In Stokes flow, the stream function associated with the velocity of the fluid satisfies the biharmonic equation. The detailed behavior of solutions to the biharmonic equation on regions with corners has been historically difficult to characterize. The problem was first examined by Lord Rayleigh in 1920; in 1973, the existence of infinite oscillations in the domain Green’s function was proven in the case of the right angle by S. Osher. In this paper, we observe that, when the biharmonic equation is formulated as a boundary integral equation, the solutions are representable by rapidly convergent series of the form \( \sum_j \left( c_j t^{\mu_j} \sin (\beta_j \log (t)) + d_j t^{\mu_j} \cos (\beta_j \log (t)) \right) \), where \( t \) is the distance from the corner and the parameters \( \mu_j, \beta_j \) are real, and are determined via an explicit formula depending on the angle at the corner. In addition to being analytically perspicuous, these representations lend themselves to the construction of highly accurate and efficient numerical discretizations, significantly reducing the number of degrees of freedom required for the solution of the corresponding integral equations. The results are illustrated by several numerical examples.

**On the solution of Stokes equation on regions with corners**

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1. Introduction

In classical potential theory, solutions to elliptic partial differential equations are represented by potentials on the boundaries of the regions. Over the last four decades, several well-conditioned boundary integral representations have been developed for the biharmonic equation with various boundary conditions. These integral representations have been studied extensively when the boundaries of the regions are approximated by a smooth curve (see, [1, 2, 3, 4, 5, 6, 7, 8], for example). In all of these representations, the kernels of the integral equation are at worst weakly singular and, in same cases, even smooth. The corresponding solutions to the integral equations tend to be as smooth as the boundaries of the domains and the incoming data.

However, when the boundary of the region has corners, the solutions to both the differential equation and the corresponding integral equations are known to develop singularities. The solutions to the differential equation have been studied extensively on regions with corners for both Dirichlet and gradient boundary conditions. In particular, the behavior of solutions to the biharmonic equation on a wedge enclosed by straight boundaries $\theta = 0$ and $\theta = \alpha$, and a circular arc $r = r_0$, where $(r, \theta)$ are polar coordinates, has received much attention over the years. To the best of our knowledge, this particular problem was first studied by Lord Rayleigh in 1920 [9], and over the decades, by A. Dixon [10], Dean and Montagnon [11], Szegö [12], and Moffat [13], to name a few. In 1973, S. Osher showed that the Green’s function for the biharmonic equation on a right angle wedge has infinitely many oscillations in the vicinity of the corner on all but finitely many rays [14]. The more complicated structure of the Green’s function for the biharmonic equation is explained, in part, by the fact that the biharmonic equation does not have a maximum principle associated with it, while, for example, Laplace’s equation does.

While much has been published about the solutions to the differential equation, the solutions of the corresponding integral equation have been studied much less exhaustively. Recently, a detailed analysis of solutions to integral equations corresponding to the Laplace and Helmholtz equations on regions with corners was carried out by the second author and V. Rokhlin [15, 16, 17]. They observed that the solutions to these integral equations can be expressed as rapidly convergent series of singular powers in the Laplace case and Bessel functions of non-integer order in the Helmholtz case.

In this paper, we investigate the solutions to a standard integral equation corresponding to the velocity boundary value problem for Stokes equation. For the velocity boundary value problem, the stream function associated with the velocity field satisfies the biharmonic equation with gradient boundary conditions (it turns out that the same integral equation can be used for Dirichlet boundary conditions, see, for example [8]). We show that, if the boundary data is smooth on each side of the corner, then the solutions of this integral equation can be expressed a rapidly convergent series of elementary functions of the form $t^{\mu_j} \cos (\beta_j \log |t|)$ and $t^{\mu_j} \sin (\beta_j \log |t|)$, where the parameters $\mu_j, \beta_j$ can be computed explicitly by a simple formula depending only on the angle at the corner. Furthermore, we prove that, for any $N$, there exists a linear combination of the first $N$ of these basis functions which satisfies the integral equation with error $O(|t|^N)$, where $t$ is the distance from the corner.

The detailed information about the analytical behavior of the solution in the vicinity of corners, discussed in this paper, allows for the construction of purpose-made discretizations
of the integral equation. These discretizations accurately represent the solutions near corners using far fewer degrees of freedom than graded meshes, which are commonly used in such environments; thereby leading to highly efficient numerical solvers.

The rest of the paper is organized as follows. In Section 2, we discuss the mathematical preliminaries for the governing equation and its reformulation as an integral equation. In Section 3, we derive several analytical results using techniques required for the principal results which are derived in Section 4. We illustrate the performance of the numerical scheme which utilizes the explicit knowledge of the structure of the solution to the integral equation in Section 5. In Section 6, we present generalizations and extensions of the apparatus of this paper. The proof of several results in Section 4 are technical and are presented in Sections 8 and 9.

2. Preliminaries

In this paper, vector-valued quantities are denoted by bold, lower-case letters (e.g. \( \mathbf{h} \)), while tensor-valued quantities are bold and upper-case (e.g. \( \mathbf{T} \)). Subscript indices of non-bold characters (e.g. \( h_j \) or \( T_{jkl} \)) are used to denote the entries within a vector (\( \mathbf{h} \)) or tensor (\( \mathbf{T} \)). We use the standard Einstein summation convention; in other words, there is an implied sum taken over the repeated indices of any term (e.g. the symbol \( a_j b_j \) is used to represent the sum \( \sum_j a_j b_j \)). Let \( \mathcal{C}^k \) denote the space of functions which have \( k \) continuous derivatives.

Suppose now that \( \Omega \) is a simply connected open subset of \( \mathbb{R}^2 \). Let \( \Gamma \) denote the boundary of \( \Omega \) and suppose that \( \Gamma \) is a simple closed curve of length \( L \) with \( n_c \) corners. Let \( \gamma : [0, L] \to \mathbb{R}^2 \) denote an arc length parameterization of \( \Gamma \) in the counter-clockwise direction, and suppose that the location of the corners are given by \( \gamma(s_j), j = 1, 2, \ldots n_c \) with \( 0 = s_1 < s_2 \ldots < s_{n_c} < s_{n_c+1} = L \). Furthermore, suppose that \( \gamma \) is analytic on the intervals \( (s_j, s_{j+1}) \) for each \( j = 1, 2, \ldots n_c \). Let \( \tau(x) \) and \( \nu(x) \) denote the positively-oriented unit tangent and the outward unit normal respectively, for \( x \in \Gamma \). Let \( h_{\tau} = h_j \tau_j \) and \( h_{\nu} = h_j \nu_j \) denote the tangential and normal components of the vector \( \mathbf{h} \) respectively, see Figure 1.

Figure 1: A sample domain \( \Omega \) with three corners
2.1. Velocity boundary value problem

The equations of incompressible Stokes flow with velocity boundary conditions on a domain $\Omega$ with boundary $\Gamma$ are

\[\begin{align*}
-\Delta u + \nabla p &= 0 \quad \text{in } \Omega, \\
\nabla \cdot u &= 0 \quad \text{in } \Omega, \\
\quad u &= h \quad \text{on } \Gamma,
\end{align*}\]

where $u$ is the velocity of the fluid, $p$ is the fluid pressure and $h$ is the prescribed velocity on the boundary. For any $h$ which satisfies

\[\int_{\Gamma} h \cdot \nu \, dS = 0,
\]

there exists a unique velocity field $u$ and a pressure $p$, defined uniquely up to a constant, that satisfy (1) – (3). We summarize the result in the following lemma (see [18] for a proof).

**Lemma 1.** Suppose $h \in L^2(\Gamma)$ and satisfies $\int_{\Gamma} h \cdot \nu \, dS = 0$, then there exists a unique velocity $u$ and a pressure $p$ which is unique up to a constant which satisfy the velocity boundary value problem (1) – (3).

**Remark 2.** The Stokes equation with velocity boundary conditions can be reformulated as a biharmonic equation with gradient boundary conditions. First, we represent the velocity $u: \Omega \to \mathbb{R}^2$ as $u = \nabla^\perp w$, where $w: \Omega \to \mathbb{R}$ is the stream function associated with the velocity field and $\nabla^\perp$ is the operator given by

\[\nabla^\perp w = \left[ -\frac{\partial w}{\partial x_2}, \frac{\partial w}{\partial x_1} \right].\]

Next, we observe that $u = \nabla^\perp w$ automatically satisfies the divergence free condition (2). Finally, taking the dot product of $\nabla^\perp$ with (1), we observe that $w$ satisfies the biharmonic equation with gradient boundary conditions given by

\[\begin{align*}
\Delta^2 w &= 0 \quad \text{in } \Omega, \\
\nabla^\perp w &= h \quad \text{on } \Gamma.
\end{align*}\]

2.2. Integral equation formulation

Following the treatment of [19, 5], the fundamental solution to the Stokes equations (the Stokeslet) is given by

\[G_{j,k}(x, y) = \frac{1}{4\pi} \left[ -\log |x - y| \delta_{ij} + \frac{(x_j - y_j)(x_k - y_k)}{|x - y|^2} \right], \quad j, k \in 1, 2,
\]

for $x, y \in \mathbb{R}^2$, and $x \neq y$, where $\delta_{ij}$ is the Kronecker delta function. The stress tensor $T_{j,k,\ell}(x, y)$ associated with the Green’s function, or the stresslet, is given by

\[T_{j,k,\ell}(x, y) = -\frac{1}{\pi} \frac{(x_j - y_j)(x_k - y_k)(x_\ell - y_\ell)}{|x - y|^4}, \quad j, k, \ell \in 1, 2,
\]
\( x, y \in \mathbb{R}^2 \) and \( x \neq y \). The stresslet \( T \) is roughly analogous to a dipole in electrostatics. The double layer Stokes potential is the velocity field due to a surface density of stresslets \( \mu \) and is defined by

\[
(D_\Gamma[\mu](x))_j = \int_\Gamma T_{k,j,\ell}(y, x) \mu_k(y) \nu_\ell(y) \, dS_y, \quad j, k, \ell \in 1, 2,
\]

for \( x \in \mathbb{R}^2 \). Clearly, \( D_\Gamma[\mu](x) \) satisfies Stokes equation for \( x \in \Omega \).

The following lemma describes the behavior of the double layer Stokes potential as \( x \to x_0 \) where \( x_0 \in \Gamma \).

**Lemma 3.** Suppose that \( \mu : \Gamma \to \mathbb{R}^2 \) and let \( D_\Gamma[\mu](x) \) denote a double layer Stokes potential (10). Then \( D_\Gamma[\mu](x) \) satisfies the jump relation:

\[
\lim_{x \to x_0} D_\Gamma[\mu](x) = -\frac{1}{2}\mu(x_0) + \text{p.v.} \int_\Gamma K_0(x_0, y)\mu(y) \, dS_y,
\]

where \( x_0, y \in \Gamma \), and \( K_0 \) is given by

\[
K_0(x, y) = -\frac{1}{\pi} \frac{(y - x) \cdot \nu(y)}{|x - y|^4} \left[ \begin{array}{c|c} (y_1 - x_1)^2 & (y_1 - x_1)(y_2 - x_2) \\ \hline (y_1 - x_1)(y_2 - x_2) & (y_2 - x_2)^2 \end{array} \right]
\]

for \( x, y \in \Gamma \) and \( \text{p.v.} \int \) denotes a principal value integral.

The following lemma states that the kernel \( K_0 \) is smooth if the boundary \( \Gamma \) is smooth.

**Lemma 4.** The kernel \( K_0 \) is \( C^{k-2} \) if \( \gamma \) is \( C^k \) with limiting values

\[
\lim_{t \to s} K_0(\gamma(t), \gamma(s)) = -\frac{1}{\pi} \kappa(\gamma(t)) \begin{bmatrix} \gamma_1'(t)^2 & \gamma_1'(t)\gamma_2'(t) \\ \gamma_1'(t)\gamma_2'(t) & \gamma_2'(t)^2 \end{bmatrix},
\]

where \( \kappa(\gamma(t)) \) is the curvature at \( \gamma(t) \). Furthermore, \( K_0 \) is analytic if \( \gamma \) is analytic.

The following theorem reduces the velocity boundary value problem (1) – (3) to an integral equation on the boundary by representing \( u \) as double layer Stokes potential with unknown density \( \mu \), i.e.

\[
u(x) = D_\Gamma[\mu](x) \quad x \in \Omega.
\]

**Lemma 5.** Suppose \( h \in L^2(\Gamma) \) and that \( \int_\Gamma h \cdot \nu \, dS = 0 \). Then there exists a unique solution \( \mu \in L^2(\Gamma) \) which satisfies

\[
-\frac{1}{2}\mu(x) + \text{p.v.} \int_\Gamma K_0(x, y)\mu(y) \, dS_y = h(x), \quad x \in \Gamma,
\]

and \( \int_\Gamma \mu \cdot \nu \, dS = 0 \). Furthermore, \( u(x) = D_\Gamma[\mu](x) \) satisfies Stokes equations (1), (2), along with the boundary conditions \( u(x) = h(x) \) for \( x \in \Gamma \).

**Proof.** See, for example, [5] for a proof.

The following lemma extends Lemma 5 to the case where the boundary \( \Gamma \) is an open arc.

**Lemma 6.** Suppose \( h \in L^2(\Gamma) \) then there exists a unique solution \( \mu \in L^2(\Gamma) \) which satisfies

\[
-\frac{1}{2}\mu(x) + \text{p.v.} \int_\Gamma K_0(x, y)\mu(y) \, dS_y = h(x) \quad x \in \Gamma,
\]

where \( K_0 \) is defined by (12).
2.3. Integral equation in tangential and normal coordinates

It turns out that it is convenient to represent both the velocity on the boundary $\mathbf{h}$ and the solution of the integral equation $\mathbf{\mu}$ in terms of their tangential and normal coordinates, denoted by $\mathbf{h} = (h_\tau, h_\nu)$ and $\mathbf{\mu} = (\mu_\tau, \mu_\nu)$ respectively, as opposed to their Cartesian coordinates. In this section, we discuss the representation in the tangential and normal coordinates of the double layer Stokes potential, and the corresponding integral equation for the velocity boundary value problem.

Let $\mathbf{R}(x)$ denote the unitary transformation that converts vectors expressed in Cartesian coordinates to vectors expressed in tangential and normal components, i.e.

$$\mathbf{R}(x) = \begin{bmatrix} \tau_1(x) & \tau_2(x) \\ n_1(x) & n_2(x) \end{bmatrix} \quad x \in \Gamma. \quad (17)$$

Let $\mathbf{R}^*(x)$ denote the adjoint of $\mathbf{R}(x)$. Suppose $\mathbf{u}(x) = (u_1(x), u_2(x))$ is the double layer Stokes potential with density $\mathbf{\mu}(x) = (\mu_\tau(x), \mu_\nu(x))$, given by

$$\mathbf{u}(x) = \mathcal{D}_\Gamma[\mathbf{\mu}](x) \quad x \in \Omega, \quad (18)$$

where

$$\left(\mathcal{D}_\Gamma[\mathbf{\mu}](x)\right)_j = \int_\Gamma T_{k,j,\ell}(y, x) \left(\mathbf{R}^*(y) \cdot \begin{bmatrix} \mu_\tau(y) \\ \mu_\nu(y) \end{bmatrix} \right)_k \nu_\ell(y) dS_y, \quad j, k, \ell = 1, 2, \quad (19)$$

where $x \in \Omega$. The following theorem reduces the velocity boundary value problem (1) – (3) to an integral equation on the boundary in the rotated frame.

**Lemma 7.** Suppose $\mathbf{h} = (h_\tau, h_\nu) \in \mathbb{L}^2(\Gamma)$ and that $\int_\Gamma h_\nu dS = 0$. Then there exists a unique solution $\mathbf{\mu} = (\mu_\tau, \mu_\nu) \in \mathbb{L}^2(\Gamma)$ which satisfies

$$-\frac{1}{2} \begin{bmatrix} \mu_\tau(x) \\ \mu_\nu(x) \end{bmatrix} + p.v. \int_\Gamma \mathbf{K}(x, y) \begin{bmatrix} \mu_\tau(y) \\ \mu_\nu(y) \end{bmatrix} dS_y = \begin{bmatrix} h_\tau(x) \\ h_\nu(x) \end{bmatrix}, \quad x \in \Gamma, \quad (20)$$

along with $\int_\Gamma \mu_\nu dS = 0$, where

$$\mathbf{K}(x, y) = \mathbf{R}(x)\mathbf{K}_0(x, y)\mathbf{R}^*(y), \quad x, y \in \Gamma. \quad (21)$$

Furthermore, $\mathbf{u}(x) = \mathcal{D}_\Gamma[\mathbf{\mu}](x)$ satisfies Stokes equations (1), (2), along with the boundary conditions $\mathbf{u}(x) = \mathbf{h}(x)$ for $x \in \Gamma$.

**Proof.** The result is a straightforward consequence of Lemma 5. \hfill \blacksquare

The following lemma extends Lemma 7 to the case where the boundary $\Gamma$ is an open arc.

**Lemma 8.** Suppose $\mathbf{h} = (h_\tau, h_\nu) \in \mathbb{L}^2(\Gamma)$ then there exists a unique solution $\mathbf{\mu} = (\mu_\tau, \mu_\nu) \in \mathbb{L}^2(\Gamma)$ which satisfies

$$-\frac{1}{2} \begin{bmatrix} \mu_\tau(x) \\ \mu_\nu(x) \end{bmatrix} + p.v. \int_\Gamma \mathbf{K}(x, y) \begin{bmatrix} \mu_\tau(y) \\ \mu_\nu(y) \end{bmatrix} dS_y = \begin{bmatrix} h_\tau(x) \\ h_\nu(x) \end{bmatrix}, \quad x \in \Gamma, \quad (22)$$

where $\mathbf{K}$ is defined by (21).
Figure 2: A wedge with interior angle $\pi \theta$

2.4. Integral equations on the wedge

Suppose $\gamma(t): [-1, 1] \to \mathbb{R}^2$ is a wedge with interior angle $\pi \theta$ and side length 1 on either side of the corner, parametrized by arc-length (see, Figure 2).

In a slight misuse of notation, let $\mu_\tau(t)$ denote $\mu_\tau(\gamma(t))$ for all $-1 < t < 1$. Likewise, let $\mu_\nu(t), h_\tau(t), \text{ and } h_\nu(t)$ denote $\mu_\nu(\gamma(t)), h_\tau(\gamma(t)) \text{ and } h_\nu(\gamma(t))$, respectively. The integral equation (22) for the velocity boundary value problem is then given by

$$\int_{-1}^{1} \mathbf{K}(\gamma(t), \gamma(s)) \begin{bmatrix} \mu_\tau(t) \\ \mu_\nu(t) \end{bmatrix} ds = \begin{bmatrix} h_\tau(t) \\ h_\nu(t) \end{bmatrix}, -1 < t < 1,$$

where $\mathbf{K}$ is defined by (21). In this case, the kernel $\mathbf{K}$ has a simple form which is given by the following lemma.

**Lemma 9.** Suppose $\gamma : [-1, 1] \to \mathbb{R}^2$ is defined by the formula

$$\gamma(t) = \begin{cases} -t \cdot (\cos(\pi \theta), \sin(\pi \theta)) & \text{if } -1 < t < 0, \\ (t, 0) & \text{if } 0 < t < 1. \end{cases}$$

Suppose further that $k_j(s,t), j = 1, 2, 3, 4$, are defined by

$$k_1(s,t) = \frac{t \sin(\pi \theta)}{\pi (s^2 + t^2 + 2st \cos(\pi \theta))^2} \quad \begin{cases} (s + t \cos(\pi \theta))(s \cos(\pi \theta) + t) & \text{if } -1 < s < 0, \\ 0 & \text{if } 0 < s < 1. \end{cases}$$

$$k_2(s,t) = \frac{t \sin(\pi \theta)}{\pi (s^2 + t^2 + 2st \cos(\pi \theta))^2} \quad \begin{cases} t \sin(\pi \theta)(t + s \cos(\pi \theta)) & \text{if } -1 < s < 0, \\ 0 & \text{if } 0 < s < 1. \end{cases}$$

$$k_3(s,t) = -\frac{t \sin(\pi \theta)}{\pi (s^2 + t^2 + 2st \cos(\pi \theta))^2} \quad \begin{cases} (s + t \cos(\pi \theta))s \sin(\pi \theta) & \text{if } -1 < s < 0, \\ 0 & \text{if } 0 < s < 1. \end{cases}$$

$$k_4(s,t) = -\frac{t \sin(\pi \theta)}{\pi (s^2 + t^2 + 2st \cos(\pi \theta))^2} \quad \begin{cases} st \sin^2(\pi \theta) & \text{if } -1 < s < 0, \\ 0 & \text{if } 0 < s < 1. \end{cases}$$

for all $-1 < s, t < 1$. If $0 < t < 1$, then

$$\mathbf{K}(\gamma(t), \gamma(s)) = \begin{cases} \begin{bmatrix} k_1(s,t) & k_2(s,t) \\ k_3(s,t) & k_4(s,t) \end{bmatrix} & \text{if } -1 < s < 0, \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} & \text{if } 0 < s < 1. \end{cases}$$
Likewise, if \(-1 < t < 0\), then

\[
\begin{align*}
\mathbf{K}(\gamma(t), \gamma(s)) &= \begin{cases} 
0 & \text{if } -1 < s < 0, \\
\begin{bmatrix}
-k_1(s, t) & k_2(s, t) \\
k_3(s, t) & -k_4(s, t)
\end{bmatrix} & \text{if } 0 < s < 1.
\end{cases}
\end{align*}
\tag{30}
\]

In the following theorem, we show that, when \(\Gamma\) is a wedge, the integral equation \((23)\) decouples into two independent integral equations on the interval \([0, 1]\).

**Theorem 10.** Suppose \(\mu_{\tau}, \mu_{\nu}\) are functions in \(L^2[-1, 1]\). Let \(h_{\tau}, h_{\nu}\) be defined by \((23)\). We denote the odd and the even parts of a function \(f\) by \(f_o\) and \(f_e\) respectively, where

\[
f_o(s) = \frac{1}{2} (f(s) - f(-s)), \quad \text{and}, \quad f_e(s) = \frac{1}{2} (f(s) + f(-s)),
\tag{31}
\]

for \(-1 < s < 1\). Then for \(0 < t < 1\),

\[
\begin{align*}
\begin{bmatrix} h_{\tau,e}(t) \\ h_{\nu,o}(t) \end{bmatrix} &= -\frac{1}{2} \begin{bmatrix} \mu_{\tau,e}(t) \\ \mu_{\nu,o}(t) \end{bmatrix} - \int_0^1 \begin{bmatrix} k_{1,1}(s, t) & k_{1,2}(s, t) \\
k_{2,1}(s, t) & k_{2,2}(s, t) \end{bmatrix} \begin{bmatrix} \mu_{\tau,e}(s) \\ \mu_{\nu,o}(s) \end{bmatrix} ds, \\
\begin{bmatrix} h_{\tau,o}(t) \\ h_{\nu,e}(t) \end{bmatrix} &= -\frac{1}{2} \begin{bmatrix} \mu_{\tau,o}(t) \\ \mu_{\nu,e}(t) \end{bmatrix} + \int_0^1 \begin{bmatrix} k_{1,1}(s, t) & k_{1,2}(s, t) \\
k_{2,1}(s, t) & k_{2,2}(s, t) \end{bmatrix} \begin{bmatrix} \mu_{\tau,o}(s) \\ \mu_{\nu,e}(s) \end{bmatrix} ds,
\end{align*}
\tag{32, 33}
\]

where \(k_{j,\ell}(s, t), j, \ell = 1, 2,\) are given by

\[
\begin{align*}
k_{1,1}(s, t) &= \frac{1}{2} (k_1(s, -t) - k_1(-s, t)) = \frac{t \sin(\pi \theta)(s - t \cos(\pi \theta))(t - s \cos(\pi \theta))}{\pi(s^2 + t^2 - 2st \cos(\pi \theta))^2}, \\
k_{1,2}(s, t) &= \frac{1}{2} (-k_2(s, -t) + k_2(-s, t)) = \frac{t^2 \sin^2(\pi \theta)(t - s \cos(\pi \theta))}{\pi(s^2 + t^2 - 2st \cos(\pi \theta))^2}, \\
k_{2,1}(s, t) &= \frac{1}{2} (k_3(s, -t) - k_3(-s, t)) = \frac{st \sin^2(\pi \theta)(s - t \cos(\pi \theta))}{\pi(s^2 + t^2 - 2st \cos(\pi \theta))^2}, \\
k_{2,2}(s, t) &= \frac{1}{2} (-k_4(s, -t) + k_4(-s, t)) = \frac{st^2 \sin^3(\pi \theta)}{\pi(s^2 + t^2 - 2st \cos(\pi \theta))^2},
\end{align*}
\tag{34, 35, 36, 37}
\]

for all \(0 < s, t < 1\).

**Proof.** Substituting the expression for the kernel \(\mathbf{K}\) given by \((29), (30)\) in \((23)\), we get

\[
\begin{align*}
\begin{bmatrix} h_{\tau}(t) \\ h_{\nu}(t) \end{bmatrix} &= -\frac{1}{2} \begin{bmatrix} \mu_{\tau}(t) \\ \mu_{\nu}(t) \end{bmatrix} + \int_{-1}^0 \begin{bmatrix} k_1(s, t) & k_2(s, t) \\
k_3(s, t) & k_4(s, t) \end{bmatrix} \begin{bmatrix} \mu_{\tau}(s) \\ \mu_{\nu}(s) \end{bmatrix} ds, \\
\begin{bmatrix} h_{\tau}(t) \\ h_{\nu}(t) \end{bmatrix} &= -\frac{1}{2} \begin{bmatrix} \mu_{\tau}(t) \\ \mu_{\nu}(t) \end{bmatrix} + \int_0^1 \begin{bmatrix} -k_1(s, t) & k_2(s, t) \\
k_3(s, t) & -k_4(s, t) \end{bmatrix} \begin{bmatrix} \mu_{\tau}(s) \\ \mu_{\nu}(s) \end{bmatrix} ds,
\end{align*}
\tag{38, 39}
\]
for $-1 < t < 0$. Then, making the change of variable $s \to -s$ in (38) and the change of variable $t \to -t$ in (39), we get
\[
\begin{bmatrix}
h_\tau(t) \\
h_\nu(t)
\end{bmatrix} = -\frac{1}{2} \begin{bmatrix}
\mu_\tau(t) \\
\mu_\nu(t)
\end{bmatrix} + \int_0^1 \begin{bmatrix}
k_1(-s, t) & k_2(-s, t) \\
k_3(-s, t) & k_4(-s, t)
\end{bmatrix} \begin{bmatrix}
\mu_\tau(-s) \\
\mu_\nu(-s)
\end{bmatrix} ds ,
\]
for $0 < t < 1$ and
\[
\begin{bmatrix}
h_\tau(-t) \\
h_\nu(-t)
\end{bmatrix} = -\frac{1}{2} \begin{bmatrix}
\mu_\tau(-t) \\
\mu_\nu(-t)
\end{bmatrix} + \int_0^1 \begin{bmatrix}
-k_1(s, -t) & k_2(s, -t) \\
k_3(s, -t) & -k_4(s, -t)
\end{bmatrix} \begin{bmatrix}
\mu_\tau(s) \\
\mu_\nu(s)
\end{bmatrix} ds ,
\]
for $0 < t < 1$. Finally, combining (40), (41), we get
\[
\begin{bmatrix}
h_{\tau,e}(t) \\
h_{\nu,e}(t)
\end{bmatrix} = -\frac{1}{2} \begin{bmatrix}
\mu_{\tau,e}(t) \\
\mu_{\nu,e}(t)
\end{bmatrix} - \int_0^1 \begin{bmatrix}
k_{1,1}(s, t) & k_{1,2}(s, t) \\
k_{2,1}(s, t) & k_{2,2}(s, t)
\end{bmatrix} \begin{bmatrix}
\mu_{\tau,e}(s) \\
\mu_{\nu,e}(s)
\end{bmatrix} ds ,
\]
and
\[
\begin{bmatrix}
h_{\tau,o}(t) \\
h_{\nu,o}(t)
\end{bmatrix} = -\frac{1}{2} \begin{bmatrix}
\mu_{\tau,o}(t) \\
\mu_{\nu,o}(t)
\end{bmatrix} + \int_0^1 \begin{bmatrix}
k_{1,1}(s, t) & k_{1,2}(s, t) \\
k_{2,1}(s, t) & k_{2,2}(s, t)
\end{bmatrix} \begin{bmatrix}
\mu_{\tau,o}(s) \\
\mu_{\nu,o}(s)
\end{bmatrix} ds .
\]

Thus, the integral equation (23) clearly splits into two cases:

- Tangential odd, normal even: the tangential components of both the velocity field $h$ and the density $\mu$, $h_\tau(t)$ and $\mu_\tau(t)$, are odd functions of $t$ and the normal components $h_\nu(t)$ and $\mu_\nu(t)$ are even functions of $t$.

- Tangential even, normal odd: the tangential components of both the velocity field $h$ and the density $\mu$, $h_\tau(t)$ and $\mu_\tau(t)$, are even functions of $t$ and the normal components $h_\nu(t)$ and $\mu_\nu(t)$ are odd functions of $t$.

3. Analytical Apparatus

In this section, we investigate integrals of the form
\[
\int_0^1 k_{j,\ell}(s, t)s^z ds , \quad j, \ell = 1, 2 ,
\]
for $0 < t < 1$, where $z \in \mathbb{C}$ with $\text{Re}(z) > -1$, and $k_{j,\ell}$ are defined in (34) – (37). We use the principal branch of log for the definition of $s^z$, i.e., $\text{Arg}(s) \in [0, 2\pi)$. The principal result of this section is Theorem 15.

On inspecting the kernels $k_{j,\ell}(s, t)$, $j, \ell = 1, 2$, we observe that it suffices to evaluate integrals of the form
\[
I(z, \theta, t) = \frac{1}{\pi} \int_0^1 \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} ds , \quad \text{for } 0 < t < 1 ,
\]
where $\text{Re}(z) > -1$. Using standard techniques in complex analysis, we first derive an expression for the above integral from 0 to $\infty$ in the following lemma.
Lemma 11. Suppose $z \in \mathbb{C}$, $-1 < \text{Re}(z) < 3$, $z \neq 0, 1, 2$, and $\theta \in \mathbb{C}$. Then
\[
I_1(z, \theta, t) = \frac{1}{\pi} \int_0^\infty \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \, ds = a(z, \theta)t^{z-3},
\] (46)
for $0 < t < 1$ where
\[
a(z, \theta) = \frac{z \sin(2\pi \theta) \cos(\pi(1 - \theta)z) + 2\sin(\pi(1 - \theta)z)(1 - z \sin^2(\pi \theta))}{4\sin(\pi z)\sin^3(\pi \theta)}.
\] (47)

Proof. Let $\Gamma_{\text{key}} = \Gamma_\varepsilon \cup \Gamma_1 \cup \Gamma_R \cup \Gamma_2$ denote a keyhole contour where
\[
\Gamma_\varepsilon = \{-\varepsilon e^{ix}, -\frac{\pi}{2} < x < \frac{\pi}{2}\}, \quad \Gamma_1 = \{x + i\varepsilon, 0 \leq x \leq \sqrt{R^2 - \varepsilon^2}\},
\]
\[
\Gamma_R = \{Re^{ix}, x_0 < x < 2\pi - x_0\}, \quad \Gamma_2 = \{x - i\varepsilon, 0 \leq x \leq \sqrt{R^2 - \varepsilon^2}\},
\]
where $x_0 = \arctan \varepsilon/R$, see Figure 3. Using Cauchy’s integral theorem,
\[
\left| \int_{\Gamma_\varepsilon} \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \, ds \right| = \left| \left( \int_{\Gamma_1} + \int_{\Gamma_R} + \int_{\Gamma_2} + \int_{\Gamma_\varepsilon} \right) \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \, ds \right|,
\] (48)
\[
= 2\pi i \left. \frac{d}{ds} \left( \frac{s^z}{(s - te^{i\pi(2-\theta)})^2} \right) \right|_{s=te^{i\pi \theta}} + \left. \frac{d}{ds} \left( \frac{s^z}{(s - te^{i\pi \theta})^2} \right) \right|_{s=te^{i\pi(2-\theta)}} = -\frac{2\pi t^{z-3}}{\sin^3(\pi \theta)} \left( (z - 2) \left( e^{i\pi \theta z} - e^{2\pi iz} e^{-i\pi \theta z} \right) - z \left( e^{i\pi \theta(z-2)} - e^{2\pi iz} e^{-i\pi \theta(z-2)} \right) \right),
\] (49)
\[
for 0 < t < 1. Suppose 0 < t < 1, then there exists a constant $M_1 < \infty$ such that
\[
\left| \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \right| \leq M_1 \varepsilon^{\text{Re}(z)},
\] (50)
for all $s \in \Gamma_\varepsilon$. Taking the limit as $\varepsilon \to 0$, we get
\[
\lim_{\varepsilon \to 0} \left| \int_{\Gamma_\varepsilon} \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \, ds \right| \leq \lim_{\varepsilon \to 0} \pi M_1 \varepsilon^{\text{Re}(z)+1} = 0,
\] (52)
since \( \text{Re}(z) > -1 \). Similarly, for \( 0 < t < 1 \), there exists a constant \( M_2 < \infty \) such that
\[
\left| \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \right| \leq \frac{M_2}{R^{4 - \text{Re}(z)}},
\]
for all \( s \in \Gamma_R \). Taking the limit as \( R \to \infty \), we get
\[
\lim_{R \to \infty} \left| \int_{\Gamma_R} \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \, ds \right| \leq \lim_{R \to \infty} \frac{2\pi M}{R^{3 - \text{Re}(z)}} = 0,
\]
since \( \text{Re}(z) < 3 \). On \( \Gamma_1 \) and \( \Gamma_2 \), taking the limits as \( \varepsilon \to 0 \) and \( R \to \infty \), we get
\[
\lim_{\varepsilon \to 0} \lim_{R \to \infty} \int_{\Gamma_1} \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \, ds = I_1(z, \theta, t),
\]
\[
\lim_{\varepsilon \to 0} \lim_{R \to \infty} \int_{\Gamma_2} \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \, ds = -e^{2\pi iz} I_1(z, \theta, t).
\]
The result follows by taking the limit \( \varepsilon \to 0 \) and \( R \to \infty \) in (50), and using (52) and (54) to (56).

Suppose now that \( I_2(z, \theta, t) \) is defined by
\[
I_2(z, \theta) = \int_1^\infty \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} \, ds,
\]
for \( 0 < t < 1 \). Clearly,
\[
I(z, \theta, t) = I_1(z, \theta, t) - I_2(z, \theta, t),
\]
where \( I(z, \theta, t) \) is given by (45) and \( I_1(z, \theta, t) \) by (46). In the following lemma, we compute an expression for \( I_2(z, \theta, t) \) by deriving a Taylor expansion for
\[
f(s, t) = \frac{1}{(s^2 + t^2 - 2st \cos(\pi \theta))^2}, \quad \text{for } |s| > 1, |t| < 1.
\]

Lemma 12. Suppose that \( \theta \in \mathbb{C} \). Then for all \( |t| < 1 \) and \( |s| > 1 \)
\[
f(s, t) = \frac{1}{(s^2 + t^2 - 2st \cos(\pi \theta))^2} = \sum_{n=0}^{\infty} a_n(s) t^n,
\]
where
\[
a_n(s) = \frac{1}{4s^{n+4} \sin^3(\pi \theta)} ((n + 3) \sin((n + 1)\pi \theta) - (n + 1) \sin((n + 3)\pi \theta)).
\]
Furthermore, for \(-1 < \text{Re}(z) < 3\), and \( z \neq 0, 1, 2\),
\[
I_2(z, \theta, t) = \frac{1}{\pi} \int_1^\infty s^z f(s, t) \, ds = \sum_{n=0}^{\infty} F(n, z, \theta) t^n,
\]
where
\[
F(n, z, \theta) = \frac{(n + 1) \sin((n + 3)\pi \theta) - (n + 3) \sin((n + 1)\pi \theta)}{4\pi(-z + n + 3) \sin^3(\pi \theta)}.
\]
Proof. For fixed \(|s| > 1\), \(f(s, t)\) is analytic in \(|t| < 1\). Using Cauchy’s integral formula, the coefficients of the Taylor series are given by

\[ a_n(s) = \frac{1}{2\pi i} \int_{|\xi|=1} \frac{f(s, \xi)}{\xi^{n+1}} d\xi = \frac{1}{2\pi} \int_0^{2\pi} e^{-inx} f(s, e^{ix}) \, dx, \tag{63} \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \frac{(s - e^{i(\pi\theta + x)})^2(s - e^{i(-\pi\theta + x)})^2}{dx}, \tag{64} \]

\[ = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{-inx}}{(1 - e^{i(n+1)x})^2} \frac{e^{-inx}}{(1 - e^{i(-n+1)x})^2} \, dx. \tag{65} \]

Using the Taylor expansion of \(1/(1-x)^2\) given by

\[ \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} (n+1)x^n, \tag{66} \]

for \(0 < x < 1\), equation (65) simplifies to,

\[ a_n(s) = \frac{1}{2\pi} \frac{1}{s^{n+4}} \int_0^{2\pi} e^{-inx} \sum_{\ell=0}^{\infty} \sum_{m=0}^{\infty} (\ell + 1)(m + 1) \frac{e^{i(\ell-m)x + (\ell-m)\pi\theta}}{s^{\ell+m}} \, dx \tag{67} \]

Due to the orthogonality of the Fourier basis \(\{e^{inx}\}\) in \(L^2[0, 2\pi]\), the only terms that contribute to the integral in (67) are when \(\ell + m = n\). Thus,

\[ a_n(s) = \frac{1}{s^{n+4}} \sum_{\ell+m=n} (\ell + 1)(m + 1) e^{i(\ell-m)\pi\theta} = e^{-in\pi\theta} \sum_{\ell=0}^{n} (\ell + 1)(n - \ell + 1) e^{2i\ell\pi\theta}. \tag{68} \]

The result for the Taylor series for \(f(s, t)\) then follows by using

\[ \sum_{\ell=0}^{n} p^\ell e^{i\ell x} = \frac{d^p}{dx^p} \left( \frac{1 - e^{(n+1)x}}{1 - e^{ix}} \right), \quad p = 0, 1, 2. \tag{69} \]

Given the Taylor expansion for \(f(s, t)\), the integral \(I_2(z, \theta, t)\) can be computed by switching the order of summation and integration and using

\[ \int_1^{\infty} s^n a_n(s) \, ds = F(n, z, \theta), \tag{70} \]

which concludes the proof.

Using Lemmas 11 and 12, we compute \(I(z, \theta, t)\) in the following theorem.

Theorem 13. Suppose \(-1 < \text{Re}(z) < 3\), and \(z \neq 0, 1, 2\), then

\[ I(z, \theta, t) = \int_0^{1} \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi\theta))^2} = a(z, \theta)t^{z-3} - \sum_{n=1}^{\infty} F(n, z, \theta)t^n, \tag{71} \]

for \(0 < t < 1\).
We observe that both the expressions on the left and right of (71) are analytic functions of \( z \) for \( \text{Re}(z) > -1 \) and \( z \) not an integer. In the following theorem, we extend the definition of \( I(z, \theta, t) \) to \( \text{Re}(z) > -1 \) and \( z \) not an integer.

**Theorem 14.** Suppose that \( z \in \mathbb{C}, \text{Re}(z) > -1, \ z \neq 0, 1, 2, \ldots \), and \( \theta \in \mathbb{C} \). Then

\[
I(z, \theta, t) = \int_0^1 \frac{s^z}{(s^2 + t^2 - 2st \cos(\pi\theta))^2} = a(z, \theta)t^{z-3} - \sum_{n=1}^{\infty} F(n, z, \theta)t^n, \tag{72}
\]

for \( 0 < t < 1 \).

**Proof.** The result follows from (71) and using analytic continuation. \( \blacksquare \)

We now present the principal result of this section in the following theorem.

**Theorem 15.** Suppose \( z \in \mathbb{C}, \text{Re}(z) > -1, \ z \neq 0, 1, 2, \ldots \), and \( \theta \in \mathbb{C} \). Recall that,

\[
k_{1,1}(s, t) = \frac{t \sin(\pi\theta)(s - t \cos(\pi\theta))(t - s \cos(\pi\theta))}{\pi(s^2 + t^2 - 2st \cos(\pi\theta))^2}, \tag{73}
\]

\[
k_{1,2}(s, t) = \frac{t^2 \sin^2(\pi\theta)(t - s \cos(\pi\theta))}{\pi(s^2 + t^2 - 2st \cos(\pi\theta))^2}, \tag{74}
\]

\[
k_{2,1}(s, t) = \frac{st \sin^2(\pi\theta)(s - t \cos(\pi\theta))}{\pi(s^2 + t^2 - 2st \cos(\pi\theta))^2}, \tag{75}
\]

\[
k_{2,2}(s, t) = \frac{st^2 \sin^3(\pi\theta)}{\pi(s^2 + t^2 - 2st \cos(\pi\theta))^2}, \tag{76}
\]

for \( 0 < s, t < 1 \). Then

\[
\int_0^1 k_{1,1}(s, t)s^zds = a_{1,1}(z, \theta) \cdot t^z + \sum_{n=1}^{\infty} F_{1,1}(n, z, \theta) \cdot t^n, \tag{77}
\]

\[
\int_0^1 k_{1,2}(s, t)s^zds = a_{1,2}(z, \theta) \cdot t^z + \sum_{n=1}^{\infty} F_{1,2}(n, z, \theta) \cdot t^n, \tag{78}
\]

\[
\int_0^1 k_{2,1}(s, t)s^zds = a_{2,1}(z, \theta) \cdot t^z + \sum_{n=1}^{\infty} F_{2,1}(n, z, \theta) \cdot t^n, \tag{79}
\]

\[
\int_0^1 k_{2,2}(s, t)s^zds = a_{2,2}(z, \theta) \cdot t^z + \sum_{n=1}^{\infty} F_{2,2}(n, z, \theta) \cdot t^n, \tag{80}
\]

for \( 0 < t < 1 \), where

\[
a_{1,1}(z, \theta) = \frac{1}{2 \sin(\pi z)} \left[ (z + 1) \sin(\pi \theta) \cos(\pi z(1 - \theta)) - \sin(\pi(z(1 - \theta) + \theta)) \right], \tag{81}
\]

\[
a_{1,2}(z, \theta) = -\frac{1}{2 \sin(\pi z)}(z - 1) \sin(\pi \theta) \sin(\pi z(1 - \theta)), \tag{82}
\]

\[
a_{2,1}(z, \theta) = \frac{1}{2 \sin(\pi z)}(z + 1) \sin(\pi \theta) \sin(\pi z(1 - \theta)), \tag{83}
\]

\[
a_{2,2}(z, \theta) = \frac{1}{2 \sin(\pi z)} \left[ (z + 1) \sin(\pi \theta) \cos(\pi z(1 - \theta)) + \sin(\pi(z(1 - \theta) - \theta)) \right], \tag{84}
\]
and

\[ F_{1,1}(n, z, \theta) = \frac{n \sin (\pi \theta) \cos (n \pi \theta) + \cos (\pi \theta) \sin (n \pi \theta)}{2\pi(n - z)}, \]
\[ F_{1,2}(n, z, \theta) = \frac{(n - 1) \sin (\pi \theta) \sin (n \pi \theta)}{2\pi(n - z)}, \]
\[ F_{2,1}(n, z, \theta) = -\frac{(n + 1) \sin (\pi \theta) \sin (n \pi \theta)}{2\pi(n - z)}, \]
\[ F_{2,2}(n, z, \theta) = \frac{n \sin (\pi \theta) \cos (n \pi \theta) - \cos (\pi \theta) \sin (n \pi \theta)}{2\pi(n - z)}. \]

**Proof.** The result follows from observing that

\[
\int_{0}^{1} k_{1,1}(s, t) s^2 \, ds = -\frac{\sin (2\pi \theta)}{2} \left( t \cdot I(z + 2, \theta, t) + t^3 \cdot I(z, \theta, t) \right) + t^2(1 + \cos^2 (\pi \theta)) \cdot I(z + 1, \theta, t),
\]
\[
\int_{0}^{1} k_{1,2}(s, t) s^2 \, ds = t^2 \sin^2 (\pi \theta) (t \cdot I(z, \theta, t) - \cos (\pi \theta) I(z + 1, \theta, t)),
\]
\[
\int_{0}^{1} k_{2,1}(s, t) s^2 \, ds = t \sin^2 (\pi \theta) (I(z + 2, \theta, t) - t \cos (\pi \theta) \cdot I(z + 1, \theta, t)),
\]
\[
\int_{0}^{1} k_{2,2}(s, t) s^2 \, ds = t^2 \sin^3 (\pi \theta) \cdot I(z + 1, \theta, t),
\]

and using the formula for \( I(z, \theta, t) \) derived in Theorem 14. \( \blacksquare \)

### 4. Analysis of the integral equation

Suppose that \( \gamma(t) : [-1, 1] \to \mathbb{R}^2 \) is a wedge with interior angle \( \pi \theta \) and side length 1 on either side of the corner, parametrized by arc length (see Figure 2). Suppose further that the odd and the even parts of a function \( f \) are denoted by \( f_o \) and \( f_e \) respectively (see (31)). In Section 2.4, we observed that the integral equation

\[
-\frac{1}{2} \begin{bmatrix} \mu_{\tau}(t) \\ \mu_{\nu}(t) \end{bmatrix} + \text{p.v.} \int_{-1}^{1} \mathbf{K}(\gamma(t), \gamma(s)) \begin{bmatrix} \mu_{\tau}(s) \\ \mu_{\nu}(s) \end{bmatrix} \, ds = \begin{bmatrix} h_{\tau}(t) \\ h_{\nu}(t) \end{bmatrix}, \quad -1 < t < 1,
\]

simplifies into two uncoupled integral equations on the interval \([0,1]\), given by

\[
-\frac{1}{2} \begin{bmatrix} \mu_{\tau,o}(t) \\ \mu_{\nu,e}(t) \end{bmatrix} + \int_{0}^{1} \begin{bmatrix} k_{1,1}(s, t) & k_{1,2}(s, t) \\ k_{2,1}(s, t) & k_{2,2}(s, t) \end{bmatrix} \begin{bmatrix} \mu_{\tau,o}(s) \\ \mu_{\nu,e}(s) \end{bmatrix} \, ds = \begin{bmatrix} h_{\tau,o}(t) \\ h_{\nu,e}(t) \end{bmatrix},
\]

for \( 0 < t < 1 \), and

\[
-\frac{1}{2} \begin{bmatrix} \mu_{\tau,e}(t) \\ \mu_{\nu,o}(t) \end{bmatrix} - \int_{0}^{1} \begin{bmatrix} k_{1,1}(s, t) & k_{1,2}(s, t) \\ k_{2,1}(s, t) & k_{2,2}(s, t) \end{bmatrix} \begin{bmatrix} \mu_{\tau,e}(s) \\ \mu_{\nu,o}(s) \end{bmatrix} \, ds = \begin{bmatrix} h_{\tau,e}(t) \\ h_{\nu,o}(t) \end{bmatrix},
\]

for \( 0 < t < 1 \).
for $0 < t < 1$, where $K$ is defined in (21) and the kernels $k_{j,\ell}(s, t)$, $j, \ell = 1, 2$, are defined in (34)–(37). As in Section 2.4, we refer to (94) as the tangential odd, normal even case and to (95) as the tangential even, normal odd case.

In Section 4.1, we investigate equation (94), i.e., the tangential odd, normal even case. We observe that, if $\mu_{\tau,\nu}(t)$ and $\mu_{\nu,\nu}(t)$ are of the form

$$
\mu_{\tau,\nu}(t) = p \cdot |t|^z \text{sgn}(t), \quad \text{and} \quad \mu_{\nu,\nu}(t) = q \cdot |t|^z,
$$

for $-1 < t < 1$, where $p, q, z \in \mathbb{C}$, then for certain values of $p, q,$ and $z$ depending on the angle $\pi \theta$, the corresponding components of the velocity $h_{\tau,\nu}(t)$ and $h_{\nu,\nu}(t)$, defined by (94), are smooth. We also prove the converse. Suppose that $N$ is a positive integer. Then for all but countably many $0 < \theta < 2$, there exist $p_{n,j}, q_{n,j}, z_{n,j} \in \mathbb{C}$, $n = 1, 2, \ldots N$ and $j = 1, 2$, such that the following holds. Suppose $\alpha_n, \beta_n \in \mathbb{C}$, $n = 0, 1, \ldots N$, and $h_{\tau,\nu}(t)$ and $h_{\nu,\nu}(t)$ are given by

$$
h_{\tau,\nu}(t) = \left( \sum_{n=0}^{N} \alpha_n |t|^n \right) \text{sgn}(t), \quad \text{and} \quad h_{\nu,\nu}(t) = \sum_{n=0}^{N} \beta_n |t|^n,
$$

for $-1 < t < 1$. Then, there exist unique numbers $c_n, d_n \in \mathbb{C}$, $n = 0, 1, \ldots N$, such that, if

$$
\mu_{\tau,\nu}(t) = \left( c_0 + \sum_{n=1}^{N} c_n p_{n,1} |t|^{z_{n,1}} + d_n p_{n,2} |t|^{z_{n,2}} \right) \text{sgn}(t), \quad (98)
$$

$$
\mu_{\nu,\nu}(t) = d_0 + \sum_{n=1}^{N} c_n q_{n,1} |t|^{z_{n,1}} + d_n q_{n,2} |t|^{z_{n,2}}, \quad (99)
$$

for $-1 < t < 1$, then $\mu_{\tau,\nu}(t), \mu_{\nu,\nu}(t)$ satisfy equation (94) with error $O(|t|^{N+1})$. We prove this result in Theorem 27.

Similarly, in Section 4.2, we investigate equation (95), i.e., the tangential even, normal odd case. We observe that, if $\mu_{\tau,\nu}(t)$ and $\mu_{\nu,\nu}(t)$ are of the form

$$
\mu_{\tau,\nu}(t) = p \cdot |t|^z, \quad \text{and} \quad \mu_{\nu,\nu}(t) = q \cdot |t|^z \text{sgn}(t),
$$

for $-1 < t < 1$, where $p, q, z \in \mathbb{C}$, then for certain values of $p, q,$ and $z$ depending on the angle $\pi \theta$, the corresponding components of the velocity $h_{\tau,\nu}(t)$ and $h_{\nu,\nu}(t)$, defined by (95), are smooth. We also prove the converse in the following sense. Suppose that $N$ is a positive integer. Then for all but countably many $0 < \theta < 2$, there exist $p_{n,j}, q_{n,j}, z_{n,j} \in \mathbb{C}$, $n = 1, 2, \ldots N$ and $j = 1, 2$, such that the following holds. Suppose $\alpha_n, \beta_n \in \mathbb{C}$, $n = 0, 1, \ldots N$, and $h_{\tau,\nu}(t)$ and $h_{\nu,\nu}(t)$ are given by

$$
h_{\tau,\nu}(t) = \sum_{n=0}^{N} \alpha_n |t|^n, \quad \text{and} \quad h_{\nu,\nu}(t) = \left( \sum_{n=0}^{N} \beta_n |t|^n \right) \text{sgn}(t) \quad (101)
$$

for $-1 < t < 1$. Then, there exist unique numbers $c_n, d_n \in \mathbb{C}$ for $n = 0, 1, \ldots N$, such that, if

$$
\mu_{\tau,\nu}(t) = c_0 + \sum_{n=1}^{N} c_n p_{n,1} |t|^{z_{n,1}} + d_n p_{n,2} |t|^{z_{n,2}} \quad (102)
$$

$$
\mu_{\nu,\nu}(t) = \left( d_0 + \sum_{n=1}^{N} c_n q_{n,1} |t|^{z_{n,1}} + d_n q_{n,2} |t|^{z_{n,2}} \right) \text{sgn}(t), \quad (103)
$$

17
for $-1 < t < 1$, then $\mu_{r,e}(t), \mu_{v,o}(t)$ satisfy equation (95) with error $O(|t|^{N+1})$. We prove this result in Theorem 34.

**Remark 16.** We note that the real and imaginary parts of the function $|t|^z$ are given by $|t|^a \cos(\beta \log |t|)$ and $|t|^a \sin(\beta \log |t|)$ respectively, where $z = \alpha + i\beta$. Analogous results where the density $\mu$ is expressed in terms of the functions $|t|^a \cos(\beta \log |t|)$ and $|t|^a \sin(\beta \log |t|)$, as opposed to $|t|^z$, can be derived for both the tangential odd, normal even case, and the tangential even, normal odd case.

### 4.1. Tangential odd, normal even case

In this section, we investigate the tangential odd, normal even case (see equation (94)). In Section 4.1.1, we investigate the values of $p, q$ and $z$ in (96) for which the resulting components of the velocity are smooth functions. In Section 4.1.2, we show that, for every $h$ of the form (97), there exists a density $\mu$ of the form (98), (99), which satisfies the integral equation (94) to order $N$.

#### 4.1.1. The values of $p, q, z$ in (96)

Suppose that $\mu_{v,o}(t)$ and $\mu_{v,e}(t)$ are given by

$$\mu_{v,o}(t) = p \cdot |t|^z \text{sgn}(t), \quad \text{and} \quad \mu_{v,e}(t) = q \cdot |t|^z,$$

for $-1 < t < 1$, where $p, q, z \in \mathbb{C}$. In this section, we determine the values of $p, q$ and $z$ such that $h_{v,o}(t)$ and $h_{v,e}(t)$ defined by

$$\begin{bmatrix} h_{v,o}(t) \\ h_{v,e}(t) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} \mu_{v,o}(t) \\ \mu_{v,e}(t) \end{bmatrix} + \int_{0}^{1} \begin{bmatrix} k_{1,1}(s,t) & k_{1,2}(s,t) \\ k_{2,1}(s,t) & k_{2,2}(s,t) \end{bmatrix} \begin{bmatrix} \mu_{v,o}(s) \\ \mu_{v,e}(s) \end{bmatrix} ds,$$

are smooth functions of $t$ for $0 < t < 1$, where the kernels $k_{j,\ell}(s,t), j, \ell = 1, 2,$ are defined in (34) – (37). The principal result of this section is Theorem 24.

The following lemma describes sufficient conditions for $p, q,$ and $z$ such that, if $\mu$ is defined by (104), then the velocity $h$ given by (105) is smooth.

**Theorem 17.** Suppose that $\theta \in (0, 2)$, $z$ is not an integer, and $z$ satisfies $\det A(z, \theta) = 0$, where

$$A(z, \theta) = -\frac{1}{2} \mathbf{I} + \begin{bmatrix} a_{1,1}(z, \theta) & a_{1,2}(z, \theta) \\ a_{2,1}(z, \theta) & a_{2,2}(z, \theta) \end{bmatrix},$$

$I$ is the $2 \times 2$ identity matrix, and $a_{j,\ell}(z, \theta), j, \ell = 1, 2,$ are given by (81) – (84). Furthermore, suppose that $(p, q) \in \mathcal{N}\{A(z, \theta)\}$, where $\mathcal{N}\{A\}$ denotes the null space of the matrix $A$.

Suppose finally that

$$\mu_{v,o}(t) = p \cdot t^z, \quad \text{and} \quad \mu_{v,e}(t) = q \cdot t^z,$$

for $0 < t < 1$. Then $h_{v,o}(t)$ and $h_{v,e}(t)$ defined by (105) satisfy

$$\begin{bmatrix} h_{v,o}(t) \\ h_{v,e}(t) \end{bmatrix} = \sum_{n=1}^{\infty} F(n, z, \theta) \begin{bmatrix} p \\ q \end{bmatrix} \cdot t^n,$$

for $0 < t < 1$, where

$$F(n, z, \theta) = \begin{bmatrix} F_{1,1}(n, z, \theta) & F_{1,2}(n, z, \theta) \\ F_{2,1}(n, z, \theta) & F_{2,2}(n, z, \theta) \end{bmatrix},$$

and $F_{j,\ell}(n, z, \theta), j, \ell = 1, 2,$ are given by (85) – (88).
Proof. Substituting $\mu_{\tau,o}(t) = p \cdot t^z$ and $\mu_{\nu,e}(t) = q \cdot t^z$ in (105) and using Theorem 15, we get

$$
\begin{bmatrix}
    h_{\tau,o}(t) \\
    h_{\nu,e}(t)
\end{bmatrix} =
\begin{bmatrix}
    -1/2p \cdot t^z + \int_0^1 (p \cdot k_{1,1}(s,t) + q \cdot k_{1,2}(s,t)) s^2 ds \\
    -1/2q \cdot t^z + \int_0^1 (p \cdot k_{2,1}(s,t) + q \cdot k_{2,2}(s,t)) s^2 ds
\end{bmatrix},
$$

(110)

$$
= A(z,\theta) \begin{bmatrix} p \\ q \end{bmatrix} t^z + \sum_{n=1}^{\infty} F(n,z,\theta) \begin{bmatrix} p \\ q \end{bmatrix} t^n,
$$

(111)

Since $(p,q) \in \mathcal{N}(A(z,\theta))$, we note that $A(z,\theta) \cdot (p,q) = (0,0)$ and thus

$$
\begin{bmatrix}
    h_{\tau,o}(t) \\
    h_{\nu,e}(t)
\end{bmatrix} = \sum_{n=1}^{\infty} \begin{bmatrix}
    F_{1,1}(n,z,\theta) & F_{1,2}(n,z,\theta) \\
    F_{2,1}(n,z,\theta) & F_{2,2}(n,z,\theta)
\end{bmatrix} \begin{bmatrix} p \\ q \end{bmatrix} t^n.
$$

(112)

A straightforward calculation shows that

$$
\det A(z,\theta) = \frac{(z \sin (\pi\theta) - \sin (\pi z \theta)) (z \sin (\pi\theta) - \sin (\pi z (2 - \theta)))}{4 \sin^2 (\pi z)}
$$

(113)

Thus, if $z$ is not an integer and either

$$
z \sin (\pi\theta) - \sin (\pi z (2 - \theta)) = 0,
$$

(114)

or

$$
z \sin (\pi\theta) - \sin (\pi z \theta) = 0,
$$

(115)

then $\det A(z,\theta) = 0$.

Remark 18. If $z$ satisfies the implicit relations (114), (115), we have that $\det A(z,\theta) = 0$. It is then straightforward to determine $p$ and $q$ via an explicit formula in terms of the entries of $A(z,\theta)$, since $(p,q) \in \mathcal{N}(A(z,\theta))$.

In the following theorem, we prove the existence of the implicit functions $z(\theta)$ defined by (114), (115) on the interval $\theta \in (0,2)$.

Theorem 19. Suppose that $N \geq 2$ is an integer. Then there exists $3N - 2$ real numbers $\theta_1, \theta_2, \ldots, \theta_{3N-2} \in (0,2)$ such that the following holds. Suppose that $D$ is the strip in the upper half plane with $0 < Re(\theta) < 2$ that includes the interval $(0,2) \setminus \{\theta_j\}_{j=1}^{3N-2}$, i.e.

$$
D = \{ \theta \in \mathbb{C} : Re(\theta) \in (0,2), \quad 0 \leq Im(\theta) < \infty \} \setminus \{\theta_j\}_{j=1}^{3N-2}.
$$

(116)

Then, there exists a simply connected open set $D \subset V \subset \mathbb{C}$ and analytic functions $z_{n,1}(\theta) : V \to \mathbb{C}$, $n = 1,2,\ldots,N$, which satisfy

$$
z \sin (\pi\theta) - \sin (\pi z (2 - \theta)) = 0, \quad z(1) = n,
$$

(117)
for $\theta \in V$, and analytic functions $z_{n,2}(\theta) : V \to \mathbb{C}$, $n = 2, 3, \ldots N$, which satisfy
\begin{equation}
 z \sin(\pi \theta) - \sin(\pi z \theta) = 0, \quad z(1) = n, 
\end{equation}
for $\theta \in V$ (see Figure 4 for an illustrative domain $V$). Moreover, the functions $z_{n,1}(\theta)$, $n = 1, 2 \ldots N$, do not take integer values for all $\theta \in V \setminus \{1\}$, and satisfy $\det A(z_{n,1}(\theta), \theta) = 0$, $n = 1, 2 \ldots N$, for all $\theta \in V$ (see (106), (113)). Similarly, the functions $z_{n,2}(\theta)$, $n = 2, 3, \ldots N$, do not take integer values for all $\theta \in V \setminus \{1\}$, and satisfy $\det A(z_{n,2}(\theta), \theta) = 0$, $n = 2, 3 \ldots N$, for all $\theta \in V$ (see (106), (113)).

**Proof.** The proof is technical and is contained in Section 8. \hfill \blacksquare

**Remark 20.** In fact, the real numbers $\theta_1, \theta_2 \ldots \theta_{3N-2}$ are the combined branch points of the functions $z_{n,1}(\theta)$, $n = 1, 2, \ldots N$, and $z_{n,2}(\theta)$, $n = 2, 3, \ldots N$. (see Section 8).

**Remark 21.** As shown in Section 8, the functions $z_{n,1}(\theta)$, $n = 2 \ldots N$, and $z_{n,2}(\theta)$, $n = 3, \ldots N$, have exactly three branch singularities each, and the functions $z_{1,1}(\theta)$ and $z_{2,2}(\theta)$ have exactly one branch singularity each. We plot $z_{2,1}(\theta)$, $z_{2,2}(\theta)$, $z_{3,1}(\theta)$, and $z_{3,2}(\theta)$ in Figure 5.

Suppose now that, as in Theorem 19, $z_{n,1}(\theta)$, $n = 1, 2, \ldots N$, are analytic functions which satisfy $\det A(z_{n,1}(\theta), \theta) = 0$, and $z_{n,2}(\theta)$, $n = 2, 3, \ldots N$, are analytic functions which satisfy $\det A(z_{n,2}(\theta), \theta) = 0$, $n = 2, 3, \ldots N$. We observed in Remark 18 that $p_{n,1}(\theta)$ and $q_{n,1}(\theta)$ are determined explicitly by $z_{n,1}(\theta)$ for $n = 1, 2 \ldots N$, and, similarly $p_{n,2}(\theta)$ and $q_{n,2}(\theta)$ are determined explicitly by $z_{n,2}(\theta)$ for $n = 2, 3 \ldots N$. We recall that, if
\begin{equation}
 \mu_{\tau,o}(t) = p_{n,j} \cdot |t|^{z_{n,j}}, \quad \mu_{\nu,e}(t) = q_{n,j} \cdot |t|^{z_{n,j}},
\end{equation}
then the corresponding components of the velocity $h_{\tau,o}(t)$ and $h_{\nu,e}(t)$ defined by (105) satisfy
\begin{equation}
 \begin{bmatrix} h_{\tau,o}(t) \\ h_{\nu,e}(t) \end{bmatrix} = \sum_{m=1}^{\infty} \mathbf{F}(m, z_{n,j}, \theta) \begin{bmatrix} p_{n,j} \\ q_{n,j} \end{bmatrix} \cdot t^{m},
\end{equation}
for $0 < t < 1$, $n = 1, 2, \ldots N$ when $j = 1$, and $n = 2, 3 \ldots N$ when $j = 2$, where $\mathbf{F}$ is defined by (109) (see Theorem 17).
Figure 5: Plots for the real and imaginary parts of the functions $z_{2,1}(\theta)$ (top left), $z_{2,2}(\theta)$ (top right), $z_{3,1}(\theta)$ (bottom left), and $z_{3,2}(\theta)$ (bottom right) for $0 < \theta < 2$. The solid lines represent the real part of $z$, the dashed line represents the imaginary part of $z$, and the vertical dotted lines indicate the locations of the branch points.

We note that the implicit functions $z_{n,2}(\theta)$, satisfying (118), are defined for $n \geq 2$, as opposed to the implicit functions $z_{n,1}(\theta)$, satisfying (117), which are defined for $n \geq 1$. We observe that the function $z_{1,2}(\theta)$, defined by $z_{1,2}(\theta) \equiv 1$, satisfies (118), since when $z = 1$,

$$z \sin (\pi \theta) - \sin (\pi z \theta) = \sin (\pi \theta) - \sin (\pi \theta) = 0,$$

(121)

for all $\theta$. In the following lemma, we compute the velocity field when $(\mu_{\tau,o}(t), \mu_{\nu,e}(t)) = (0, 1) \cdot t$.

**Lemma 22.** Suppose that $\theta \in \mathbb{C}$, $\mu_{\tau,o}(t) = 0$ and $\mu_{\nu,e}(t) = t$, for $0 < t < 1$. Then $h_{\tau,o}(t)$
and $h_{\nu,e}(t)$ defined by (105) satisfy

$$
\begin{bmatrix}
    h_{\tau,o}(t) \\
    h_{\nu,e}(t)
\end{bmatrix}
= F_1(\theta) t + \sum_{n=2}^{\infty} F(n,1,\theta) \begin{bmatrix}
    0 \\
    1
\end{bmatrix} \cdot t^n,
$$

(122)

for $0 < t < 1$, where $F$ is defined in (109) and

$$
F_1(\theta) = -\frac{1}{2\pi} \begin{bmatrix}
    -\sin^2(\pi\theta) \\
    -\sin(\pi\theta) \cos(\pi\theta)
\end{bmatrix}.
$$

(123)

**Proof.** Substituting $\mu_{\tau,o}(t) = 0$ and $\mu_{\nu,e}(t) = t^2$ in (105), where $z$ is not an integer, the corresponding components on the boundary $h_{\tau,o}(t)$ and $h_{\nu,e}(t)$ using Theorem 15 are given by

$$
\begin{bmatrix}
    h_{\tau,o}(t) \\
    h_{\nu,e}(t)
\end{bmatrix}
= \begin{bmatrix}
    a_{1,2}(z,\theta) \\
    -1/2 + a_{2,2}(z,\theta)
\end{bmatrix} t^2 + \sum_{n=2}^{\infty} \begin{bmatrix}
    F_{1,2}(n,z,\theta) \\
    F_{2,2}(n,z,\theta)
\end{bmatrix} t^n,
$$

(124)

for $0 < t < 1$, where $a_{1,2}(z,\theta), a_{2,2}(z,\theta)$ are defined in (82), (84) respectively, and $F_{1,2}(n,z,\theta), F_{2,2}(n,z,\theta)$ are defined in (86), (88) respectively. The result then follows from taking the limit as $z \to 1$.

It is clear from (120), (122) that there is no constant term in the Taylor series of the components of the velocity $h_{\tau,o}(t)$ and $h_{\nu,e}(t)$. The following lemma computes the velocity field when the components of the density $\mu_{\tau,o}(t)$ and $\mu_{\nu,e}(t)$ are constants.

**Lemma 23.** Suppose that $\theta \in \mathbb{C}$, $\mu_{\tau,o}(t) = p_0$ and $\mu_{\nu,e}(t) = q_0$, where $p_0,q_0$ are constants. Then $h_{\tau,o}(t)$ and $h_{\nu,e}(t)$ defined by (105) satisfy

$$
\begin{bmatrix}
    h_{\tau,o}(t) \\
    h_{\nu,e}(t)
\end{bmatrix}
= F_0(\theta) \begin{bmatrix}
    p_0 \\
    q_0
\end{bmatrix} + \sum_{n=1}^{\infty} F(n,0,\theta) \begin{bmatrix}
    p_0 \\
    q_0
\end{bmatrix} \cdot t^n,
$$

(125)

for $0 < t < 1$, where $F$ is defined in (109) and

$$
F_0(\theta) = -\frac{1}{2\pi} \begin{bmatrix}
    \pi - \sin(\pi\theta) + \pi(1-\theta) \cos(\pi\theta) & -\pi(1-\theta) \sin(\pi\theta) \\
    -\pi(1-\theta) \sin(\pi\theta) & \pi - \sin(\pi\theta) - \pi(1-\theta) \cos(\pi\theta)
\end{bmatrix}.
$$

(126)

**Proof.** The result follows from taking the limit $z \to 0$ in (111).

In the following theorem, we describe the matrix $B(\theta)$ that maps the coefficients of the basis functions $(p_{n,j}|t|\omega_j, q_{n,j}|t|\omega_j)$ to the Taylor expansion coefficients of the corresponding velocity field.

**Theorem 24.** Suppose $N \geq 2$ is an integer. Suppose further that, as in Theorem 19, $\theta_1,\theta_2,\ldots,\theta_{3N-2}$ are real numbers on the interval $(0,2)$, and that $z_{n,1}(\theta), n = 1,2,\ldots,N$, are analytic functions satisfying $\det A(z_{n,1}(\theta),\theta) = 0$ for $\theta \in V \subset \mathbb{C}$, where $V$ is a simply connected open set containing the strip $D$ with $\text{Re}(\theta) \in (0,2)$ and the interval $(0,2) \setminus \{\theta_j\}_{j=1}^{3N-2}$. Similarly, suppose that $z_{n,2}(\theta), n = 2,3,\ldots,N$, are analytic functions satisfying $\det A(z_{n,2}(\theta),\theta) = 0$ for $\theta \in V$. Let $(p_{n,1},q_{n,1}) \in \mathcal{N}(A(z_{n,1}(\theta),\theta)), n = 1,2,\ldots,N,
and \((p_{n,2}, q_{n,2}) \in \mathcal{N}\{A(z_{n,2}(\theta), \theta)\}, \ n = 2, 3, \ldots N\). Suppose that \(z_{1,2}(\theta) \equiv 1\), \(p_{1,2} = 0\), and \(q_{1,2} = 1\). Finally, suppose that

\[
\mu_{\tau,o}(t) = \left(c_0 + \sum_{n=1}^{N} c_n p_{n,1} t^{z_{n,1}} + d_n p_{n,2} t^{z_{n,2}}\right) \text{sgn}(t),
\]

\[
\mu_{\nu,e}(t) = \left(d_0 + \sum_{n=1}^{N} c_n q_{n,1} t^{z_{n,1}} + d_n q_{n,2} t^{z_{n,2}}\right),
\]

for \(-1 < t < 1\), where \(c_j, d_j \in \mathbb{C}\), \(j = 0, 1 \ldots N\). Then

\[
h_{\tau,o}(t) = \left(\sum_{n=0}^{N} \alpha_n t^n\right) \text{sgn}(t) + O(|t|^{N+1})
\]

\[
h_{\nu,e}(t) = \left(\sum_{n=0}^{N} \beta_n t^n\right) + O(|t|^{N+1}),
\]

for \(-1 < t < 1\), where

\[
\begin{bmatrix}
\alpha_0 \\
\beta_0 \\
\vdots \\
\alpha_N \\
\beta_N
\end{bmatrix} = B(\theta)
\begin{bmatrix}
c_0 \\
d_0 \\
\vdots \\
c_N \\
d_N
\end{bmatrix},
\]

\(B(\theta)\) is a \((2N + 2) \times (2N + 2)\) matrix, and \(\theta \in V\). The \(2 \times 2\) block of \(B(\theta)\) which maps \(c_n, d_n\) to \(\alpha_\ell, \beta_\ell\) is given by

\[
B_{\ell,n}(\theta) =
\begin{bmatrix}
F(\ell, z_{n,1}(\theta), \theta) & p_{n,1}(\theta) \\
q_{n,1}(\theta)
\end{bmatrix}
\begin{bmatrix}
F(\ell, z_{n,2}(\theta), \theta) & p_{n,2}(\theta) \\
q_{n,2}(\theta)
\end{bmatrix},
\]

for \(\ell, n = 1, 2, \ldots N\), where \(F\) is defined in (109), except for the case \(\ell = n = 1\). In the case \(\ell = n = 1\), the matrix \(B_{1,1}(\theta)\) is given by

\[
B_{1,1}(\theta) =
\begin{bmatrix}
F(1, z_{1,1}(\theta), \theta) & p_{1,1}(\theta) \\
q_{1,1}(\theta)
\end{bmatrix}
\begin{bmatrix}
F(1, z_{1,2}(\theta), \theta) & p_{1,2}(\theta) \\
q_{1,2}(\theta)
\end{bmatrix}
\]

where \(F\) is defined in (109), and \(F_1\) is defined in (123). Finally, if either \(\ell = 0\) or \(n = 0\), then the matrices \(B_{\ell,n}(\theta)\) are given by

\[
B_{\ell,0}(\theta) =
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix},
\]

\[
B_{0,0}(\theta) = F_0(\theta),
\]

\[
B_{0,n}(\theta) = F(n, 0, \theta),
\]

for \(\ell, n = 1, 2, \ldots N\), where \(F\) is defined in (109), and \(F_0\) is defined in (126).
using Theorem 69 in Section 9, we observe that \( B \) branch points of the functions \( \{z_n, \theta\} \). Suppose that \( \phi \in \text{Theorem 19} \). Since \((p_{n,1},q_{n,1}) \in N\{A(z_{n,1},\theta)\}, n = 1,2,\ldots,N,\) we observe that \( p_{n,1},q_{n,1}, \) and \( z_{n,1}, n = 1,2,\ldots,N \) are not integers for \( \theta \in V \setminus \{1\} \). Since \((p_{n,2},q_{n,2}) \in N\{A(z_{n,2},\theta)\}, n = 2,3,\ldots,N,\) we observe that \( p_{n,2},q_{n,2}, \) and \( z_{n,2}, n = 2,3,\ldots,N \) satisfy the conditions of Theorem 17 for \( \theta \in V \setminus \{1\} \) and the corresponding entries of the matrix \( B(\theta) \) can be derived from (108) in Theorem 17.

Furthermore, we observe that the density corresponding to \( p_{1,2},q_{1,2} \) and \( z_{1,2} \) satisfy the conditions for Lemma 22, and thus the corresponding entries of the matrix \( B(\theta) \) can be derived from (122) in Lemma 22. Finally, the entries of \( B(\theta) \) corresponding to \( \mu_{t,o}(t),\mu_{r,e}(t) = (c_0,d_0) \) can be obtained from Lemma 23, with which the result follows for \( \theta \in V \setminus \{1\} \).

The result for \( \theta = 1 \) follows by taking the limit \( \theta \to 1 \) in (131) and from the observation that the limit \( B(\theta) \) as \( \theta \to 1 \) exists (see Theorem 69 in Section 9).

\subsection*{4.1.2. Invertibility of \( B \) in (131)}

The matrix \( B(\theta) \) is a mapping from coefficients of the basis functions \((p_{n,j}t|z_{n,j},q_{n,j}t|z_{n,j})\) to the Taylor expansion coefficients of the corresponding velocity field (see (131)). In this section, we observe that \( B(\theta) \) is invertible for all \( \theta \in (0,2) \) except for countably many values of \( \theta \). We then use this result to derive a converse of Theorem 24. The principal result of this section is Theorem 27.

In the following lemma, we show that \( B(\theta) \) is invertible for all \( \theta \in (0,2) \) except for countably many values of \( \theta \).

**Lemma 25.** Suppose that \( N \geq 2 \) is an integer. There exists a countable set \( \{\phi_m\}_{m=1}^{\infty} \subset (0,2) \), such that \( B(\theta) \) is an invertible matrix for \( \theta \in (0,2) \setminus \{\phi_m\}_{m=1}^{\infty} \). Moreover, the limit points of \( \{\phi_m\}_{m=1}^{\infty} \) are a subset of \( \{\theta_j\}_{j=1}^{3N-2} \cup \{0,2\} \), where \( \theta_j, j = 1,2,\ldots,3N-2, \) are the branch points of the functions \( z_{n,1}(\theta), n = 1,2,\ldots,N, \) and \( z_{n,2}(\theta), n = 2,3,\ldots,N, \) defined in Theorem 19.

**Proof.** Suppose that \( V \) is as defined in Theorem 19. Recall that the interval \( (0,2) \setminus \{\theta_j\}_{j=1}^{3N-2} \subset V \). Clearly, the entries of \( B(\theta) \) are analytic functions of \( \theta \in V \setminus \{1\} \). Furthermore, using Theorem 69 in Section 9, we observe that \( B(\theta) \) is analytic at \( \theta = 1 \) as well, since

\[
\lim_{\theta \to 1} B_{\ell,j}(\theta) = \begin{cases} 
-1/2 & 0 \\
0 & -1/2 \\
-1/2 & 0 \\
0 & -1/2 \\
\ell = j = 0 \\
\ell = j = 2m \neq 0 \\
\ell = j = 2m + 1 \\
\ell \neq j 
\end{cases}
\]

Thus, \( \det B(\theta) \) is an analytic function for \( \theta \in V \). Moreover, using (137), \( \det(B(1)) \neq 0 \). Using standard results in complex analysis, since \( \det B(\theta) \) is not identically zero, we conclude

\[
\begin{cases} 
-1/2 & 0 \\
0 & -1/2 \\
-1/2 & 0 \\
0 & -1/2 \\
\ell = j = 0 \\
\ell = j = 2m \neq 0 \\
\ell = j = 2m + 1 \\
\ell \neq j 
\end{cases}
\]
that the matrix $B(\theta)$ is invertible for all $\theta \in (0, 2)$ except for a countable set of values of $\theta = \phi_m$, $m = 1, 2, \ldots$. Moreover, the set of limit points of $\det B(\theta) = 0$, i.e. the values of $\theta$ for which $B(\theta)$ is not invertible, is a subset of $\partial V \cap (0, 2)$, where $\partial V$ is the boundary of the set $V$. Clearly, $\partial V \cap (0, 2) = \{\theta_j\}_{j=1}^{3N-2} \cup \{0, 2\}$.

\textbf{Remark 26.} In Lemma 25, we show that $\det \{B(\theta)\}$ has countably many zeros on the interval $(0, 2)$. In fact, it is possible to show that there are finitely many zeros of $\det \{B(\theta)\}$ on the interval $(0, 2)$. On inspecting the form of the entries of $B(\theta)$, we note that $\det \{B(\theta)\}$ is a linear combination of $T(\theta)/P(\theta)$ where $T(\theta)$ is a trigonometric polynomial of degree less than or equal to $N$ and $P(\theta)$ is a finite product of functions $(z_j,\ell(\theta) - k_j,\ell)$ for some integer $k_j,\ell$. A detailed analysis shows that the functions $z_j,\ell(\theta)$ are essentially non-oscillatory for $\theta \in (0, 2)$. Since both the functions $T(\theta)$ and $P(\theta)$ are band-limited functions, $\det \{B(\theta)\}$ cannot have infinitely many zeros for $\theta \in (0, 2)$.

The following theorem is the principal result of this section.

\textbf{Theorem 27.} Suppose that $N \geq 2$ is an integer. Then for each $\theta \in (0, 2)$ except for countably many values, there exist $p_{n,j}, q_{n,j}, z_{n,j} \in \mathbb{C}$, $n = 1, 2, \ldots, N$ and $j = 1, 2$, such that the following holds. Suppose $\alpha_n, \beta_n \in \mathbb{C}$, $n = 0, 1, \ldots, N$, and $h_{\tau, o}(t)$ and $h_{\nu, e}(t)$ are given by

$$h_{\tau, o}(t) = \left(\sum_{n=0}^{N} \alpha_n |t|^n\right) \text{sgn}(t), \quad \text{and} \quad h_{\nu, e}(t) = \left(\sum_{n=0}^{N} \beta_n |t|^n\right), \quad (138)$$

for $-1 < t < 1$. Then there exist unique numbers $c_n, d_n \in \mathbb{C}$, $n = 0, 1, \ldots, N$, such that, if $\mu_{\tau, o}(t)$ and $\mu_{\nu, e}(t)$ defined by

$$\mu_{\tau, o}(t) = \left(c_0 + \sum_{n=1}^{N} c_n p_{n,1} |t|^{z_{n,1}} + d_n p_{n,2} |t|^{z_{n,2}}\right) \text{sgn}(t), \quad (139)$$

$$\mu_{\nu, e}(t) = d_0 + \sum_{n=1}^{N} c_n q_{n,1} |t|^{z_{n,1}} + d_n q_{n,2} |t|^{z_{n,2}},$$

for $-1 < t < 1$, then $\mu_{\tau, o}(t)$ and $\mu_{\nu, e}(t)$ satisfy

$$\begin{bmatrix} h_{\tau, o}(t) \\ h_{\nu, e}(t) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} \mu_{\tau, o}(t) \\ \mu_{\nu, e}(t) \end{bmatrix} + \int_{0}^{1} \begin{bmatrix} k_{1,1}(s, t) & k_{1,2}(s, t) \\ k_{2,1}(s, t) & k_{2,2}(s, t) \end{bmatrix} \begin{bmatrix} \mu_{\tau, o}(s) \\ \mu_{\nu, e}(s) \end{bmatrix} ds, \quad (140)$$

for $-1 < t < 1$ with error $O(|t|^{N+1})$, where $k_{j,\ell}(s, t)$, $j, \ell = 1, 2$, are defined in (34) – (37).

\textbf{Proof.} Suppose $z_{n,1}(\theta), n = 1, 2, \ldots, N$, and $z_{n,2}(\theta), n = 2, 3 \ldots, N$, are the implicit functions that satisfy $\det A(z, \theta) = 0$ as defined in Theorem 19. Let $(p_{n,1}, q_{n,1}) \in \mathcal{N}\{A(z_{n,1}(\theta), \theta)\}$, $n = 1, 2 \ldots, N$, and $(p_{n,2}, q_{n,2}) \in \mathcal{N}\{A(z_{n,2}(\theta), \theta)\}$. Let $z_{1,2} = 1$, $p_{1,2} = 0$, and $q_{1,2} = 1$. Given $p_{n,j}, q_{n,j}$, and $z_{n,j}, n = 1, 2 \ldots, N$ and $j = 1, 2$, let $B(\theta)$ be the $(2N + 2) \times (2N + 2)$ matrix defined in Theorem 24. Suppose further that $\{\phi_m\}_{m=1}^{\infty} \subset (0, 2)$ are the values of
θ for which \( B(\theta) \) is not invertible (see Lemma 25). Finally, since \( B(\theta) \) is invertible for all \( \theta \in (0, 2) \setminus \{ \phi_m \}_{m=1}^{\infty} \), let
\[
\begin{bmatrix}
c_0 \\
d_0 \\
\vdots \\
c_N \\
d_N
\end{bmatrix} = B^{-1}(\theta) \begin{bmatrix}
c_0 \\
d_0 \\
\vdots \\
c_N \\
d_N
\end{bmatrix}.
\]
(141)
The result then follows from using Theorem 24.

Remark 28. In Remark 16, we noted that the components of the density \( \mu \) can be expressed in terms of functions of the form \( |t|^{\alpha} \cos (\beta \log |t|) \) and \( |t|^{\alpha} \sin (\beta \log |t|) \), as opposed to \( |t|^z \), where \( z = \alpha + i\beta \). The precise statement is as follows. We observe that, when \( \theta \) is real, if \( \det A(z, \theta) = 0 \), then \( \det \bar{A}(\bar{z}, \theta) = 0 \). Moreover, if \( (p, q) \in \mathcal{N}\{A(z, \theta)\} \), then \( (\bar{p}, \bar{q}) \in \mathcal{N}\{A(\bar{z}, \theta)\} \). Thus, the numbers \( p, q, \) and \( z \) occur in complex conjugates. Results analogous to Theorem 24 and Theorem 27 can be derived for the case when the components of \( \mu \) are given by
\[
\begin{align*}
\mu_{r,o}(t) &= \left( c_0 + \sum_{n=1}^{N} c_n (r_n |t|^{\alpha_n} \cos (\beta_n \log |t|) - s_n |t|^{\alpha_n} \sin (\beta_n \log |t|)) \\
&+ \sum_{n=1}^{N} d_n (s_n |t|^{\alpha_n} \cos (\beta_n \log |t|) + r_n |t|^{\alpha_n} \sin (\beta_n \log |t|)) \right) \text{sgn}(t) \\
\mu_{\nu,e}(t) &= \left( d_0 + \sum_{n=1}^{N} c_n (v_n |t|^{\alpha_n} \cos (\beta_n \log |t|) - w_n |t|^{\alpha_n} \sin (\beta_n \log |t|)) \\
&+ \sum_{n=1}^{N} d_n (w_n |t|^{\alpha_n} \cos (\beta_n \log |t|) + v_n |t|^{\alpha_n} \sin (\beta_n \log |t|)) \right),
\end{align*}
\]
for \(-1 < t < 1\), where \( z_n = \alpha_n + i\beta_n \), \( p_n = r_n + is_n \), and \( q_n = v_n + iw_n \). The advantage for using the representation (142), (143) for the density \( \mu \) is the following. If the components of the velocity \( h_{r,o}(t) \) and \( h_{\nu,e}(t) \) are real, then the solution \( c_n, d_n \) when \( \mu \) defined by (142), (143) which satisfies (105) to order \( N \) accuracy, is also real.

4.2. Tangential even, normal odd case

In this section, we investigate tangential even, normal odd case (see equation (95)). In Section 4.2.1, we investigate the values of \( p, q \) and \( z \) in (96) for which the resulting components of the velocity are smooth functions. In Section 4.2.2, we show that, for every \( h \) of the form (101), there exists a density \( \mu \) of the form (102), (103), which satisfies the integral equation (95) to order \( N \). The proofs of the results in this section are essentially identical to the corresponding proofs in Section 4.1. For brevity, we present the statements of the theorems without proof.
4.2.1. The values of $p$, $q$, and $z$ in (100)

Suppose that $\mu_{\tau,e}(t)$ and $\mu_{\nu,o}(t)$ are given by

$$
\begin{align*}
\mu_{\tau,e}(t) &= p \cdot |t|^z, \\
\mu_{\nu,o}(t) &= q \cdot |t|^z \text{sgn}(t),
\end{align*}
$$

for $-1 < t < 1$, where $p, q, z \in \mathbb{C}$. In this section, we determine the values of $p, q$, and $z$ such that $h_{\tau,e}(t)$ and $h_{\nu,o}(t)$ defined by

$$
\begin{bmatrix}
h_{\tau,e}(t) \\
h_{\nu,o}(t)
\end{bmatrix} = -\frac{1}{2} \begin{bmatrix}
\mu_{\tau,e}(t) \\
\mu_{\nu,o}(t)
\end{bmatrix} - \int_0^t \begin{bmatrix}
k_{1,1}(s,t) & k_{1,2}(s,t) \\
k_{2,1}(s,t) & k_{2,2}(s,t)
\end{bmatrix} \begin{bmatrix}
\mu_{\tau,e}(s) \\
\mu_{\nu,o}(s)
\end{bmatrix} ds,
$$

are smooth functions of $t$ for $0 < t < 1$. The principal result of this section is Theorem 33.

The following lemma describes sufficient conditions for $p, q$, and $z$ such that, if $\mu$ is defined by (144), then the velocity $h$ given by (145) is smooth.

**Theorem 29.** Suppose that $\theta \in (0, 2)$, $z$ is not an integer, and $z$ satisfies $\det A(z, \theta) = 0$ where

$$
A(z, \theta) = -\frac{1}{2} I - \begin{bmatrix}
a_{1,1}(z, \theta) & a_{1,2}(z, \theta) \\
a_{2,1}(z, \theta) & a_{2,2}(z, \theta)
\end{bmatrix},
$$

$I$ is the $2 \times 2$ identity matrix and $a_{j,\ell}(z, \theta)$, $j, \ell = 1, 2$, are given by (81) – (84). Furthermore, suppose that $(p, q) \in \mathcal{N}\{A(z, \theta)\}$, where $\mathcal{N}\{A\}$ denotes the null space of the matrix $A$. Suppose finally that

$$
\mu_{\tau,e}(t) = p \cdot t^z, \quad \text{and} \quad \mu_{\nu,o}(t) = q \cdot t^z,
$$

for $0 < t < 1$. Then $h_{\tau,e}(t)$ and $h_{\nu,o}(t)$ defined by (145) satisfy

$$
\begin{bmatrix}
h_{\tau,o}(t) \\
h_{\nu,e}(t)
\end{bmatrix} = -\sum_{n=1}^{\infty} F(n, z, \theta) \begin{bmatrix}
p \\
q
\end{bmatrix} \cdot t^n,
$$

for $0 < t < 1$, where

$$
F(n, z, \theta) = \begin{bmatrix}
F_{1,1}(n, z, \theta) & F_{1,2}(n, z, \theta) \\
F_{2,1}(n, z, \theta) & F_{2,2}(n, z, \theta)
\end{bmatrix},
$$

and $F_{j,\ell}(n, z, \theta)$, $j, \ell = 1, 2$, are given by (85) – (88).

A straightforward calculation shows that

$$
\det A(z, \theta) = \frac{(z \sin(\pi \theta) + \sin(z \pi \theta))(z \sin(\pi \theta) + \sin(z(2 - \theta)))}{4 \sin^2(\pi z)}
$$

Thus, if $z$ is not an integer and either

$$
(z \sin(\pi \theta) + \sin(z(2 - \theta))) = 0,
$$

or

$$
(z \sin(\pi \theta) + \sin(z \pi \theta)) = 0,
$$

then $\det A(z, \theta) = 0$.

In the following theorem, we prove the existence of the implicit functions $z(\theta)$ defined by (151), (152) on the interval $(0, 2)$. 

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Theorem 30. Suppose that \( N \geq 2 \) is an integer. Then there exists \( 3N - 2 \) real numbers \( \theta_1, \theta_2, \ldots, \theta_{3N-2} \in (0, 2) \) such that the following holds. Suppose that \( D \) is the strip in the upper half plane with \( 0 < \text{Re}(\theta) < 2 \) that includes the interval \( (0, 2) \setminus \{ \theta_j \}_{j=1}^{3N-2} \), i.e.

\[
D = \{ \theta \in \mathbb{C} : \text{Re}(\theta) \in (0, 2), \quad 0 \leq \text{Im}(\theta) < \infty \} \setminus \{ \theta_j \}_{j=1}^{3N-2}.
\] (153)

Then, there exists a simply connected open set \( D \subset V \subset \mathbb{C} \) and analytic functions \( z_{n,1}(\theta) : V \to \mathbb{C}, n = 2, 3 \ldots N \), which satisfy

\[
z \sin (\pi \theta) + \sin (\pi z(2 - \theta)) = 0, \quad z(1) = n,
\] (154)

for \( \theta \in V \), and analytic functions \( z_{n,2}(\theta) : V \to \mathbb{C}, n = 1, 2, \ldots N \), which satisfy

\[
z \sin (\pi \theta) + \sin (\pi z \theta) = 0, \quad z(1) = n,
\] (155)

for \( \theta \in V \) (see Figure 4 for an illustrative domain \( V \)). Moreover, the functions \( z_{n,1}(\theta) \), \( n = 2, 3 \ldots N \), do not take integer values for all \( \theta \in V \setminus \{ 1 \} \) and satisfy \( \det A(z_{n,1}(\theta), \theta) = 0 \), \( n = 2, 3 \ldots N \), for all \( \theta \in V \) (see (146), (150)). Similarly, the functions \( z_{n,2}(\theta) \), \( n = 1, 2, \ldots N \), do not take integer values for all \( \theta \in V \setminus \{ 1 \} \) and satisfy \( \det A(z_{n,2}(\theta), \theta) = 0 \), \( n = 1, 2, \ldots N \), for all \( \theta \in V \) (see (146), (150)).

We note that the implicit functions \( z_{n,1}(\theta) \), satisfying (154), are defined for \( n \geq 2 \), as opposed to, the implicit functions \( z_{n,2}(\theta) \), satisfying (155), which are defined for \( n \geq 1 \). We observe that the function \( z_{1,1}(\theta) \) defined by \( z_{1,1}(\theta) \equiv 1 \), satisfies (154), since when \( z = 1 \),

\[
z \sin (\pi \theta) + \sin (\pi z(2 - \theta)) = \sin (\pi \theta) - \sin (\pi \theta) = 0,
\] (156)

for all \( \theta \). In the following lemma, we compute the velocity field when \( (\mu_{\tau,e}(t), \mu_{\nu,o}(t)) = (0, 1)t \).

Lemma 31. Suppose that \( \theta \in \mathbb{C} \), \( \mu_{\tau,e}(t) = 0 \) and \( \mu_{\nu,o}(t) = t \), for \( 0 < t < 1 \). Then \( h_{\tau,e}(t) \) and \( h_{\nu,o}(t) \) defined by (145) satisfy

\[
\begin{bmatrix} h_{\tau,e}(t) \\ h_{\nu,o}(t) \end{bmatrix} = F_1(\theta) t - \sum_{n=2}^{\infty} F(n, 1, \theta) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \cdot t^n,
\] (157)

for \( 0 < t < 1 \), where \( F \) is defined in (109) and

\[
F_1(\theta) = -\frac{1}{2\pi} \left[ \sin^2(\pi \theta) \right] - \frac{1}{\pi (2 - \theta) + \sin(\pi \theta) \cos(\pi \theta)}.
\] (158)

It is clear from (148), (157) that there is no constant term in the Taylor series of the components of the velocity \( h_{\tau,e} \) and \( h_{\nu,o} \). The following lemma computes the velocity field when the components of the density \( \mu_{\tau,e} \) and \( \mu_{\nu,o} \) are constants.

Lemma 32. Suppose that \( \theta \in \mathbb{C} \), \( \mu_{\tau,e}(t) = p_0 \) and \( \mu_{\nu,o}(t) = q_0 \), where \( p_0, q_0 \) are constants. Then \( h_{\tau,e}(t) \) and \( h_{\nu,o}(t) \) defined by (145) satisfy

\[
\begin{bmatrix} h_{\tau,e}(t) \\ h_{\nu,o}(t) \end{bmatrix} = F_0(\theta) \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} - \sum_{n=1}^{\infty} F(n, 0, \theta) \begin{bmatrix} p_0 \\ q_0 \end{bmatrix} \cdot t^n,
\] (159)

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for $0 < t < 1$, where $F$ is defined in (149) and

$$
F_0(\theta) = -\frac{1}{2\pi} \begin{bmatrix}
\pi + \sin(\pi\theta) - \pi(1 - \theta)\cos(\pi\theta) & \pi(1 - \theta)\sin(\pi\theta) \\
\pi(1 - \theta)\sin(\pi\theta) & \pi + \sin(\pi\theta) + \pi(1 - \theta)\cos(\pi\theta)
\end{bmatrix}.
$$

(160)

In the following theorem, we describe the matrix $B(\theta)$ that maps the coefficients of the basis functions $(p_{n,j}|t|^{z_{n,j}}, q_{n,j}|t|^{z_{n,j}})$ to the Taylor expansion coefficients of the corresponding velocity field.

**Theorem 33.** Suppose $N \geq 2$ is an integer. Suppose further that, as in Theorem 30, $\theta_1, \theta_2, \ldots, \theta_{3N-2}$ are real numbers on the interval $(0, 2)$ $z_{n,1}(\theta)$, $n = 2, 3, \ldots, N$, are analytic functions satisfying $\det A(z_{n,1}(\theta), \theta) = 0$ for $\theta \in V \subset \mathbb{C}$, where $V$ is a simply connected open set containing the strip $D$ with $\text{Re}(\theta) \in (0, 2)$ and the interval $(0, 2) \backslash \{\theta_j\}_{j=1}^{3N-2}$. Similarly, suppose that $z_{n,2}(\theta)$, $n = 1, 2, \ldots, N$, are analytic functions satisfying $\det A(z_{n,2}(\theta), \theta) = 0$ for $\theta \in V$. Let $(p_{n,1}, q_{n,1}) \in N\{A(z_{n,1}(\theta), \theta)\}$, $n = 2, 3, \ldots, N$, and $(p_{n,2}, q_{n,2}) \in N\{A(z_{n,2}(\theta), \theta)\}$, $n = 1, 2, \ldots, N$. Suppose that $z_{1,1}(\theta) \equiv 1$, $p_{1,1} = 0$, and $q_{1,1} = 1$. Finally, suppose that

$$
\mu_{\tau,o}(t) = \left( c_0 + \sum_{n=1}^{N} c_np_{n,1}|t|^{z_{n,1}} + d_np_{n,2}|t|^{z_{n,2}} \right),
$$

(161)

$$
\mu_{\nu,o}(t) = \left( d_0 + \sum_{n=1}^{N} c_nq_{n,1}|t|^{z_{n,1}} + d_nq_{n,2}|t|^{z_{n,2}} \right) \text{sgn}(t),
$$

(162)

for $-1 < t < 1$, where $c_j, d_j \in \mathbb{C}$, $j = 0, 1, \ldots, N$. Then

$$
h_{\tau,o}(t) = \left( \sum_{n=0}^{N} \alpha_n|t|^n \right) + O(|t|^{N+1})
$$

(164)

$$
h_{\nu,o}(t) = \left( \sum_{n=0}^{N} \beta_n|t|^n \right) \text{sgn}(t) + O(|t|^{N+1}),
$$

(165)

for $-1 < t < 1$, where

$$
\begin{bmatrix}
\alpha_0 \\
\beta_0 \\
\vdots \\
\alpha_N \\
\beta_N
\end{bmatrix} = B(\theta)\begin{bmatrix}
c_0 \\
d_0 \\
\vdots \\
c_N \\
d_N
\end{bmatrix},
$$

(166)

$B(\theta)$ is a $(2N + 2) \times (2N + 2)$ matrix, and $\theta \in V$. The $2 \times 2$ block of $B(\theta)$ which maps $c_n, d_n$ to $\alpha_\ell, \beta_\ell$ is given by

$$
B_{\ell,n}(\theta) = -\begin{bmatrix}
F(\ell, z_{n,1}(\theta), \theta) \left[ p_{n,1}(\theta) \right] \\
F(\ell, z_{n,1}(\theta), \theta) \left[ q_{n,1}(\theta) \right]
\end{bmatrix} \begin{bmatrix}
c_{n,1}(\theta) \\
q_{n,1}(\theta)
\end{bmatrix} - \begin{bmatrix}
F(\ell, z_{n,2}(\theta), \theta) \left[ p_{n,2}(\theta) \right] \\
F(\ell, z_{n,2}(\theta), \theta) \left[ q_{n,2}(\theta) \right]
\end{bmatrix},
$$

(167)
for \( \ell, n = 1, 2, \ldots N \), where \( \mathbf{F} \) is defined in (109), except for the case \( \ell = n = 1 \). In the case \( \ell = n = 1 \), the matrix \( \mathbf{B}_{1,1}(\theta) \) is given by

\[
\mathbf{B}_{1,1}(\theta) = \begin{bmatrix} \mathbf{F}_1(\theta) & -\mathbf{F}(1, z_{1,2}(\theta), \theta) [p_{1,2}(\theta) / q_{1,2}(\theta)] \end{bmatrix},
\]

with error \( O(|t|^{N+1}) \), the matrix \( \mathbf{B}(\theta) \) is defined in (109), and \( \mathbf{F}_1 \) is defined in (158). Finally, if either \( \ell = 0 \) or \( n = 0 \), then the matrices \( \mathbf{B}_{\ell,n}(\theta) \) are given by

\[
\mathbf{B}_{\ell,0}(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
\mathbf{B}_{0,0}(\theta) = \mathbf{F}_0(\theta),
\]

\[
\mathbf{B}_{0,n}(\theta) = -\mathbf{F}(n, 0, \theta),
\]

for \( \ell, n = 1, 2, \ldots N \), where \( \mathbf{F} \) is defined in (109), and \( \mathbf{F}_0 \) is defined in (160).

### 4.2.2. Invertibility of \( \mathbf{B} \) in (166)

The matrix \( \mathbf{B}(\theta) \) is a mapping from coefficients of the basis functions \((p_{n,j}|t|^{z_{n,j}}, q_{n,j}|t|^{z_{n,j}})\), to the corresponding Taylor expansion coefficients of the velocity field on the boundary \( \mathbf{h} \). In this section, we observe that \( \mathbf{B}(\theta) \) is invertible for all \( \theta \in (0, 2) \) except for countably many values of \( \theta \). We then use this result to derive a converse of Theorem 33. The following theorem is the principal result of this section.

**Theorem 34.** Suppose that \( N \geq 2 \) is an integer. Then for each \( \theta \in (0, 2) \) except for countably many values, there exist \( p_{n,j}, q_{n,j}, z_{n,j} \in \mathbb{C}, n = 1, 2, \ldots N \) and \( j = 1, 2 \), such that the following holds. Suppose \( \alpha_n, \beta_n \in \mathbb{C}, n = 0, 1, \ldots N \), and \( h_{\tau,e}(t) \) and \( h_{\nu,o}(t) \) are given by

\[
h_{\tau,e}(t) = \left( \sum_{n=0}^{N} \alpha_n |t|^n \right), \quad \text{and} \quad h_{\nu,o}(t) = \left( \sum_{n=0}^{N} \beta_n |t|^n \right) \text{sgn}(t),
\]

for \(-1 < t < 1\). Then there exist unique numbers \( c_n, d_n \in \mathbb{C}, n = 0, 1, \ldots N \), such that, if \( \mu_{\tau,e}(t) \) and \( \mu_{\nu,o}(t) \) defined by

\[
\mu_{\tau,e}(t) = \left( c_0 + \sum_{n=1}^{N} c_n p_{n,1} |t|^{z_{n,1}} + d_n p_{n,2} |t|^{z_{n,2}} \right),
\]

\[
\mu_{\nu,o}(t) = \left( d_0 + \sum_{n=1}^{N} c_n q_{n,1} |t|^{z_{n,1}} + d_n q_{n,2} |t|^{z_{n,2}} \right) \text{sgn}(t),
\]

for \(-1 < t < 1\), then \( \mu_{\tau,e}(t) \) and \( \mu_{\nu,o}(t) \) satisfy

\[
\begin{bmatrix} h_{\tau,e}(t) \\ h_{\nu,o}(t) \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} \mu_{\tau,e}(t) \\ \mu_{\nu,o}(t) \end{bmatrix} - \int_0^1 \begin{bmatrix} k_{1,1}(s, t) & k_{1,2}(s, t) \\ k_{2,1}(s, t) & k_{2,2}(s, t) \end{bmatrix} \begin{bmatrix} \mu_{\tau,e}(s) \\ \mu_{\nu,o}(s) \end{bmatrix} ds,
\]

for \(-1 < t < 1\) with error \( O(|t|^{N+1}) \), where \( k_{j,\ell}, j, \ell = 1, 2, \) are defined in (34) – (37).
5. Numerical Results

To solve the integral equations (94), (95) on polygonal domains, we use the representations (139) and (173) to construct purpose made discretizations using standard techniques (see, for example [20, 21, 22]). A detailed description of this part of the procedure will be published at a later date.

We illustrate the performance of the algorithm with several numerical examples. The interior velocity boundary problem was solved on each of the domains in Figures 6 to 10, where the boundary data is generated by five Stokeslets located outside the respective domains. We then compute the error \( E \) given by
\[
E = \sqrt{\frac{\sum_{m=1}^{5} |u_{\text{comp}}(t_m) - u_{\text{exact}}(t_m)|^2}{\sum_{m=1}^{5} |u_{\text{exact}}(t_m)|^2}},
\]
where \( t_m \) are targets in the interior of the domain, \( u_{\text{comp}}(t) \) is velocity computed numerically using the algorithm, and \( u_{\text{exact}}(t) \) is the exact velocity at the target \( t \). We plot the spectrum for the discretized linear systems corresponding to the associated integral equations in Figures 6 to 10. In Table 1, we report the number of discretization nodes \( n \), the condition number of the discrete linear system \( \kappa \), and the error \( E \), for each domain.

<table>
<thead>
<tr>
<th>( \Gamma )</th>
<th>( n )</th>
<th>( E )</th>
<th>( \kappa )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Gamma_1 )</td>
<td>220</td>
<td>( 3.0 \times 10^{-15} )</td>
<td>( 4.5 \times 10^2 )</td>
</tr>
<tr>
<td>( \Gamma_2 )</td>
<td>285</td>
<td>( 2.1 \times 10^{-14} )</td>
<td>( 2.9 \times 10^5 )</td>
</tr>
<tr>
<td>( \Gamma_3 )</td>
<td>489</td>
<td>( 4.8 \times 10^{-14} )</td>
<td>( 8.7 \times 10^5 )</td>
</tr>
<tr>
<td>( \Gamma_4 )</td>
<td>968</td>
<td>( 1.1 \times 10^{-12} )</td>
<td>( 7.9 \times 10^5 )</td>
</tr>
<tr>
<td>( \Gamma_5 )</td>
<td>1343</td>
<td>( 6.7 \times 10^{-13} )</td>
<td>( 6.1 \times 10^6 )</td>
</tr>
</tbody>
</table>

Table 1: Condition number and error for polygonal domains \( \Gamma_1 - \Gamma_5 \)

Figure 6: A cone \( \Gamma_1 \) (left) and the spectrum of the discretized linear system for \( \Gamma_1 \) (right)

Remark 35. We note that the condition numbers reported for the boundaries \( \Gamma_j, \ j = 2, 3, 4, 5 \), are larger than the condition numbers of the underlying integral equations. This issue can be remedied by using a slightly more involved scaling of the discretization scheme (see, for example, [23]).
6. Conclusions and extensions

In this paper, we construct an explicit basis for the solution of a standard integral equation corresponding to the biharmonic equation with gradient boundary conditions, on polygonal domains. The explicit and detailed knowledge of the behavior of solutions to the integral equation in the vicinity of the corner was used to create purpose-made discretizations, resulting in efficient numerical schemes accurate to essentially machine precision. In this section, we discuss several directions in which these results are generalizable.
6.1. Infinite oscillations of the biharmonic Green’s function

In 1973, S. Osher showed that the domain Green’s function for the biharmonic equation has infinite oscillations for a right-angled wedge, and conjectured that this result holds for any domain with corners. Earlier, we observed that the representations of solutions to the associated integral equations also exhibit infinite oscillations near the corner (see Remark 28). The oscillatory behavior of these basis functions suggests that Osher’s conjecture may be amenable to an analysis similar to the one presented in this paper.

6.2. Curved boundaries

In this paper, we derive a representation for the solutions of the integral equations associated with the biharmonic equation, on polygonal domains. In the more general case of curved boundaries with corners, the apparatus of this paper also leads to detailed representations of the solutions near corners. Specifically, the solutions to the associated integral equations are representable by rapidly convergent series of products of complex powers of $t$ and logarithms of $t$, where $t$ is the distance from the corner. This analysis closely mirrors the generalization of the authors’ analysis of Laplace’s equation on polygonal domains (see [15]) to the authors’ analysis on domains having curved boundaries with corners (see [24]).

6.3. Generalization to three dimensions

The apparatus of this paper admits a straightforward generalization to surfaces with edge singularities, where the parts of the surface on either side of the edge meet at a constant angle along on the edge. The generalization to the case of edges with more complicated geometries is more involved and will be presented at a later date.

6.4. Other boundary conditions

In this paper, we analyze the integral equations associated with the velocity boundary value problem for Stokes equation. The approach of this paper extends to a number of other boundary conditions, including the traction boundary value problem, and the mobility problem. In particular, the traction boundary value problem and the mobility problem can be formulated as boundary integral equations that are the adjoints of the integral equations for the velocity boundary value problem.
6.5. Modified biharmonic equation

The modified biharmonic equation for a potential $\psi$ is given by $\Delta^2 \psi - \alpha \Delta \psi = 0$. The equation naturally arises when mixed implicit-explicit schemes are used for the time discretization of the incompressible Navier-Stokes equation (see, for example [25]). A preliminary analysis indicates that the solutions of the integral equations associated with the modified biharmonic equation are representable as rapidly convergent series of Bessel functions of certain non-integer complex orders. The generalization of the analysis of the biharmonic equation, presented in this paper, to the analysis of the modified biharmonic equation closely mirrors the generalization of the authors’ analysis of Laplace’s equation (see [15]) to the authors’ analysis of the Helmholtz equation (see [17]).

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8. Appendix A

In this appendix, we prove theorems 19, 30 which are restated here as theorems 65, and 66, respectively. Section 8.1 deals with the tangential odd, normal even case (see (94)), and Section 8.2 deals with the tangential even, normal odd case (see (95)).

8.1. Tangential odd, normal even case

Suppose that $A(z, \theta)$ is the $2 \times 2$ matrix defined in (106). We recall that

$$\begin{align*}
\det A(z, \theta) &= \frac{(z \sin (\pi \theta) - \sin (\pi z \theta)) (z \sin (\pi \theta) - \sin (\pi z (2 - \theta)))}{4 \sin^2 (\pi z)}.
\end{align*}$$

(176)

If $z$ is not an integer, and satisfies either

$$z \sin (\pi \theta) - \sin (\pi z (2 - \theta)) = 0,$$

(177)

or

$$z \sin (\pi \theta) - \sin (\pi z \theta) = 0,$$

(178)
then $\det (A(z, \theta)) = 0$. Section 8.1.1 deals with the implicit functions defined by (177) and, similarly, Section 8.1.2 deals with the implicit functions defined by (178). The principal result of this section is Theorem 65, which is a restatement of Theorem 19.
8.1.1. Analysis of implicit function \( z \) in (177)

Suppose that \( H : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) is the entire function defined by

\[
H(z, \theta) = z \sin (\pi \theta) - \sin (\pi z(2 - \theta)).
\]

(179)

In this section, we investigate the implicit functions \( z(\theta) \) which satisfy \( H(z(\theta), \theta) = 0 \).

We begin by stating the connection between \( \text{sinc}(z) \) and the function \( H(z, \theta) \) defined in (179).

**Lemma 36.** Suppose that \( G : \mathbb{C} \times \mathbb{C} \to \mathbb{C} \) is the entire function defined by

\[
G(w, \alpha) = \text{sinc}(w) + \text{sinc}(\alpha).
\]

(180)

Then \( G(w, \alpha) = 0 \) if and only if \( H(z, \theta) = 0 \) where \( z = \frac{w}{\alpha} \) and \( \theta = 2 - \frac{\alpha}{\pi} \).

**Proof.** Since \( z = \frac{w}{\alpha} \) and \( \theta = 2 - \frac{\alpha}{\pi} \),

\[
H(z, \theta) = z \sin (\pi \theta) - \sin (\pi z(2 - \theta)) = 0 \iff (181)
\]

\[
-w \sin (\alpha) + \sin w = 0 \iff (183)
\]

\[
G(w, \alpha) = 0.
\]

(184)

A simple calculation shows that

\[
\frac{d}{dz} \text{sinc}(z) = \tan(z) - z.
\]

(185)

In the following lemma, we discuss the zeros of \( \tan(z) - z \).

**Lemma 37.** There exists a countable collection of real \( \lambda_j > 0, j = 1, 2, \ldots \) such that all the zeros of \( \tan(z) - z \) are given by \( \{-\lambda_j\}_{j=1}^\infty \cup \{\lambda_j\}_{j=1}^\infty \cup \{0\} \), where \( \lambda_j \in (j\pi + \frac{\pi}{4}, (j+1)\pi - \frac{\pi}{2}) \).

**Proof.** We first show that all the zeros of \( \tan(z) - z \) are real. We observe that, if \( z = x + iy \), then

\[
\tan(z) = \frac{\sin(2x)}{\cos(2x) + \cosh(2y)} + i \frac{\sinh(2y)}{\cos(2x) + \cosh(2y)},
\]

(186)

If \( \tan z = z \), then

\[
\frac{\sin(2x)}{2x} = \frac{\sinh(2y)}{2y}.
\]

(187)

For all \( y \neq 0, |\frac{\sinh(2y)}{2y}| > 1 \) and for all \( x, |\frac{\sin(2x)}{2x}| \leq 1 \). Thus, if \( \tan(z) - z = 0 \), then \( z \in \mathbb{R} \).

Next, we observe that \( x = 0 \) clearly satisfies \( \tan(x) = x \). Furthermore, we note that if \( \lambda \in \mathbb{R} \) satisfies \( \tan(\lambda) = \lambda \), then \( -\lambda \) also satisfies \( \tan(-\lambda) = -\lambda \) since since \( \tan(x) \) is an odd function of \( x \). Thus, we restrict our attention to the roots of \( \tan(x) = x \) for \( x > 0 \). There are no roots of \( \tan(x) = x \) on the intervals \((j\pi + \pi/2, (j+1)\pi), j = 0, 1, 2 \ldots \), since \( \tan(x) < 0 \) for all \( x \in (j\pi + \pi/2, (j+1)\pi), j = 0, 1, 2 \ldots \). Moreover, for all \( j \geq 1 \),
there are no roots of \( \tan(x) = x \) on the intervals \([j\pi, j\pi + \pi/4]\), since \( 0 \leq \tan(x) \leq 1 \) for \( x \in [j\pi, j\pi + \pi/4] \), and \( x > 1 \) on \([j\pi, j\pi + \pi/4]\). Also, there are no roots of \( \tan(x) = x \) for \( x \in (0, \pi/2) \), since \( \frac{d}{dx} \tan(x) > 1 \) for all \( x \in (0, \pi/2) \). Finally, for each \( j = 1, 2, \ldots \), \( \tan(x) : (j\pi + \pi/4, j\pi + \pi/2) \to (1, \infty) \) is a bijection. Thus, there exists exactly one value \( \lambda_j \in (j\pi + \pi/4, j\pi + \pi/2) \) such that \( \tan(x) = x \).

The following lemma describes some elementary properties of \( \lambda_j \), \( j = 1, 2, \ldots \).

**Lemma 38.** Suppose that \( \lambda_j \) are as defined in Lemma 37. Then

- \( \text{sinc}(\lambda_j) = \cos(\lambda_j) \)
- \( \cos(\lambda_{2j}) > 0, \ j = 1, 2, \ldots \) and \( \cos(\lambda_{2j} - 1) < 0, \ j = 1, 2, \ldots \)
- \( |\cos(\lambda_j)| > |\cos(\lambda_{j+1})| \)

**Proof.** Recall that \( \lambda_j \), satisfy \( \tan(\lambda_j) = \lambda_j \). Thus,

\[
\text{sinc}(\lambda_j) = \frac{\sin(\lambda_j)}{\lambda_j} = \frac{\tan(\lambda_j) \cos(\lambda_j)}{\lambda_j} = \cos(\lambda_j).
\]

Since \( \lambda_j \in (j\pi + \pi/4, j\pi + \pi/2), \ j = 1, 2, \ldots \), it follows that \( \cos(\lambda_{2n}) > 0 \) and \( \cos(\lambda_{2n-1}) < 0 \).

Finally, suppose \( \lambda_j = j\pi + \theta_j \), where \( \theta_j \in (\pi/4, \pi/2), \ j = 1, 2, \ldots \). We first show that \( \theta_{j+1} > \theta_j \). We note that \( \tan(\lambda_j) = \tan(\theta_j + j\pi) = \tan(\theta_j) \). Thus,

\[
\tan(\theta_{j+1}) = \lambda_{j+1} > \lambda_j = \tan(\theta_j).
\]

Since, \( \tan(\theta) \) is a strictly monotonically increasing function for \( \theta \in (0, \pi/2) \), we conclude that \( \theta_{j+1} > \theta_j \). Then, \( |\cos(\lambda_j)| = |\cos(j\pi + \theta_j)| = \cos(\theta_j) \). Since \( \cos(\theta) \) is a strictly monotonically decreasing function for \( \theta \in (0, \pi/2) \), we conclude that \( |\cos(\lambda_j)| = \cos(\theta_j) > \cos(\theta_{j+1}) = |\cos(\lambda_{j+1})|, \ j = 1, 2, \ldots \).

In the following lemma, we describe contours in the complex plane for which \( \text{sinc}(z) \) is a real number.
Lemma 39. Suppose that \( j \) is a positive integer and that \( \lambda_j \) is defined in Lemma 37. Then there exists a function \( x_j : \mathbb{R} \to (j\pi, \lambda_j] \) which satisfies
\[
x_j(y) = \tan(x_j(y)) \cdot y \cdot \coth(y), \quad x_j(0) = \lambda_j,
\]
for all \( y \in \mathbb{R} \). Furthermore, if \( z = x_j(y) + iy \), then \( \text{sinc}(z) \in \mathbb{R} \).

Proof. It suffices to show existence of the function \( x_j(y) \) which satisfies (190) for \( y \geq 0 \), since if \( (x_j(y), y) \) satisfies (190), then \( (x_j(-y), -y) \) also satisfies (190), i.e. the function \( x_j(y) \) defined for \( y \in [0, \infty) \) can be extended to \( y \in (-\infty, \infty) \) using an even extension.

We observe that \( y \cdot \coth(y) : [0, \infty) \to [1, \infty) \) is a strictly monotonically increasing function and a bijection. Furthermore, an argument similar to the proof of Lemma 37 shows that for each \( m \in [1, \infty) \), there exists a unique solution \( x_j \) of the equation \( x/m = \tan(x) \) contained in the interval \((j\pi, \lambda_j] \). Moreover, the mapping from \( m \to x_j \) is monotonically decreasing as a function of \( m \) and maps \( m \in [1, \infty) \to x_j \in (j\pi, \lambda_j] \) (see Figure 12). Combining both of these statements, it follows that there exists a unique \( x_j(y) \) for each \( y \) which satisfies (190) and moreover, \( x_j(y) : [0, \infty) \to (j\pi, \lambda_j] \) is a bijection.

![Figure 12: An illustrative figure to demonstrate the solutions \( x_1(m) \) and \( x_2(m) \) of \( \tan(x) = x/m \) as a function of \( m \).](image)

Finally, a simple calculation shows that, if \( z = x + iy \), then
\[
\text{sinc}(z) = \frac{1}{x^2 + y^2} \left( x \sin(x) \cosh(y) + y \cos(x) \sinh(y) + i(x \cos(x) \sinh(y) - y \sin(x) \cosh(y)) \right),
\]
from which it follows that \( \text{sinc}(x_j(y) + iy) \) is real for all \( y \in \mathbb{R} \).

In the following lemma, we describe the behavior of the sinc function along the curve \( (x_j(y), y), j = 1, 2 \ldots \).

Lemma 40. Suppose that \( j \) is a positive integer. Suppose \( x_j : \mathbb{R} \to (j\pi, \lambda_j] \) is as defined in Lemma 39. Suppose further that \( z_j : \mathbb{R} \to \mathbb{C} \) is defined by \( z_j(y) = x_j(y) + iy \). Then the following holds
Thus, sinc is a strictly monotonically increasing function of $y$ for all $y > 0$, and sinc($z_j(y)$) : $(0, \infty) \to (\cos(\lambda_j), \infty)$ is a bijection. Likewise, sinc($z_j(y)$) is a strictly monotonically decreasing function of $y$ for all $y < 0$, and sinc($z_j(y)$) : $(-\infty, 0) \to (\cos(\lambda_j), \infty)$ is a bijection.

**Case 2, $j$ is odd:** sinc($z_j(y)$) is a strictly monotonically decreasing function of $y$ for all $y > 0$, and sinc($z_j(y)$) : $(0, \infty) \to (-\infty, \cos(\lambda_j))$ is a bijection. Likewise, sinc($z_j(y)$) is a strictly monotonically decreasing function of $y$ for all $y < 0$, and sinc($z_j(y)$) : $(-\infty, 0) \to (-\infty, \cos(\lambda_j))$ is a bijection.

**Proof.** We prove the result for the case when $j$ is even. The proof for the case when $j$ is odd follows in a similar manner.

A simple calculation shows that

$$
sinc(z_j(y)) = \frac{\cos(x_j(y)) \sinh(y)}{y}
$$

(192)

Recall that $x_j(y) = x_j(-y)$, and hence, sinc($z_j(y)$) = sinc($z_j(-y)$). Thus, it suffices to prove the result when $y > 0$.

Using Lemma 37, we note that $d/dz(sinc(z_j(y))) \neq 0$ for all $y > 0$. Hence for every $z_j(y)$, there exists a $\delta > 0$ such that sinc($z$) is one-one for all $z \in |z - z_j(y)| < \delta$. It then follows that sinc($z_j(y)$) is either a strictly monotonically increasing function or a strictly monotonically decreasing function for all $y > 0$.

When $j$ is even, using (192) and that $\lim_{y \to \infty} x_j(y) = j\pi$, we conclude that $\lim_{y \to \infty} \text{sinc}(z_j(y)) = \infty$. Finally, from Lemmas 38 and 39, we note that $\text{sinc}(z_j(0)) = \text{sinc}(\lambda_j) = \cos(\lambda_j)$. Thus, sinc($z_j(y)$) is a strictly monotonically increasing function and sinc($z_j(y)$) : $(0, \infty) \to (\cos(\lambda_j), \infty)$ is a bijection.

We note that $x(y) = 0$ satisfies (190) for all $y \in \mathbb{R}$. Moreover, if $z = x(y) + iy = iy$, we note that sinc($z$) is real. Thus, it is natural to define $x_0(y) \equiv 0$ for all $y$.

In the following lemma, we discuss the inverse of sinc($z$).

**Lemma 41.** Suppose that $j$ is a positive integer. Suppose $x_j(y)$, $j = 1, 2, \ldots$ for $y \in \mathbb{R}$, are as defined in Lemma 39. Suppose further that we define $x_0(y) = 0$ for all $y \in \mathbb{R}$. Let $\mathbb{H}^+$ denote the upper half plane and $\mathbb{H}^-$ denote the lower half plane. Furthermore, for any set $A \subset \mathbb{C}$, we denote the closure of $A$ by $\overline{A}$. Suppose $\Gamma_{j,+}$ is the open set in the upper half plane bounded by the curves $x_j(y)$ and $x_{j+1}(y)$, i.e.

$$
\Gamma_{j,+} = \{(x, y) : x_j(y) < x < x_{j+1}(y), \quad \text{and} \quad y > 0\},
$$

(193)

for $j = 0, 1, 2, \ldots$ (see Figure 13). Similarly suppose that $\Gamma_{j,-}$ is the open set in the lower half plane bounded by the curves $x_j(y)$ and $x_{j+1}(y)$, i.e.

$$
\Gamma_{j,-} = \{(x, y) : x_j(y) < x < x_{j+1}(y), \quad \text{and} \quad y < 0\},
$$

(194)

for $j = 1, 2, \ldots$ (see Figure 13). Then the following holds.
• Case 1, $j$ is even: $\text{sinc}(z) : \Gamma_{j,+} \to \mathbb{H}^-$ is a bijection which maps $\Gamma_{j,+} \to \mathbb{R}$. Moreover, the inverse function, which we denote by $\text{sinc}^{-1}_{j,+}(z)$, is a bijection from $\mathbb{H}^- \to \Gamma_{j,+}$ and is analytic for $z \in \mathbb{H}^-$. Similarly, $\text{sinc}(z) : \Gamma_{j,-} \to \mathbb{H}^+$ is a bijection which maps $\Gamma_{j,-} \to \mathbb{R}$. The inverse function, which we denote by $\text{sinc}^{-1}_{j,-}(z)$, is a bijection from $\mathbb{H}^+ \to \Gamma_{j,-}$ and is analytic for $z \in \mathbb{H}^+$.

• Case 2, $j$ is odd: $\text{sinc}(z) : \Gamma_{j,+} \to \mathbb{H}^+$ is a bijection which maps $\Gamma_{j,+} \to \mathbb{R}$. Moreover, the inverse function, which we denote by $\text{sinc}^{-1}_{j,+}(z)$, is a bijection from $\mathbb{H}^+ \to \Gamma_{j,+}$ and is analytic for $z \in \mathbb{H}^+$. Similarly, $\text{sinc}(z) : \Gamma_{j,-} \to \mathbb{H}^-$ is a bijection which maps $\Gamma_{j,-} \to \mathbb{R}$. The inverse function, which we denote by $\text{sinc}^{-1}_{j,-}(z)$, is a bijection from $\mathbb{H}^- \to \Gamma_{j,-}$ and is analytic for $z \in \mathbb{H}^-$. 

![Figure 13: The regions $\Gamma_{j,+}$ and $\Gamma_{j,-}$]

**Proof.** We prove the result for the case $\text{sinc}(z) : \Gamma_{j,+} \to \mathbb{H}^-$, when $j$ is even. The results for the other cases follows in a similar manner. First, it follows from Lemma 37 that $\frac{d}{dz}\text{sinc}(z) \neq 0$ for all $z \in \Gamma_{j,+}$. Thus, $\text{sinc}(z)$ is conformal for $z \in \Gamma_{j,+}$. Using Lemma 40, we note that $\text{sinc}(z) : \partial \Gamma_{j,+} \to \mathbb{R}$ is a bijection. Thus, $\text{sinc}(z)$ either maps $\Gamma_{j,+}$ to either the lower half plane or the upper half plane. A simple calculation shows that when $j$ is even, $\text{sinc}(z)$ maps $\Gamma_{j,+}$ to the lower half plane. Since $\text{sinc}(z)$ is conformal for $z \in \Gamma_{j,+}$, the inverse $\text{sinc}^{-1}_{j,+}(z)$ exists, is a bijection from $\mathbb{H}^- \to \Gamma_{j,+}$ and is analytic for $z \in \mathbb{H}^-$.

In the following lemma, we discuss the solutions $\alpha \in [0, 2\pi]$ of $\text{sinc}(\alpha) = -\cos(\lambda_j)$.

**Lemma 42.** Suppose $j$ is a positive integer and $\lambda_j$ is defined in Lemma 37.

• Case 1, $j$ is even: the equation $\text{sinc}(\alpha) = -\cos(\lambda_j)$ has only two solutions $\alpha_1^j$, and $\alpha_2^j$ on the interval $\alpha \in [0, 2\pi]$ where $\pi < \alpha_1^j < \lambda_1 < \alpha_2^j < 2\pi$. 

39
• Case 2, \( j \) is odd: the equation \( \text{sinc}(\alpha) = -\cos(\lambda) \) has only one solution \( \alpha_1^j \) on the interval \( \alpha \in [0, 2\pi] \), where \( 0 < \alpha_1^j < \pi \).

**Proof.** Suppose that \( j \) is even. We note that \( -\cos(\lambda_j) < 0 \) and furthermore \( -\cos(\lambda_j) > -\cos(\lambda_1) \) (see Lemma 38). Firstly, we note that \( \text{sinc}(\alpha) \geq 0 \) for all \( \alpha \in [0, \pi] \). Thus, there are no solutions to \( \text{sinc}(\alpha) = -\cos(\lambda_j) \) for \( \alpha \in [\pi, 2\pi] \). Refering to Figure 14, we observe that \( \text{sinc}(\alpha) : (\pi, \lambda_1) \to (-\cos(\lambda_1), 0) \) is a bijection. Thus, there exists a unique \( \alpha_1^j \in (\pi, \lambda_1) \) such that \( \text{sinc}(\alpha_1^j) = -\cos(\lambda_j) \). Similarly, \( \text{sinc}(\alpha) : (\lambda_1, 2\pi) \to (-\cos(\lambda_1), 0) \) is also a bijection. Thus, there exists a unique \( \alpha_2^j \in (\lambda_1, 2\pi) \) such that \( \text{sinc}(\alpha_2^j) = -\cos(\lambda_j) \).

Suppose now that \( j \) is odd. We note that \( -\cos(\lambda_j) > 0 \) (see Lemma 38). \( \text{sinc}(\alpha) \leq 0 \) for \( \alpha \in [\pi, 2\pi] \). Thus, there are no solutions to \( \text{sinc}(\alpha) = -\cos(\lambda_j) \) for \( \alpha \in [\pi, 2\pi] \). Finally, referring to Figure 14, we observe that \( \text{sinc}(\alpha) : (0, \pi) \to (0, 1) \) is a bijection. Thus, there exists a unique \( \alpha_1^j \in (0, \pi) \) such that \( \text{sinc}(\alpha_1^j) = -\cos(\lambda_j) \). ■

![Figure 14: Solutions of \( \text{sinc}(\alpha) = -\cos(\lambda_j) \). Case 1, \( j \) is even \((-\cos(\lambda_j) < 0\) left), and case 2, \( j \) is odd \((-\cos(\lambda_j) > 0\) right).](image)

### 8.1.1.1 Even case, \( z(1) = 2m \)

In this section, we analyze the implicit functions \( z(\theta) \) which satisfy \( H(z, \theta) = 0 \), with \( z(1) = 2m \) where \( m \) is a positive integer. The principal result of this section is Lemma 46.

**Lemma 43.** Suppose that \( m \) is a positive integer, and that \( G(w, \alpha) \) is as defined in (180). Suppose the regions \( \Gamma_{j,+, j, -}, j = 0, 1, \ldots \) are as defined in Lemma 41. Suppose that \( \alpha_1^{2m-1}, \alpha_1^{2m}, \) and \( \alpha_2^{2m} \) are as defined in Lemma 42. As before, for any set \( A \) let \( \overline{A} \) denote the closure of \( A \). Furthermore, suppose that \( D \) is the strip in the lower half plane with \( 0 < \text{Re}(\alpha) < 2\pi \), i.e.

\[
D = \{ \alpha \in \mathbb{C} : 0 < \text{Re}(\alpha) < 2\pi, \quad \text{Im}(\alpha) < 0 \}. \tag{195}
\]

Suppose that \( D_1 \) is the region \( \overline{D} \cap \overline{\Gamma_{0, -}} \) and \( D_2 \) is the region \( \overline{D} \setminus \overline{\Gamma_{0, -}} \) (see Figure 15).

Suppose finally that \( w(\alpha) : \overline{D} \to \mathbb{C} \) is defined by

\[
w(\alpha) = \begin{cases} \text{sinc}_1^{2m-1,-}\left(-\text{sinc}(\alpha)\right) & \alpha \in D_1, \\ \text{sinc}_1^{2m,-}\left(-\text{sinc}(\alpha)\right) & \alpha \in D_2. \end{cases} \tag{196}
\]
Then for all $\alpha \in D$, $w(\alpha)$ satisfies $G(w(\alpha), \alpha) = 0$ and is an analytic function for $\alpha \in D$. Moreover, $w(\pi) = 2m\pi$.

**Proof.** Suppose, as before, that $\mathbb{H}^+$ denotes the upper half plane and $\mathbb{H}^-$ denotes the lower half plane. We first note that for all $\alpha \in \mathcal{D}$, the function $w$ is well defined and satisfies $G(w(\alpha), \alpha) = 0$. For $\alpha \in D_1$, $w(\alpha) = \text{sinc}_{2m-1,-}(\text{sinc}(\alpha))$. The domain of definition for $\text{sinc}_{2m-1,-}(z)$ is $z \in \mathbb{H}^+$, and using Lemma 41, $-\text{sinc}(\alpha) \in \mathbb{H}^+$ for $\alpha \in D_1$. Moreover, for $\alpha \in \mathcal{D}_1$,

$$G(w(\alpha), \alpha) = \text{sinc}(w(\alpha)) + \text{sinc}(\alpha) \quad (197)$$

$$= \text{sinc}(\text{sinc}_{2m-1,-}(\text{sinc}(\alpha)) + \text{sinc}(\alpha) \quad (198)$$

$$= -\text{sinc}(\alpha) + \text{sinc}(\alpha) = 0. \quad (199)$$

Similarly, for $\alpha \in D_2$, $w(\alpha) = \text{sinc}_{2m,-}(\text{sinc}(\alpha))$. The domain of definition for $\text{sinc}_{2m,-}(z)$ is $z \in \mathbb{H}^-$, and using Lemma 41, $-\text{sinc}(\alpha) \in \mathbb{H}^-$ for all $\alpha \in D_2$. Moreover, for $\alpha \in \mathcal{D}_2$,

$$G(w(\alpha), \alpha) = \text{sinc}(w(\alpha)) + \text{sinc}(\alpha) \quad (200)$$

$$= \text{sinc}(\text{sinc}_{2m,-}(\text{sinc}(\alpha)) + \text{sinc}(\alpha) \quad (201)$$

$$= -\text{sinc}(\alpha) + \text{sinc}(\alpha) = 0. \quad (202)$$

Clearly, $w(\alpha)$ is analytic for $\alpha \in D_1 \cap D$ since both $\text{sinc}_{2m-1,-}(\alpha) \text{sinc}(\alpha))$ and $\text{sinc}(\alpha)$ are analytic functions on their respective domains of definition. Similarly, $w(\alpha)$ is analytic for $\alpha \in D_2 \cap D$. In order to show that $w(\alpha)$ is analytic for $\alpha \in \mathcal{D}$, it suffices to show that $w$ is continuous across $\mathcal{D}_1 \cap \mathcal{D}_2$. It follows from the definitions of the regions $D_1, D_2$, that $\mathcal{D}_1 \cap \mathcal{D}_2$ is precisely the curve $(x_1(y), y)$ for $y \in (-\infty, 0]$. For each $y \in (-\infty, 0)$, let $\alpha(y) = x_1(y) + iy$. Then

$$\{-\text{sinc}(\alpha(y)) : -\infty < y < 0\} = (-\cos(\lambda_1), \infty). \quad (203)$$

Let $w(y) = x_{2m}(y) + iy$, for $y \in (-\infty, 0)$, then $\text{sinc}(w(y)) \in (\cos(\lambda_{2m}), \infty)$. Moreover, $\text{sinc}(w(y))$ is a monotonically decreasing function of $y$ for $y < 0$ (see Lemma 40). Furthermore, using Lemma 38, we note that $-\cos(\lambda_1) > \cos(\lambda_{2m})$. Thus, there exists a unique $y_1 \in (-\infty, 0)$ such that $\text{sinc}(w(y_1)) = -\cos(\lambda_1)$.

Figure 15: The regions $D_1 = \overline{D} \cap \Gamma_{0,-}$ and $D_2 = \overline{D} \setminus \Gamma_{0,-}$.
Referring to Figure 16, we observe that
\[
\{\text{sinc}_{2m-1,-}(y) : -\cos(\lambda_1) < y < \infty\} = \{x_{2m}(y) + iy, \ -\infty < y < y_1\}. 
\]  
(204)

Similarly,
\[
\{\text{sinc}_{2m-1,-}(y) : -\cos(\lambda_1) < y < \infty\} = \{x_{2m}(y) + iy, \ -\infty < y < y_1\}. 
\]  
(205)

Combining (203) – (205), we conclude that \(w(\alpha)\) is continuous across \(\bar{D}_1 \cap \bar{D}_2\). It then follows from Morera’s theorem that \(w(\alpha)\) is analytic for \(\alpha \in D\).

Finally, \(\pi \in D_1\), and it follows from the definition of \(\text{sinc}_{2m-1,-}(z)\), that
\[
\begin{align*}
w(\pi) &= \text{sinc}_{2m-1,-}(\pi) = \text{sinc}_{2m-1,-}(0) = 2m\pi, \\
&
\end{align*}
\]  
(206)

from which the result follows. 

\[ \blacksquare \]

\textbf{Remark 44.} \textit{In Figure 16, we provide a more detailed description of the values of \(w(\alpha) \in \mathbb{C}\), which satisfies \(G(w(\alpha), \alpha) = 0\) and \(w(\pi) = 2m\pi\), for \(\alpha \in (0, 2\pi)\).}

In the following lemma, we further extend the domain of definition of \(w(\alpha)\) defined in Lemma 43 to a simply connected open set containing the strip in the lower half plane with \(0 < \text{Re}(\alpha) < 2\pi\) that includes the interval \((0, 2\pi) \setminus \{\alpha_1^{2m-1}, \alpha_1^{2m}, \alpha_2^{2m}\}\).

\textbf{Lemma 45.} \textit{Suppose that \(m\) is a positive integer, and that \(G(w, \alpha)\) is as defined in (180). Suppose that \(\alpha_1^{2m-1}, \alpha_1^{2m}, \text{ and } \alpha_2^{2m}\) are as defined in Lemma 42. As before, let \(A\) denote the closure of the set \(A\). Furthermore, suppose that the region \(D\) and the analytic function \(w(\alpha) : \bar{D} \to \mathbb{C}\) is as defined in Lemma 43. Suppose now that \(\tilde{D}\) is the strip in the lower half plane with \(0 < \text{Re}(\alpha) < 2\pi\) that includes the interval \((0, 2\pi) \setminus \{\alpha_1^{2m-1}, \alpha_1^{2m}, \alpha_2^{2m}\}\), i.e.,}
\[
\tilde{D} = \{\alpha \in \mathbb{C} : 0 < \text{Re}(\alpha) < 2\pi, \quad \text{Im}(\alpha) \geq 0\} \setminus \{\alpha_1^{2m-1}, \alpha_1^{2m}, \alpha_2^{2m}\}. 
\]  
(207)

\textit{Then there exists a simply connected open set \(\tilde{D} \subset \tilde{V} \subset \mathbb{C}\) (see Figure 17 and an analytic function \(\tilde{w}(\alpha) : \tilde{V} \to \mathbb{C}\) which satisfies \(G(\tilde{w}(\alpha), \alpha) = 0\) for all \(\alpha \in \tilde{V}\) and \(\tilde{w}(\pi) = 2m\pi\). Moreover \(\tilde{w}(\alpha) = w(\alpha)\) for all \(\alpha \in \bar{D} \cap \tilde{V}\).}

\textbf{Proof.} For all \(\alpha \in \bar{D} \cap \tilde{V}\), we define \(\tilde{w}(\alpha) = w(\alpha)\). We also note that the interval \((0, 2\pi) \setminus \{\alpha_1^{2m-1}, \alpha_1^{2m}, \alpha_2^{2m}\}\) is as defined in \(\bar{D} \cap \tilde{V}\). Furthermore, \(\tilde{w}(\alpha)\) also satisfies \(G(\tilde{w}(\alpha), \alpha) = 0\) for all \(\alpha \in \tilde{V}\), since \(w(\alpha)\) satisfies \(G(w(\alpha), \alpha) = 0\). A simple calculation shows that \(\partial_\alpha G(\tilde{w}(\alpha), \alpha) = \tan(\tilde{w}(\alpha)) - w(\alpha)\). Moreover, it follows from the definition of \(\tilde{w}(\alpha)\) that \(\tilde{w}(\alpha) \neq \lambda_j, \ j = 1, 2, \ldots\) for all \(\alpha \in (0, 2\pi) \setminus \{\alpha_1^{2m-1}, \alpha_1^{2m}, \alpha_2^{2m}\}\). Thus, we conclude from Lemma 37 that \(\partial_\alpha G(\tilde{w}(\alpha_0), \alpha_0) \neq 0\) for each \(\alpha_0 \in (0, 2\pi) \setminus \{\alpha_1^{2m-1}, \alpha_1^{2m}, \alpha_2^{2m}\}\). Finally, by the implicit function theorem there exists a \(\delta > 0\) and an implicit function \(\tilde{w}(\alpha) : |\alpha - \alpha_0| \to \mathbb{C}\) which satisfies \(G(\tilde{w}(\alpha), \alpha) = 0\), from which the result follows. 

\[ \blacksquare \]

We now present the principal result of this section.
Lemma 46. Suppose that $m$ is a positive integer and that $H(z, \theta)$ is as defined in (179). Suppose that $\alpha_1^{2m-1}$, $\alpha_1^{2m}$, and $\alpha_2^{2m}$ are as defined in Lemma 42. Suppose that $\theta_1, \theta_2$, and $\theta_3$ are given by

$$
\theta_1 = 2 - \frac{\alpha_2^{2m}}{\pi}, \quad \theta_2 = 2 - \frac{\alpha_1^{2m}}{\pi}, \quad \theta_3 = 2 - \frac{\alpha_1^{2m-1}}{\pi}.
$$

Suppose further that $D$ is the strip in the upper half plane with $0 < \text{Re}(\theta) < 2$ that includes the interval $(0, 2) \setminus \{\theta_1, \theta_2, \theta_3\}$, i.e.

$$
D = \{\theta \in \mathbb{C} : 0 < \text{Re}(\theta) < 2, \quad \text{Im}(\theta) \geq 0\} \setminus \{\theta_1, \theta_2, \theta_3\}.
$$
Then there exists a simply connected open set $D \subset V \subset \mathbb{C}$ (see Figure 18) and an analytic function $z(\theta) : V \to \mathbb{C}$ which satisfies $H(z(\theta), \theta) = 0$ for all $\theta \in V$ and $z(1) = 2m$.

Figure 18: An illustrative region of analyticity $V$ of the function $z(\theta)$, which satisfies $H(z(\theta), \theta) = 0$.

**Proof.** Suppose that $\tilde{V}$ and $\tilde{w}(\alpha) : \tilde{V} \to \mathbb{C}$ are as defined in Lemma 45. Recall that $\tilde{V}$ is an open set containing the strip $\tilde{D}$ defined in (207). Let

$$\theta = 2 - \frac{\alpha}{\pi}, \quad z(\theta) = \frac{w((2 - \theta)\pi)}{\pi(2 - \theta)}, \quad \text{and} \quad V = 2 - \frac{\tilde{V}}{\pi}. \quad (210)$$

For all $\alpha \in \tilde{V}$, we note that $\theta \in V$. Furthermore, $w(\pi) = 2m\pi$ implies that $z(1) = 2m$. Finally, using Lemma 36, we conclude that $z(\theta)$ satisfies $H(z(\theta), \theta) = 0$.

**Remark 47.** Using the Taylor expansion of $\text{sinc}(\alpha)$ in the neighborhood of $\alpha_1^{2m-1}, \alpha_1^{2m}$ and $\alpha_2^{2m}$, it is straightforward to show that $w(\alpha)$ in Lemma 43 has square root singularities at $\alpha = \alpha_1^{2m-1}, \alpha_1^{2m}, \alpha_2^{2m}$. It then follows from the definition of $z(\theta)$ in Lemma 46 has square root singularities at $\theta = \theta_1, \theta_2, \text{and} \theta_3$. Thus, $\theta_1, \theta_2, \text{and} \theta_3$ are branch points for the function $z(\theta)$.

**8.1.1.2 Odd case, $z(1) = 2m - 1, m \neq 1$**

In this section, we analyze the implicit functions which satisfy $H(z, \theta) = 0$, with $z(1) = 2m - 1$, where $m \geq 2$ is an integer. The principal result of this section is Lemma 50. The proofs of the results presented in this section are analogous to the corresponding proofs in Section 8.1.1.1. We present the statements of the theorem without proofs for brevity.
In the following lemma, we construct an analytic function \( w(\alpha) \) which satisfies \( G(w(\alpha), \alpha) = 0 \) with \( w(1) = (2m - 1)\pi \).

**Lemma 48.** Suppose that \( m \geq 2 \) is an integer, and that \( G(w, \alpha) \) is as defined in (180). Suppose the regions \( \Gamma_{j,+}, \Gamma_{j,-}, j = 0, 1, \ldots \) are as defined in Lemma 41. Suppose that \( \alpha_{1}^{2m-1}, \alpha_{2}^{2m-2}, \) and \( \alpha_{2}^{2m-2} \) are as defined in Lemma 42. As before, let \( \overline{A} \) denote the closure of the set \( A \). Furthermore, suppose that \( D \) is the strip in the lower half plane with \( 0 < \text{Re}(\alpha) < 2\pi \), i.e.

\[
D = \{ \alpha \in \mathbb{C} : 0 < \text{Re}(\alpha) < 2\pi , \quad \text{Im}(\alpha) < 0 \}. \tag{211}
\]

Suppose that \( D_{1} \) is the region \( \overline{D} \cap \overline{\Gamma}_{0,-} \) and \( D_{2} \) is the region \( \overline{D} \setminus D_{1} \). Suppose finally that \( w(\alpha) : \overline{D} \to \mathbb{C} \) is defined by

\[
w(\alpha) = \begin{cases} 
\text{sinc}_{2m-2, +}(\text{sinc}(\alpha)) & \alpha \in D_{1} \\
\text{sinc}_{2m-3, +}(\text{sinc}(\alpha)) & \alpha \in D_{2}.
\end{cases} \tag{212}
\]

Then for all \( \alpha \in D, w(\alpha) \) satisfies \( G(w(\alpha), \alpha) = 0 \) and is an analytic function for \( \alpha \in D \). Moreover, \( w(\pi) = (2m - 1)\pi \).

**Remark 49.** Referring to Figure 19, we provide a detailed description for the behavior of \( w(\alpha) \) defined in Lemma 48 for \( \alpha \in (0, 2\pi) \).

We present the principal result of this section in the following lemma.

**Lemma 50.** Suppose that \( m \geq 2 \) is an integer and that \( H(z, \theta) \) is as defined in (179). Suppose that \( \alpha_{1}^{2m-1}, \alpha_{1}^{2m-2}, \) and \( \alpha_{2}^{2m-2} \) are as defined in Lemma 42. Suppose that \( \theta_{1}, \theta_{2}, \) and \( \theta_{3} \) are given by

\[
\theta_{3} = 2 - \frac{\alpha_{2}^{2m-2}}{\pi} , \quad \theta_{2} = 2 - \frac{\alpha_{1}^{2m-2}}{\pi} , \quad \theta_{1} = 2 - \frac{\alpha_{1}^{2m-1}}{\pi} . \tag{213}
\]

Suppose further that \( D \) is the strip in the upper half plane with \( 0 < \text{Re}(\theta) < 2 \) that includes the interval \( (0, 2) \setminus \{ \theta_{1}, \theta_{2}, \theta_{3} \} \), i.e.

\[
D = \{ \theta \in \mathbb{C} : 0 < \text{Re}(\theta) < 2 , \quad \text{Im}(\theta) \geq 0 \} \setminus \{ \theta_{1}, \theta_{2}, \theta_{3} \}. \tag{214}
\]

Then there exists a simply connected open set \( D \subset V \subset \mathbb{C} \) and an analytic function \( z(\theta) : V \to \mathbb{C} \) which satisfies \( H(z(\theta), \theta) = 0 \) for all \( \theta \in V \) and \( z(1) = 2m - 1 \).

8.1.1.3 Odd case, \( z(1) = 1 \)

In this section, we analyze the implicit functions which satisfy \( H(z, \theta) = 0 \), with \( z(1) = 1 \). The principal result of this section is Lemma 53. The proofs of the results presented in this section are analogous to the corresponding proofs in Section 8.1.1.1. We present the statements of the theorem without proofs for brevity.

In the following lemma, we construct an analytic function \( w(\alpha) \) which satisfies \( G(w(\alpha), \alpha) = 0 \) with \( w(1) = \pi \).
Lemma 51. Suppose that $G(w, \alpha)$ is as defined in (180). Suppose the regions $\Gamma_{0,+}, \Gamma_{0,-}$ are as defined in Lemma 41. Suppose that $\alpha_1^1$ is as defined in Lemma 42. As before, let $\overline{A}$ denote the closure of the set $A$. Furthermore, suppose that $D$ is the strip in the lower half plane with $0 < \text{Re}(\alpha) < 2\pi$, i.e.

$$D = \{ \alpha \in \mathbb{C} : 0 < \text{Re}(\alpha) < 2\pi, \quad \text{Im}(\alpha) < 0 \}.$$  \hfill (215)

Suppose that $D_1$ is the region $\overline{D} \cap \Gamma_{0,-}$ and $D_2$ is the region $\overline{D} \setminus D_1$. Suppose finally that
$w(\alpha) : \mathbb{D} \to \mathbb{C}$ is defined by

$$w(\alpha) = \begin{cases} \text{sinc}^{-1}_0(-\text{sinc}(\alpha)) & \alpha \in D_1 \\ \text{sinc}^{-1}_0(-\text{sinc}(\alpha)) & \alpha \in D_2. \end{cases}$$  \hspace{1cm} (216)$$

Then for all $\alpha \in D$, $w(\alpha)$ satisfies $G(w(\alpha), \alpha) = 0$ and is an analytic function for $\alpha \in D$. Moreover, $w(\pi) = \pi$.

**Remark 52.** Referring to Figure 20, we provide a detailed description for the behavior of $w(\alpha)$ defined in Lemma 51 for $\alpha \in (0, 2\pi)$. 
We present the principal result of this section in the following lemma.

**Lemma 53.** Suppose that $H(z, \theta)$ is as defined in (179). Suppose that $\alpha_1$ is as defined in Lemma 42. Furthermore, suppose that $\theta_1$ are given by

$$\theta_1 = 2 - \frac{\alpha_1}{\pi}, \quad \text{(217)}$$

Suppose further that $D$ is the strip in the upper half plane with $0 < \text{Re}(\theta) < 2$ that includes the interval $(0, 2) \setminus \{\theta_1\}$, i.e.

$$D = \{\theta \in \mathbb{C} : 0 < \text{Re}(\theta) < 2, \quad \text{Im}(\theta) \geq 0\} \setminus \{\theta_1\}. \quad \text{(218)}$$

Then there exists a simply connected open set $D \subset V \subset \mathbb{C}$ (see Figure 21) and an analytic function $z(\theta) : V \to \mathbb{C}$ which satisfies $H(z(\theta), \theta) = 0$ for all $\theta \in V$ and $z(1) = 1.$

![Figure 21: An illustrative region of analyticity $V$ of the function $z(\theta)$, which satisfies $H(z(\theta), \theta) = 0$ with $z(1) = 1.$](image)

**8.1.2. Analysis of implicit function $z$ in (178)**

In this section, we investigate the implicit functions which satisfy

$$H(z, \theta) = z \sin (\pi \theta) - \sin (\pi z \theta) = 0. \quad \text{(219)}$$

The analysis for the implicit functions $z(\theta)$ which satisfy $H(z, \theta) = 0$ is similar to the analysis of the analogous implicit functions in Section 8.1.1. For conciseness, we present the statements of the theorems in this section and omit the proofs.

We first state the connection between $	ext{sinc}(z)$ and the function $H(z, \theta)$ defined in (219).

**Lemma 54.** Suppose that $G : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ is the entire function defined by

$$G(w, \alpha) = \text{sinc}(w) - \text{sinc}(\alpha). \quad \text{(220)}$$

Then $G(w, \alpha) = 0$ if and only if $H(z, \theta) = 0$ where $z = \frac{w}{\alpha}$ and $\theta = \frac{\alpha}{\pi}$.

In the following lemma, we discuss the solutions $\alpha \in [0, 2\pi]$ of $\text{sinc}(\alpha) = \cos (\lambda_j)$. 

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Figure 22: Solutions of $\text{sinc}(\alpha) = \cos(\lambda_j)$. Case 1, $j$ is even ($\cos(\lambda_j) > 0$) (left), and case 2, $j$ is odd ($\cos(\lambda_j) < 0$) (right).

**Lemma 55.** Suppose $j$ is a positive integer and $\lambda_j$ is defined in Lemma 37.

- **Case 1**, $j$ is even: the equation $\text{sinc}(\alpha) = \cos(\lambda_j)$ has only one solution $\beta_1^{j}$ on the interval $\alpha \in [0, 2\pi]$ where $0 < \beta_1^{j} < \pi$ (see Figure 22).

- **Case 2**, $j$ is odd: the equation $\text{sinc}(\alpha) = -\cos(\lambda_j)$ has only two solutions $\beta_1^{j}$, and $\beta_2^{j}$ on the interval $\alpha \in [0, 2\pi]$, where $\pi < \beta_1^{j} < \lambda_1 < \beta_2^{j} < 2\pi$ (see Figure 22).

As before, the analysis of the implicit functions $w(\alpha)$ which satisfy $G(w, \alpha) = 0$ (see (220)), and the analogous functions $z(\theta)$ which satisfy $H(z, \theta)$ (see (219)), is split into three cases. In Section 8.1.2.1, we analyze the functions implicit functions $z(\theta)$ for the case $z(1) = 2m - 1$, where $m$ is a positive integer, in Section 8.1.2.2, we analyze the implicit functions $z(\theta)$ for the case $z(1) = 2m$, where $m \geq 2$ is an integer, and finally in Section 8.1.2.3, we analyze the implicit function $z(\theta)$ for the case $z(1) = 2$.

**8.1.2.1 Odd case, $z(1) = 2m - 1$**

In this section, we analyze the implicit functions which satisfy $H(z, \theta) = 0$, with $z(1) = 2m - 1$, where $m$ is a positive integer. The principal result of this section is Lemma 58.

In the following lemma, we construct an analytic function $w(\alpha)$ which satisfies $G(w(\alpha), \alpha) = 0$ with $w(1) = (2m - 1)\pi$.

**Lemma 56.** Suppose that $m$ is a positive integer, and that $G(w, \alpha)$ is as defined in (220). Suppose the regions $\Gamma_{j,+}, \Gamma_{j,-}$, $j = 0, 1, \ldots$ are as defined in Lemma 41. Suppose that $\beta_{1,2m-1}^{2m}$, $\beta_{1,2m-1}^{2m-1}$, and $\beta_{2,2m-1}^{2m-1}$ are as defined in Lemma 55. As before, let $\overline{A}$ denote the closure of the set $A$. Furthermore, suppose that $D$ is the strip in the upper half plane with $0 < \text{Re}(\alpha) < 2\pi$, i.e.

$$D = \{ \alpha \in \mathbb{C} : 0 < \text{Re}(\alpha) < 2\pi, \quad \text{Im}(\alpha) > 0 \}. \quad (221)$$

Suppose that $D_1$ is the region $\overline{D} \cap \Gamma_{0,+}$ and $D_2$ is the region $\overline{D} \setminus \Gamma_{0,+}$ (see Figure 23).

Suppose finally that $w(\alpha) : \overline{D} \rightarrow \mathbb{C}$ is defined by

$$w(\alpha) = \begin{cases} 
\text{sinc}_{2m-2,-}^{-1}(\text{sinc}(\alpha)) & \alpha \in D_1 \\
\text{sinc}_{2m-1,-}^{-1}(\text{sinc}(\alpha)) & \alpha \in D_2.
\end{cases} \quad (222)$$
Then for all $\alpha \in D$, $w(\alpha)$ satisfies $G(w(\alpha), \alpha) = 0$ and is an analytic function for $\alpha \in D$. Moreover, $w(\pi) = (2m - 1)\pi$.

**Remark 57.** Referring to Figure 24, we provide a detailed description for the behavior of $w(\alpha)$ defined in Lemma 56 for $\alpha \in (0, 2\pi)$.

We present the principal result of this section in the following lemma.

**Lemma 58.** Suppose that $m$ is a positive integer and that $H(z, \theta)$ is as defined in (219). Suppose that $\beta_1^{2m-2}$, $\beta_1^{2m-1}$, and $\beta_2^{2m-1}$ are as defined in Lemma 55. Suppose that $\theta_1$, $\theta_2$, and $\theta_3$ are given by

$$
\theta_1 = \frac{\beta_1^{2m-2}}{\pi}, \quad \theta_2 = \frac{\beta_1^{2m-1}}{\pi}, \quad \theta_3 = \frac{\beta_2^{2m-1}}{\pi}.
$$

Suppose further that $D$ is the strip in the upper half plane with $0 < \text{Re}(\alpha) < 2\pi$ that includes the interval $(0, 2) \setminus \{\theta_1, \theta_2, \theta_3\}$, i.e.

$$
D = \{\theta \in \mathbb{C} : 0 < \text{Re}(\theta) < 2, \quad \text{Im}(\theta) \geq 0\} \setminus \{\theta_1, \theta_2, \theta_3\}.
$$

Then there exists a simply connected open set $D \subset V \subset \mathbb{C}$ and an analytic function $z(\theta) : V \rightarrow \mathbb{C}$ which satisfies $H(z(\theta), \theta) = 0$ for all $\theta \in V$ and $z(1) = 2m - 1$.

**8.1.2.2 Even case, $z(1) = 2m$, $m \neq 1$**

In this section, we analyze the implicit functions which satisfy $H(z, \theta) = 0$, with $z(1) = 2m$, where $m \geq 2$ is an integer. The principal result of this section is Lemma 61.

In the following lemma, we construct an analytic function $w(\alpha)$ which satisfies $G(w(\alpha), \alpha) = 0$ with $w(1) = 2m\pi$.

**Lemma 59.** Suppose that $m \geq 2$ is an integer, and that $G(w, \alpha)$ is as defined in (220). Suppose the regions $\Gamma_{j,+}$, $\Gamma_{j,-}$, $j = 0, 1, \ldots$ are as defined in Lemma 41. Suppose that $\beta_1^{2m}$, $\beta_1^{2m-1}$, and $\beta_2^{2m-1}$ are as defined in Lemma 55. As before, let $\overline{A}$ denote the closure of the set $A$. Furthermore, suppose that $D$ is the strip in the upper half plane with $0 < \text{Re}(\alpha) < 2\pi$, i.e.

$$
D = \{\alpha \in \mathbb{C} : 0 < \text{Re}(\alpha) < 2\pi, \quad \text{Im}(\alpha) > 0\}.
$$

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The values \( \text{sinc}(\alpha) \) for \( \alpha \in (0, 2\pi) \) (Figure 24(a)) and the corresponding values of \( w(\alpha) \) which satisfy \( G(w(\alpha), \alpha) = 0 \) with \( w(\pi) = (2m - 1)\pi \) (Figure 24(b)). In Figure 24(b), segment I represents \( w(\alpha) \) for \( \alpha \in (0, \beta_{2m-2}) \), segment II represents \( w(\alpha) \) for \( \alpha \in (\beta_2^{2m-2}, \beta_2^{2m-1}) \), segment III represents \( w(\alpha) \) for \( \alpha \in (\beta_2^{2m-1}, \lambda_1) \), segment IV represents \( w(\alpha) \) for \( \alpha \in (\lambda_1, \beta_2^{2m-1}) \), and finally segment V represents \( w(\alpha) \) for \( \alpha \in (\beta_2^{2m-1}, 2\pi) \).

Figure 24: The values \( \text{sinc}(\alpha) \) for \( \alpha \in (0, 2\pi) \) (Figure 24(a)) and the corresponding values of \( w(\alpha) \) which satisfy \( G(w(\alpha), \alpha) = 0 \) with \( w(\pi) = (2m - 1)\pi \). In Figure 24(b), segment I represents \( w(\alpha) \) for \( \alpha \in (0, \beta_{2m-2}) \), segment II represents \( w(\alpha) \) for \( \alpha \in (\beta_2^{2m-2}, \beta_2^{2m-1}) \), segment III represents \( w(\alpha) \) for \( \alpha \in (\beta_2^{2m-1}, \lambda_1) \), segment IV represents \( w(\alpha) \) for \( \alpha \in (\lambda_1, \beta_2^{2m-1}) \), and finally segment V represents \( w(\alpha) \) for \( \alpha \in (\beta_2^{2m-1}, 2\pi) \).

**Remark 60.** Referring to Figure 25, we provide a detailed description for the behavior of \( w(\alpha) \) defined in Lemma 59 for \( \alpha \in (0, 2\pi) \).
Figure 25: The values \( \text{sinc}(\alpha) \) for \( \alpha \in (0, 2\pi) \) (Figure 25(a)) and the corresponding values of \( w(\alpha) \) which satisfy \( G(w(\alpha), \alpha) = 0 \) with \( w(\pi) = 2m\pi \) (Figure 25(b)). In Figure 25(b), segment I represents \( w(\alpha) \) for \( \alpha \in (0, \beta_{2m}^1) \), segment II represents \( w(\alpha) \) for \( \alpha \in (\beta_{2m}^1, \beta_{2m}^{2m-1}) \), segment III represents \( w(\alpha) \) for \( \alpha \in (\beta_{2m}^{2m-1}, \lambda_1) \), segment IV represents \( w(\alpha) \) for \( \alpha \in (\lambda_1, \beta_{2m}^{2m-1}) \), and finally segment V represents \( w(\alpha) \) for \( \alpha \in (\beta_{2m}^{2m-1}, 2\pi) \).

We present the principal result of this section in the following lemma.

**Lemma 61.** Suppose that \( m \geq 2 \) is an integer and that \( H(z, \theta) \) is as defined in (219). Suppose that \( \beta_{1}^{2m}, \beta_{1}^{2m-1}, \text{ and } \beta_{2}^{2m-1} \) are as defined in Lemma 55. Suppose that \( \theta_1, \theta_2, \text{ and } \theta_3 \) are given by

\[
\theta_1 = \frac{\beta_{1}^{2m}}{\pi}, \quad \theta_2 = \frac{\beta_{1}^{2m-1}}{\pi}, \quad \theta_3 = \frac{\beta_{2}^{2m-1}}{\pi}.
\]  
(227)

Suppose further that \( D \) is the strip in the upper half plane with \( 0 < \text{Re}(\theta) < 2 \) that includes the interval \( (0,2) \setminus \{\theta_1, \theta_2, \theta_3\} \), i.e.

\[
D = \{ \theta \in \mathbb{C} : 0 < \text{Re}(\theta) < 2, \quad \text{Im}(\theta) \geq 0 \} \setminus \{\theta_1, \theta_2, \theta_3\}.
\]  
(228)
Then there exists a simply connected open set \( D \subset V \subset \mathbb{C} \) and an analytic function \( z(\theta) : V \to \mathbb{C} \) which satisfies \( H(z(\theta), \theta) = 0 \) for all \( \theta \in V \) and \( z(1) = 2m \).

### 8.1.2.3 Even case, \( z(1) = 2 \)

In this section, we analyze the implicit functions which satisfy \( H(z, \theta) = 0 \), with \( z(1) = 2 \). The principal result of this section is Lemma 64.

In the following lemma, we construct an analytic function \( w(\alpha) \) which satisfies \( G(w(\alpha), \alpha) = 0 \) with \( w(1) = 2\pi \).

**Lemma 62.** Suppose that \( G(w, \alpha) \) is as defined in (220). Suppose the regions \( \Gamma_{0,+}, \Gamma_{0,-} \) are as defined in Lemma 41. Suppose that \( \beta_{1,2}^2 \) is as defined in Lemma 55. Furthermore, suppose that \( D \) is the strip in the upper half plane with \( 0 < \text{Re}(\alpha) < 2\pi \), i.e.

\[
D = \{ \alpha \in \mathbb{C} : 0 < \text{Re}(\alpha) < 2\pi, \quad \text{Im}(\alpha) > 0 \}. \tag{229}
\]

Suppose that \( D_1 \) is the region \( \overline{D} \cap \Gamma_{0,+} \) and \( D_2 \) is the region \( \overline{D} \setminus D_1 \). Suppose finally that \( w(\alpha) : D \to \mathbb{C} \) is defined by

\[
w(\alpha) = \begin{cases} 
\text{sinc}_{1,1}^{-1}(\text{sinc}(\alpha)) & \alpha \in D_1 \\
\text{sinc}_{0,1}^{-1}(\text{sinc}(\alpha)) & \alpha \in D_2.
\end{cases} \tag{230}
\]

Then for all \( \alpha \in D \), \( w(\alpha) \) satisfies \( G(w(\alpha), \alpha) = 0 \) and is an analytic function for \( \alpha \in D \). Moreover, \( w(\pi) = 2\pi \).

**Remark 63.** Referring to Figure 26, we provide a detailed description for the behavior of \( w(\alpha) \) defined in Lemma 62 for \( \alpha \in (0, 2\pi) \).

We present the principal result of this section in the following lemma.

**Lemma 64.** Suppose that \( H(z, \theta) \) is as defined in (219). Suppose that \( \beta_{1}^2 \) is as defined in Lemma 55. Furthermore, suppose that \( \theta_1 \) is given by

\[
\theta_1 = \frac{\beta_{1}^2}{\pi}. \tag{231}
\]

Suppose further that \( D \) is the strip in the upper half plane with \( 0 < \text{Re}(\theta) < 2 \) that includes the interval \( (0, 2) \setminus \{ \theta_1 \} \), i.e.

\[
D = \{ \theta \in \mathbb{C} : 0 < \text{Re}(\theta) < 2, \quad \text{Im}(\theta) \geq 0 \} \setminus \{ \theta_1 \}. \tag{232}
\]

Then there exists a simply connected open set \( D \subset V \subset \mathbb{C} \) and an analytic function \( z(\theta) : V \to \mathbb{C} \) which satisfies \( H(z(\theta), \theta) = 0 \) for all \( \theta \in V \) and \( z(1) = 2 \).

Finally, we now present the principal result of Section 8.1.
Figure 26: The values sinc(\(\alpha\)) for \(\alpha \in (0, 2\pi)\) (Figure 26(a)) and the corresponding values of \(w(\alpha)\) which satisfy \(G(w(\alpha), \alpha) = 0\) with \(w(\pi) = 2\pi\) (Figure 26(b)). In Figure 26(b), segment I represents \(w(\alpha)\) for \(\alpha \in (0, \beta_1^2)\), segment II represents \(w(\alpha)\) for \(\alpha \in (\beta_1^2, \lambda_1)\), and finally segment III represents \(w(\alpha)\) for \(\alpha \in (\lambda_1, 2\pi)\).

**Theorem 65.** Suppose that \(N \geq 2\) is an integer. Then there exists \(3N - 2\) real numbers \(\theta_1, \theta_2, \ldots, \theta_{3N-2} \in (0, 2)\) such that the following holds. Suppose that \(D\) is the strip in the upper half plane with \(0 < \text{Re}(\theta) < 2\) that includes the interval \((0, 2) \setminus \{\theta_j\}_{j=1}^{3N-2}\), i.e.

\[
D = \{\theta \in \mathbb{C} : \text{Re}(\theta) \in (0, 2), \quad 0 \leq \text{Im}(\theta) < \infty\} \setminus \{\theta_j\}_{j=1}^{3N-2}.
\]  

Then, there exists a simply connected open set \(D \subset V \subset \mathbb{C}\) and analytic functions \(z_{n,1}(\theta) : V \to \mathbb{C}, \ n = 1, 2 \ldots N\), which satisfy

\[
z \sin (\pi \theta) - \sin (\pi z(2 - \theta)) = 0, \quad z(1) = n,
\]  

for \(\theta \in V\), and analytic functions \(z_{n,2}(\theta) : V \to \mathbb{C}, \ n = 2, 3, \ldots N\), which satisfy

\[
z \sin (\pi \theta) - \sin (\pi z \theta) = 0, \quad z(1) = n,
\]  

for \(\theta \in V\).
8.2. Tangential even, normal odd case

Proceeding in a similar manner, let \( z \) that the domain \( V \) are as defined in Lemma 42, and \( \beta \) are as defined in Lemma 55. Let \( \theta_1, \theta_2, \ldots, \theta_{3N-2} \) be the collection of numbers

\[
\{ \theta_j \}_{j=1}^{3N-2} = \left\{ 2 - \frac{\alpha_1 2^{m-1}}{\pi} \right\}_{m=1}^{2m-1 \leq N} \cup \left\{ 2 - \frac{\alpha_1 2^m}{\pi} \right\}_{m=1}^{2m \leq N} \cup \left\{ 2 - \frac{\alpha_2 2^{m-1}}{\pi} \right\}_{m=1}^{2m-1 \leq N} \cup \left\{ 2 - \frac{\alpha_2 2^m}{\pi} \right\}_{m=1}^{2m \leq N} \cup \left\{ \beta_1 \frac{2^{m+1}}{\pi} \right\}_{m=1}^{2m+1 \leq N} \cup \left\{ \beta_1 \frac{2^{m+1}}{\pi} \right\}_{m=1}^{2m+1 \leq N} \cup \left\{ \beta_2 \frac{2^m}{\pi} \right\}_{m=1}^{2m \leq N} \cup \left\{ \beta_2 \frac{2^m}{\pi} \right\}_{m=1}^{2m \leq N}.
\] (236)

Clearly, \( V = \cap_{n=1}^{N} V_{n,1} \cap_{n=2}^{N} V_{n,2} \) is a simply connected open neighborhood such that \( D \subset V \) and \( V \) is the common region of analyticity of the functions \( z_{n,1}(\theta), n = 1, 2, \ldots, N \), and \( z_{n,2}(\theta), n = 2, 3, \ldots, N \). Furthermore, it follows from Lemmas 46, 50, 53, 58, 61 and 64 that the domain \( V \), the analytic functions \( z_{n,1}(\theta), n = 1, 2, \ldots, N \), and the analytic functions \( z_{n,2}(\theta), n = 2, 3, \ldots, N \), satisfy all the conditions of the theorem.

8.2. Tangential even, normal odd case

Suppose that \( A(z, \theta) \) is the \( 2 \times 2 \) matrix defined in (146). We recall that

\[
\det A(z, \theta) = \frac{(z \sin (\pi \theta) + \sin (\pi z \theta)) (z \sin (\pi \theta) + \sin (\pi z (2 - \theta)))}{4 \sin^2 (\pi z)}.
\] (237)

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If \( z \) is not an integer, and satisfies either
\[
z \sin(\pi \theta) + \sin(\pi z(2 - \theta)) = 0, \quad (238)
\]
or
\[
z \sin(\pi \theta) + \sin(\pi z \theta) = 0, \quad (239)
\]
then \( \det(A(z, \theta)) = 0 \). The analysis of the implicit functions \( z(\theta) \) which satisfy (238), (239) for the tangential even, normal odd case is similar to the analysis of the implicit functions \( z(\theta) \) which satisfy (177), (178) for the tangential odd, normal even case. The principal result of this section is Theorem 66, which is a restatement of Theorem 30. For brevity, we omit the proof.

**Theorem 66.** Suppose that \( N \geq 2 \) is an integer. Then there exists \( 3N - 2 \) real numbers \( \theta_1, \theta_2, \ldots, \theta_{3N-2} \in (0, 2) \) such that the following holds. Suppose that \( D \) is the strip in the upper half plane with \( 0 < \text{Re}(\theta) < 2 \) that includes the interval \( (0, 2) \setminus \{\theta_j\}_{j=1}^{3N-2} \), i.e.
\[
D = \{\theta \in \mathbb{C} : \text{Re}(\theta) \in (0, 2), \ 0 \leq \text{Im}(\theta) < \infty\} \setminus \{\theta_j\}_{j=1}^{3N-2}.
\]
Then, there exists a simply connected open set \( D \subset V \subset \mathbb{C} \) and analytic functions \( z_{n,1}(\theta) : V \to \mathbb{C}, \ n = 2, 3, \ldots N \), which satisfy
\[
z \sin(\pi \theta) + \sin(\pi z(2 - \theta)) = 0, \quad z(1) = n, \quad (241)
\]
for \( \theta \in V \), and analytic functions \( z_{n,2}(\theta) : V \to \mathbb{C}, \ n = 1, 2, \ldots, N \), which satisfy
\[
z \sin(\pi \theta) + \sin(\pi z \theta) = 0, \quad z(1) = n, \quad (242)
\]
for \( \theta \in V \) (see Figure 27 for an illustrative domain \( V \)). Moreover, the functions \( z_{n,1}(\theta), \ n = 2, 3, \ldots N \), do not take integer values for all \( \theta \in V \setminus \{1\} \), and satisfy \( \det(A(z_{n,1}(\theta), \theta) = 0, \ n = 2, 3, \ldots N \), for all \( \theta \in V \) (see (146), (150)). Similarly, the functions \( z_{n,2}(\theta), \ n = 1, 2, \ldots, N \), do not take integer values for all \( \theta \in V \setminus \{1\} \), and satisfy \( \det(A(z_{n,2}(\theta), \theta) = 0, \ n = 1, 2, \ldots, N \), for all \( \theta \in V \) (see (146), (150)).

### 9. Appendix B

In this section we compute the limit \( \theta \to 1 \), of the linear transformation \( B(\theta) \), which maps coefficients of singular basis functions for the solution of the integral equation to the Taylor expansion coefficients of the velocity field. In Section 9.1, we investigate the tangential odd, normal even case (see (94)) and, in Section 9.2, we investigate the tangential even, normal odd case (see (95)).

#### 9.1. Tangential odd, normal even case

Suppose that \( A(z, \theta) \) is the \( 2 \times 2 \) matrix given by (106) defined in Section 4.1. Suppose \( N \) is a positive integer. Suppose further that, as in Theorem 19, \( z_{n,1}(\theta), \ n = 1, 2, \ldots, N \), are analytic functions satisfying \( \det(A(z_{n,1}(\theta), \theta) = 0 \) for \( \theta \in V_\delta \subset \mathbb{C} \), where \( V_\delta \) is a neighborhood of the contour \( C_\delta \). Similarly, suppose that \( z_{n,2}(\theta), \ n = 2, 3, \ldots N \), are analytic functions satisfying \( \det(A(z_{n,2}(\theta), \theta) = 0 \) for \( \theta \in V_\delta \). Let \( (p_{n,1}, q_{n,1}) \in \mathcal{N}(A(z_{n,1}(\theta), \theta)), \ n = 1, 2, \ldots, N, \)
and \((p_{n,2}, q_{n,2}) \in \mathcal{N}\{A(z_{n,2}(\theta), \theta)\}\), \(n = 2, 3, \ldots N\). We further assume that the vectors \((p_{n,1}, q_{n,1}), n = 1, 2, \ldots \) and \((p_{n,2}, q_{n,2}), n = 2, 3, \ldots \), are \(\ell^2\) normalized. Suppose that \(z_{1,2}(\theta) \equiv 1, p_{1,2} = 0,\) and \(q_{1,2} = 1\). Finally, suppose that \(B(\theta)\) is a \((2N + 2) \times (2N + 2)\) matrix defined in Theorem 24 for \(\theta \in V_3\). The \(2 \times 2\) blocks of \(B(\theta)\) are given by

\[
B_{\ell,n}(\theta) = \begin{bmatrix} F(\ell, z_{n,1}(\theta), \theta) \begin{bmatrix} p_{n,1}(\theta) \\ q_{n,1}(\theta) \end{bmatrix} & F(\ell, z_{n,2}(\theta), \theta) \begin{bmatrix} p_{n,2}(\theta) \\ q_{n,2}(\theta) \end{bmatrix} \end{bmatrix},
\]

for \(\ell, n = 1, 2, \ldots N\), where \(F\) is defined in (109), except for the case \(\ell = n = 1\). In the case \(\ell = n = 1\), the matrix \(B_{1,1}(\theta)\) is given by

\[
B_{1,1}(\theta) = \begin{bmatrix} F(1, z_{1,1}(\theta), \theta) \begin{bmatrix} p_{1,1}(\theta) \\ q_{1,1}(\theta) \end{bmatrix} & F(1) \end{bmatrix},
\]

where \(F\) is defined in (109), and \(F_1\) is defined in (123). Finally, if either \(\ell = 0\) or \(n = 0\), then the matrices \(B_{\ell,n}(\theta)\) are given by

\[
B_{\ell,0}(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

\[
B_{0,0}(\theta) = F_0(\theta),
\]

\[
B_{0,n}(\theta) = F(n, 0, \theta),
\]

for \(\ell, n = 1, 2, \ldots N\), where \(F\) is defined in (109), and \(F_0\) is defined in (126).

In the following lemma, we describe the behavior of \(z_{n,1}(\theta), p_{n,1}(\theta),\) and \(q_{n,1}(\theta), n = 1, 2, \ldots ,\) in the vicinity of \(\theta = 1\).

**Lemma 67.** Suppose that \(z_{n,1}(\theta), n = 1, 2, \ldots ,\) satisfying \(\det A(z_{n,1}(\theta), \theta) = 0\), be as defined in Theorem 19. Suppose further that \((p_{n,1}(\theta), q_{n,1}(\theta)), n = 1, 2, \ldots ,\) be an \(\ell^2\) normalized null vector of \(A(z_{n,1}, \theta)\). Then in the neighborhood of \(\theta = 1\),

\[
z_{2n,1}(\theta) = 2n - \frac{\pi^2 n (4n^2 - 1)}{3} (\theta - 1)^3 + O(|\theta - 1|^4)
\]

\[
p_{2n,1}(\theta) = -\frac{\pi (2n - 1)}{2} (\theta - 1) + O(|\theta - 1|^3)
\]

\[
q_{2n,1}(\theta) = 1 + O(|\theta - 1|^2)
\]

\[
z_{2n+1,1}(\theta) = 2n + 1 + 2(2n + 1)(\theta - 1) + O(|\theta - 1|^2)
\]

\[
p_{2n+1,1}(\theta) = 1 + O(|\theta - 1|)
\]

\[
q_{2n+1,1}(\theta) = O(|\theta - 1|).
\]

**Proof.** Let the superscript ‘ denote derivative with respect to \(\theta\) and \(\partial_t\) denote the partial derivative with respect to \(t\). Suppose that \(H(z, \theta)\) is given by

\[
H(z, \theta) = z \sin (\pi \theta) - \sin (\pi z (2 - \theta))
\]

From Theorem 19, we recall that \(z_{n,1}(\theta), n = 1, 2, \ldots ,\) satisfy \(H(z_{n,1}, \theta) = 0\). Using the implicit function theorem

\[
z'(\theta) = -\frac{\partial_{\theta} H}{\partial z} = z (1 - \sec(\pi z))
\]

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Thus,
\[ z_{2n,1}^\prime(1) = 0, \quad \text{and} \quad z_{2n+1,1}^\prime(1) = 2(2n + 1). \] (256)

Implicitly differentiating \( H \) twice we get
\[ \partial_z H \cdot z''(\theta) + \left( \frac{d}{d\theta} \partial_z H + \partial_z \partial_\theta H \right) \cdot z'(\theta) + \partial_\theta \partial_\theta H = 0. \] (257)

Since \( z_{2n,1}^\prime(1) = 0 \), we get
\[ z_{2n,1}''(1) = -\frac{\partial_\theta \partial_\theta H}{\partial_z H} = \pi z^2 \tan(\pi z) = 0. \] (258)

Proceeding in a similar fashion, we observe that
\[ z_{2n,1}'''(1) = -\frac{\partial_\theta \partial_\theta \partial_\theta H}{\partial_z H} = \pi^2 z_{2n,1}(\sec(\pi z_{2n,1}) - z_{2n,1}^2) = 2\pi^2 n(1 - 4n^2). \] (259)

Let
\[ \tilde{p}_{n,1}(\theta) = -\sin(\pi z) + 2a_{2,2}(z_{n,1}(\theta), \theta), \] (260)
\[ \tilde{q}_{n,1}(\theta) = 2\sin(\pi z)a_{2,1}(z_{n,1}(\theta), \theta), \] (261)
where \( a_{2,1} \) and \( a_{2,2} \) are defined in (83) and (84) respectively. Clearly, \((\tilde{p}_{n,1}, \tilde{q}_{n,1}) \in N(A)\).

We then set
\[ p_{n,1}(\theta) = \frac{\tilde{p}_{n,1}}{\sqrt{(\tilde{p}_{n,1}^2 + \tilde{q}_{n,1}^2)}}, \quad \text{and} \quad q_{n,1}(\theta) = \frac{\tilde{q}_{n,1}}{\sqrt{(\tilde{p}_{n,1}^2 + \tilde{q}_{n,1}^2)}} \] (262)

The required Taylor expansions for \( p_{n,1}, q_{n,1} \) are then readily obtained by using the Taylor expansions of \( z_{n,1}(\theta) \).

In the next lemma, we now describe the behavior of \( z_{n,2}(\theta), p_{n,2}(\theta) \) and \( q_{n,2}(\theta), n = 2, 3, \ldots, \) in the vicinity of \( \theta = 1 \).

**Lemma 68.** Suppose that \( z_{n,2}(\theta), n = 2, 3, \ldots, \) satisfying \( \det A(z_{n,2}(\theta), \theta) = 0, \) be as defined in Theorem 19. Suppose further that \((p_{n,2}(\theta), q_{n,2}(\theta)), n = 2, 3, \ldots, \) be an \( \ell^2 \) normalized null vector of \( A(z_{n,2}, \theta) \). Then in the neighborhood of \( \theta = 1 \),
\[ z_{2n,2}(\theta) = 2n - 4n(\theta - 1) + O(|\theta - 1|^2) \] (263)
\[ p_{2n,2}(\theta) = 1 + O(|\theta - 1|) \] (264)
\[ q_{2n,2}(\theta) = 0 + O(|\theta - 1|) \] (265)
\[ z_{2n+1,2}(\theta) = 2n + 1 + \frac{2\pi^2 n(n + 1)(2n + 1)}{3}(\theta - 1)^3 + O(|\theta - 1|^4) \] (266)
\[ p_{2n+1,2}(\theta) = -\pi n(\theta - 1) + O(|\theta - 1|^3) \] (267)
\[ q_{2n+1,2}(\theta) = 1 + O(|\theta - 1|^2) \] (268)

**Proof.** The proof proceeds in a similar manner as the proof of Lemma 67. ■

Combining Lemmas 67 and 68, we present the principal result of this section, which computes the limit \( \theta \to 1 \) of the matrix \( B(\theta) \) in the following theorem.
Theorem 69. Suppose that \( N \) is a positive integer and suppose further that \( B \) is given by (243) – (247). Then

\[
\lim_{\theta \to 1} B_{\ell,j}(\theta) = \begin{cases} 
-\frac{1}{2} & \ell = j = 0 \\
0 & \ell = j = 2m \neq 0 \\
-\frac{1}{2} & \ell = j = 2m + 1 \\
\frac{1}{2} & \ell = j = 0 \\
0 & \ell = j = 2m + 1 \\
0 & \ell = j = 0 \\
0 & \ell = j = 2m + 1 \\
\end{cases},
\]  

(269)

for all \( \ell, j = 0, 1, \ldots, N \).

Proof. Let \( \tilde{F}(n, \theta) = 2\pi F(n, z, \theta) \cdot (n - z) \) for \( j, \ell = 1, 2 \) where \( F \) is given by (109). On inspecting the entries of \( \tilde{F}(n, \theta) \), we observe that

\[
\tilde{F}(n, 1) = \begin{bmatrix} 0 & 0 \\
0 & 0 \end{bmatrix},
\]

(270)

for all \( n \in \mathbb{N} \). Since \( z_{j,\ell}(1) = j \), we conclude that

\[
\lim_{\theta \to 1} F(n, z_{j,\ell}, \theta) \begin{bmatrix} p_{j,\ell} \\
q_{j,\ell} \end{bmatrix} = \lim_{\theta \to 1} \frac{\tilde{F}(n, \theta)}{-z_{j,\ell} + n} \begin{bmatrix} p_{j,\ell} \\
q_{j,\ell} \end{bmatrix} = \begin{bmatrix} 0 \\
0 \end{bmatrix},
\]

(271)

for all \( j \neq n \), and \( \ell = 1, 2 \). \( B_{n,0}(\theta) = 0 \) is the zero matrix by definition and for \( B_{0,n}(\theta) \), we have

\[
\lim_{\theta \to 1} B_{0,n}(\theta) = \lim_{\theta \to 1} F(n, 0, \theta) = \lim_{\theta \to 1} \frac{\tilde{F}(n, \theta)}{2\pi n} = \begin{bmatrix} 0 & 0 \\
0 & 0 \end{bmatrix},
\]

(272)

for \( n = 1, 2, \ldots, N \).

We now turn our attention to the diagonal terms. For \( n = 0 \), it follows from a simple calculation that

\[
\lim_{\theta \to 1} B_{0,0}(\theta) = \lim_{\theta \to 1} F(0, \theta) = \begin{bmatrix} 0 & 0 \\
0 & 0 \end{bmatrix}.
\]

(273)

For \( n \geq 2 \), using Lemma 67 and Lemma 68, it follows from a rather tedious calculation that

\[
\tilde{F}(2n, \theta) \begin{bmatrix} p_{2n,1}(\theta) \\
q_{2n,1}(\theta) \end{bmatrix} = \begin{bmatrix} \frac{O(1)}{\theta - 1} \\
-\frac{\pi n}{\theta - 1} \end{bmatrix} + O(1)
\]

(274)

\[
\tilde{F}(2n + 1, \theta) \begin{bmatrix} p_{2n+1,1}(\theta) \\
q_{2n+1,1}(\theta) \end{bmatrix} = \begin{bmatrix} (2n + 1)\pi \theta - 1 + O(1) \\
O(1) \end{bmatrix}.
\]

(275)
and

\[
\mathbf{F}(2n, \theta) \begin{bmatrix} p_{2n, 2}(\theta) \\ q_{2n, 2}(\theta) \end{bmatrix} = -4n\pi(\theta - 1) + \mathcal{O}(\theta - 1^2) \\
\mathcal{O}(\theta - 1^2) 
\]

(276)

\[
\mathbf{F}(2n + 1, \theta) \begin{bmatrix} p_{2n+1, 2}(\theta) \\ q_{2n+1, 2}(\theta) \end{bmatrix} = \frac{2\pi n(n+1)(2n+1)(\theta - 1)^3 + \mathcal{O}(\theta - 1^4)}{3}.
\]

(277)

Finally, from the definition of \( F_1 \) in (123), we note that

\[
\lim_{\theta \to 1} \mathbf{F}_1(\theta) = \begin{bmatrix} 0 \\ -1/2 \end{bmatrix}.
\]

(278)

The result then follows from combining (274) – (278).

\[ \]

9.2. Tangential even, normal odd case

Suppose that \( A(z, \theta) \) is the \( 2 \times 2 \) matrix given by (146) defined in Section 4.2. Suppose \( N \) is a positive integer. Suppose further that, as in Theorem 30, \( z_{n, 1}(\theta), n = 2, 3, \ldots N \), are analytic functions satisfying \( \det A(z_{n, 1}(\theta), \theta) = 0 \) for \( \theta \in V_\delta \subset \mathbb{C} \), where \( V_\delta \) is a neighborhood of the contour \( C_\delta \). Similarly, suppose that \( z_{n, 2}(\theta), n = 1, 2 \ldots N \), are analytic functions satisfying \( \det A(z_{n, 2}(\theta), \theta) = 0 \) for \( \theta \in V_\delta \). Let \( (p_{0, 1}, q_{0, 1}) \in \mathcal{N}\{A(z_{n, 1}(\theta), \theta)\} \), \( n = 2, 3, \ldots N \), and \( (p_{n, 2}, q_{n, 2}) \in \mathcal{N}\{A(z_{n, 2}(\theta), \theta)\} \), \( n = 2, 3, \ldots N \). We further assume that the vectors \( (p_{n, 1}, q_{n, 1}), n = 2, 3, \ldots \) and \( (p_{n, 2}, q_{n, 2}), n = 1, 2, \ldots \), are \( \ell^2 \) normalized. Suppose that \( z_{1, 2}(\theta) \equiv 1 \), \( p_{1, 2} = 0 \), and \( q_{1, 2} = 1 \). Finally, suppose that \( B(\theta) \) is a \( (2N + 2) \times (2N + 2) \) matrix defined in Theorem 33 for \( \theta \in V_\delta \). The \( 2 \times 2 \) blocks of \( B(\theta) \) are given by

\[
B_{\ell, n}(\theta) = - \begin{bmatrix} \mathbf{F}(\ell, z_{n, 1}(\theta), \theta) & \mathbf{F}(\ell, z_{n, 2}(\theta), \theta) \\ \mathbf{F}(\ell, z_{n, 1}(\theta), \theta) & \mathbf{F}(\ell, z_{n, 2}(\theta), \theta) \end{bmatrix},
\]

(279)

for \( \ell, n = 1, 2, \ldots N \), where \( \mathbf{F} \) is defined in (109), except for the case \( \ell = n = 1 \). In the case \( \ell = n = 1 \), the matrix \( B_{1, 1}(\theta) \) is given by

\[
B_{1, 1}(\theta) = \begin{bmatrix} \mathbf{F}(1, z_{1, 1}(\theta), \theta) & \mathbf{F}(1, z_{1, 2}(\theta), \theta) \\ -\mathbf{F}(1, z_{1, 1}(\theta), \theta) & \mathbf{F}(1, z_{1, 2}(\theta), \theta) \end{bmatrix},
\]

(280)

where \( \mathbf{F} \) is defined in (109), and \( \mathbf{F}_1 \) is defined in (158). Finally, if either \( \ell = 0 \) or \( n = 0 \), then the matrices \( B_{\ell, n}(\theta) \) are given by

\[
B_{\ell, 0}(\theta) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},
\]

(281)

\[
B_{0, 0}(\theta) = \mathbf{F}_0(\theta),
\]

(282)

\[
B_{0, n}(\theta) = -\mathbf{F}(n, 0, \theta),
\]

(283)

for \( \ell, n = 1, 2, \ldots N \), where \( \mathbf{F} \) is defined in (109), and \( \mathbf{F}_0 \) is defined in (160). The proofs of the results in this section are similar to the corresponding proofs in Section 9.1. For conciseness, we state the results without proof. The principal result of this section is Theorem 72.

In the following lemma, we describe the behavior of \( z_{n, 1}(\theta), p_{n, 1}(\theta), \) and \( q_{n, 1}(\theta) \), \( n = 2, 3, \ldots \), in the vicinity of \( \theta = 1 \).
Lemma 70. Suppose that $z_{n,1}(\theta), n = 2, 3, \ldots,$ satisfying $\det A(z_{n,1}(\theta), \theta) = 0$, be as defined in Theorem 30. Suppose further that $(p_{n,1}(\theta), q_{n,1}(\theta)), n = 2, 3, \ldots,$ be an $\ell^2$ normalized null vector of $A(z_{n,1}, \theta)$.

\begin{align*}
z_{2n,1}(\theta) &= 2n + 4n(\theta - 1) + O(|\theta - 1|^2) \\
p_{2n,1}(\theta) &= 1 + O(|\theta - 1|) \\
q_{2n,1}(\theta) &= 0 + O(|\theta - 1|) \\
z_{2n+1,1}(\theta) &= 2n + 1 - \frac{2\pi^2 n(n + 1)(2n + 1)}{3} (\theta - 1)^3 + O(|\theta - 1|^4) \\
p_{2n+1,1}(\theta) &= -\pi n(\theta - 1) + O(|\theta - 1|^3) \\
q_{2n+1,1}(\theta) &= 1 + O(|\theta - 1|^2)
\end{align*}

Similarly, in the following lemma, we describe the behavior of $z_{n,2}(\theta), p_{n,2}(\theta), q_{n,2}(\theta), n = 1, 2, \ldots,$ in the vicinity of $\theta = 1$.

Lemma 71. Suppose that $z_{n,2}(\theta), n = 1, 2, \ldots,$ satisfying $\det A(z_{n,2}(\theta), \theta) = 0$, be as defined in Theorem 30. Suppose further that $(p_{n,2}(\theta), q_{n,2}(\theta)), n = 1, 2, \ldots,$ be an $\ell^2$ normalized null vector of $A(z_{n,2}, \theta)$. Then in the neighborhood of $\theta = 1$,

\begin{align*}
z_{2n,1}(\theta) &= 2n + \frac{\pi^2 n(4n^2 - 1)}{3} (\theta - 1)^3 + O(|\theta - 1|^4) \\
p_{2n,2}(\theta) &= -\frac{\pi (2n - 1)}{2} (\theta - 1) + O(|\theta - 1|^3) \\
q_{2n,2}(\theta) &= 1 + O(|\theta - 1|^2) \\
z_{2n+1,2}(\theta) &= 2n + 1 - 2(2n + 1)(\theta - 1) + O(|\theta - 1|^2) \\
p_{2n+1,2}(\theta) &= 1 + O(|\theta - 1|) \\
q_{2n+1,2}(\theta) &= 0 + O(|\theta - 1|)
\end{align*}

Finally, we present the principal result of this section in the following lemma.

Theorem 72. Suppose that $N$ is a positive integer and suppose further that $B$ is given by (279) – (283). Then

\[
\lim_{\theta \to 1} B_{\ell,j}(\theta) = \begin{cases}
\begin{bmatrix}
-1/2 & 0 \\
0 & -1/2
\end{bmatrix} & \ell = j = 2m \\
\begin{bmatrix}
0 & -1/2 \\
-1/2 & 0
\end{bmatrix} & \ell = j = 2m + 1 \\
\begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix} & \text{otherwise}
\end{cases}
\]

for all $\ell, j = 0, 1, 2, \ldots N$.


