leq \frac{1}{384} h^4 \|D_x^4 f\| + \frac{1}{72} k^4 \|D_y^4 f\| + \frac{1}{6} h^3 k \|D_x^3 D_y f\| \]

\[ + \frac{1}{6} k^3 h \|D_y^3 D_x f\| + \frac{1}{4} h^2 k^2 \|D_x^2 D_y^2 f\| \]

and (12) follows by adding (18) and (19) and dividing by 2.

Q.E.D.
REFERENCES
