In this paper, we present a new numerical method for the solution of linear two-point boundary value problems of ordinary differential equations. After reducing the differential equation to a second kind integral equation, we discretize the latter via a high order Nyström scheme. A somewhat involved analytical apparatus is then constructed which allows for the solution of the discrete system using $O(N \cdot p^2)$ operations, where N is the number of nodes on the interval and p is the desired order of convergence. Thus, the advantages of the integral equation formulation (small condition number, insensitivity to boundary layers, insensitivity to end-point singularities, etc.) are retained, while achieving a computational efficiency previously available only to finite difference or finite element methods.

On the Numerical Solution of Two-Point Boundary Value Problems

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I. Introduction

Second kind integral equations have been a popular analytical tool in the study of ordinary differential equations for nearly a century. When boundary value problems are being considered, the integral equations which arise are of the Fredholm type. From an abstract viewpoint, the advantage of this formulation is that many properties of the solution are readily apparent. From a computational viewpoint, the advantage of this formulation is that the linear systems which arise from discretization are generally well-conditioned. An ill-behaved differential equation, such as a high order Bessel equation, can often be reduced to a perfectly tractable integral equation by means of an appropriate choice of the "background" Green's function (see Example 2 in Section 5 below). Standard finite difference and finite element methods, on the other hand, which discretize the original differential equation, encounter serious numerical difficulties when the solution possesses derivatives of large magnitude (boundary layers). A second advantage is that there exist extremely stable, high order numerical methods for the solution of second kind Fredholm equations, while the order of convergence of most practical schemes for the solution of ordinary differential equations tends to be limited, even if Richardson extrapolation and deferred correction approaches are considered.

Despite all these advantages, integral equations are virtually never used as a numerical tool for the solution of two-point boundary value problems, since their discretization leads to dense systems of linear algebraic equations, and the solution of a dense linear system of dimension N requires order $O(N^3)$ arithmetic operations. Finite difference and finite element schemes lead to banded systems of linear algebraic equations, and the solution of the latter requires order $O(N)$ arithmetic operations, where N is the dimension of the problem. This makes the use of integral equations extremely unattractive as a numerical tool, despite their superior analytical properties. A similar difficulty is encountered when spectral methods are applied to boundary value problems. They yield high order accuracy, but result in dense systems of linear algebraic equations.

In [9], it is observed that while the integral operators of one-dimensional potential theory are dense, they can be applied to arbitrary functions in a "fast" manner (for a cost proportional to the number of nodes N). This observation is then used to construct iterative schemes for the numerical solution of second kind Fredholm equations resulting from two-point boundary value problems, with an asymptotic CPU time estimate proportional to N. Unfortunately, the number of iterations required by the resulting procedure depends on the problem being solved (the usual drawback of iterative schemes), leading in many cases to excessive computation times. A different approach, using Chebyshev polynomials is described in [8]. A method is constructed which solves the corresponding integral equation directly. It requires an amount of work of the order $O(N \log N)$, but applies only when the differential operator has constant coeffficients.

In this paper, we extend the results of [9] and [8] by showing that not only the integral operators of one-dimensional potential theory but also their inverses can be applied numerically to arbitrary functions for a cost proportional to N. This observation is used, in conjunction with a p^{th} order Nyström scheme, to construct a fast algorithm for the solution of the original

differential equation. While the asymptotic CPU time estimate is proportional to p^2N, the algorithm retains the flexibility and stability expected from second kind Fredholm equations.

The plan of this paper is as follows: in Section 2 we summarize the relevant properties of Green's functions for second order differential equations, in Section 3 we develop the analytical apparatus to be used, and in Section 4 we describe the numerical scheme itself. The performance of the method is illustrated in Section 5 with practical examples. Our conclusions and several generalizations are discussed in Section 6. Finally, in Appendix A, we describe the relevant portions of the theory of Chebyshev approximation and quadrature.

The algorithm of this paper is based on a set of simple observations, but involves a large amount of notation and a modest amount of algebraic manipulation. For the sake of clarity, we will attempt to use two levels of description throughout, one cursory and qualitative and the other detailed and rigorous.

II. Mathematical and Numerical Preliminaries

In this section we summarize several classical results.

2.1. Green's functions for second order ordinary differential equations.

We consider the problem of determining a function ϕ in $C^2[a,c]$ which satisfies a second order differential equation

$$\phi''(x) + p(x) \cdot \phi'(x) + q(x) \cdot \phi(x) = f(x) \tag{1}$$

on the interval $[a,c] \subset \mathbf{R}$, where $p, q : (a,c) \rightarrow \mathbf{R}$ are continuous functions, subject to boundary conditions of the form

$$\zeta_{11} \cdot \phi(a) + \zeta_{12} \cdot \phi'(a) = \epsilon_1, \tag{2}$$

$$\zeta_{21} \cdot \phi(c) + \zeta_{22} \cdot \phi'(c) = \epsilon_2 \ . \tag{3}$$

As is well-known, the original equation (1) with inhomogeneous boundary conditions of the form (2), (3) can be reduced to one with homogenous boundary conditions

$$\zeta_{11} \cdot \phi(a) + \zeta_{12} \cdot \phi'(a) = 0, \tag{4}$$

$$\zeta_{21} \cdot \phi(c) + \zeta_{22} \cdot \phi'(c) = 0, \tag{5}$$

by the addition of an appropriate linear function to the unknown function ϕ. We will, therefore, assume that the problem to be solved has been provided in the form (1), (4), (5).

Let us now consider the equation

$$\phi''(x) + p_0(x) \cdot \phi'(x) + q_0(x) \cdot \phi(x) = 0 \tag{6}$$

with the functions $p_0, q_0 \in C^1(a,c)$, and denote by G_0 the Green's function for equation (6) with the boundary conditions (4) and (5). The standard procedure for converting a two-point

boundary value problem into a second kind integral equation then consists in representing the solution ϕ of the problem (1), (4), (5) by the formula

$$\phi(x) = \int_a^c G_0(x,t) \cdot \sigma(t)\, dt \tag{7}$$

with $\sigma : [0,1] \to \mathbf{R}$ a new unknown function to be determined. Substituting (7) into (1), we obtain the desired integral equation

$$\begin{aligned} \sigma(x) \quad &+ \quad [p(x) - p_0(x)] \cdot \int_a^c G_1(x,t)\sigma(t)\, dt \\ &+ \quad [q(x) - q_0(x)] \cdot \int_a^c G_0(x,t)\sigma(t)\, dt = f(x) \end{aligned} \tag{8}$$

where the function $G_1 : [a,c] \times [a,c] \to \mathbf{R}$ is defined by the formula

$$G_1(x,t) = \frac{d}{dx} G_0(x,t) \tag{9}$$

for all $(x,t) \in [a,c] \times [a,c]$.

Of course, if $p_0(x) = p(x)$ and $q_0(x) = q(x)$, then the solution to equation (8) is trivially $\sigma = f$. Our working assumption is that for some functions p_0, q_0, the the Green's function is known or computable, but that for the original differential equation the Green's function is unavailable. Fortunately, there is a well known mechanism for constructing Green's functions from known independent solutions of a differential equation. This construction, described in the following theorem, is the principal analytical tool of the paper. For a proof, see [3].

Theorem 2.1 *Suppose that $u_l, u_r : [a,c] \to \mathbf{R}$ are two linearly independent solutions of the equation (6), such that u_l satisfies the boundary condition (4) and u_r satisfies the condition (5). Suppose further that the functions $u_l', u_r', v_l, v_r : [a,c] \to \mathbf{R}$ are defined by the formulae*

$$\begin{aligned} u_l'(t) &= \frac{d}{dt} u_l(t), \\ u_r'(t) &= \frac{d}{dt} u_r(t), \\ v_l(t) &= \frac{u_l(t)}{u_r'(t) \cdot u_l(t) - u_l'(t) \cdot u_r(t)}, \\ v_r(t) &= \frac{u_r(t)}{u_r'(t) \cdot u_l(t) - u_l'(t) \cdot u_r(t)}. \end{aligned} \tag{10}$$

Then the Green's function and its derivative for equation (1) with the boundary conditions (4) and (5) are given by

$$\begin{aligned} G_0(x,t) &= u_l(x) \cdot v_l(t) \quad \textit{for } t \le x, \\ G_0(x,t) &= u_r(x) \cdot v_r(t) \quad \textit{for } x \le t, \end{aligned} \tag{11}$$

$$
\begin{aligned}
G_1(x,t) &= u_l'(x) \cdot v_l(t) \quad for\ t \leq x, \\
G_1(x,t) &= u_r'(x) \cdot v_r(t) \quad for\ x \leq t.
\end{aligned} \tag{12}
$$

2.2. A Lemma from Linear Algebra.

The following lemma provides analytical inverses for rank one perturbations of the identity matrix. It is a particular case of the Sherman-Morrison formula (see, for example, [6]) and is easy to verify directly.

Lemma 2.1 *For any two vectors $U, V \in L^2$ such that $(U,V) \neq 1$,*

$$
(I - U \circ V^T)^{-1} = I + \frac{1}{1-(U,V)} \cdot U \circ V^T. \tag{13}
$$

III. The Analytical Apparatus

In the remainder of this paper, we assume that the solution to the differential equation (1) is being sought on the interval $[a,c]$ and that b is some intermediate point ($a < b < c$). The fundamental observation on which the fast algorithm is based is that the solution to the integral equation (8) on the entire domain $[a,c]$ can easily be constructed from the solution of two independent integral equations, one defined on $[a,b]$ and one on $[b,c]$. This leads naturally to a recursive algorithm, in which independent solutions on a large number of subintervals are successively merged until the full solution is obtained. A precise formulation of the construction and the resulting numerical scheme will require some notation.

3.1. Notation.

We will denote the subintervals $[a,b]$ and $[b,c]$ of $[a,c]$ by A and B, respectively. For convenience, we write the integral equation (8) in the form

$$
\sigma(x) + \tilde{p}(x) \cdot \int_a^c G_1(x,t)\sigma(t)\,dt + \tilde{q}(x) \cdot \int_a^c G_0(x,t)\sigma(t)\,dt = f(x)
$$

where $\tilde{p}(x) = p(x) - p_0(x)$ and $\tilde{q}(x) = q(x) - q_0(x)$. The functions $G_0, G_1 : [a,c] \times [a,c] \rightarrow \mathbf{R}$ are the Green's function and its derivative defined by formulae (11) and (12).

We define the operator $P : L^2[a,c] \rightarrow L^2[a,c]$ by

$$
P(\sigma)(x) = \sigma(x) + \tilde{p}(x) \cdot \int_a^c G_1(x,t) \cdot \sigma(t)\,dt + \tilde{q}(x) \cdot \int_a^c G_0(x,t) \cdot \sigma(t)\,dt\ , \tag{14}
$$

so that equation (8) assumes the form

$$
P\sigma = f\ . \tag{15}
$$

We will require the four operators

$$
\begin{aligned}
P_{AA} &: L^2[a,b] \rightarrow L^2[a,b]\ , \\
P_{AB} &: L^2[b,c] \rightarrow L^2[a,b]\ , \\
P_{BA} &: L^2[a,b] \rightarrow L^2[b,c]\ , \\
P_{BB} &: L^2[b,c] \rightarrow L^2[b,c]
\end{aligned}
$$

defined by

$$P_{AA}(\sigma)(x) = \sigma(x) + \tilde{p}(x) \cdot \int_a^b G_1(x,t) \cdot \sigma(t)\, dt + \tilde{q}(x) \cdot \int_a^b G_0(x,t) \cdot \sigma(t)\, dt\ , \tag{16}$$

$$P_{AB}(\sigma)(x) = \tilde{p}(x) \cdot \int_b^c G_1(x,t) \cdot \sigma(t)\, dt + \tilde{q}(x) \cdot \int_b^c G_0(x,t) \cdot \sigma(t)\, dt\ , \tag{17}$$

$$P_{BA}(\sigma)(x) = \tilde{p}(x) \cdot \int_a^b G_1(x,t) \cdot \sigma(t)\, dt + \tilde{q}(x) \cdot \int_a^b G_0(x,t) \cdot \sigma(t)\, dt\ , \tag{18}$$

$$P_{BB}(\sigma)(x) = \sigma(x) + \tilde{p}(x) \cdot \int_b^c G_1(x,t) \cdot \sigma(t)\, dt + \tilde{q}(x) \cdot \int_b^c G_0(x,t) \cdot \sigma(t)\, dt. \tag{19}$$

We will also require the functions ψ_l and ψ_r defined by

$$\psi_l(x) = \tilde{p}(x) \cdot u_l'(x) + \tilde{q}(x) \cdot u_l(x) \tag{20}$$

$$\psi_r(x) = \tilde{p}(x) \cdot u_r'(x) + \tilde{q}(x) \cdot u_r(x)\ . \tag{21}$$

Given a function $f \in L^2[a,c]$, we will follow the convention of denoting its restriction to A and B by $f_{|A}$ and $f_{|B}$ respectively. Assuming that the operators P, P_{AA}, P_{BB} are non-singular, we then define the mappings

$$\begin{aligned} \chi_l, \chi_r &: [a,c] \to \mathbf{R}\ , \\ \phi_{l_A}, \phi_{r_A} &: A \to \mathbf{R}\ , \\ \phi_{l_B}, \phi_{r_B} &: B \to \mathbf{R} \end{aligned}$$

via the formulae

$$\chi_l = P^{-1}(\psi_l), \tag{22}$$

$$\chi_r = P^{-1}(\psi_r), \tag{23}$$

$$\begin{aligned} \phi_{l_A} &= P_{AA}^{-1}(\psi_{l_{|A}}), \\ \phi_{r_A} &= P_{AA}^{-1}(\psi_{r_{|A}}), \\ \phi_{l_B} &= P_{BB}^{-1}(\psi_{l_{|B}}), \\ \phi_{r_B} &= P_{BB}^{-1}(\psi_{r_{|B}}). \end{aligned} \tag{24}$$

Finally, we will define three 2×2 matrices α^A, α^B and α by the formulae

$$\begin{aligned} \alpha_{11}^A &= (v_{l_{|A}}, \phi_{l_A}) \quad , \qquad \alpha_{12}^A = (v_{l_{|A}}, \phi_{r_A})\ , \\ \alpha_{21}^A &= (v_{r_{|A}}, \phi_{l_A}) \quad , \qquad \alpha_{22}^A = (v_{r_{|A}}, \phi_{r_A})\ , \end{aligned} \tag{25}$$

$$\begin{aligned} \alpha_{11}^B &= (v_{l_{|B}}, \phi_{l_B}) \quad , \qquad \alpha_{12}^B = (v_{l_{|B}}, \phi_{r_B})\ , \\ \alpha_{21}^B &= (v_{r_{|B}}, \phi_{l_B}) \quad , \qquad \alpha_{22}^B = (v_{r_{|B}}, \phi_{r_B})\ , \end{aligned} \tag{26}$$

$$\begin{aligned} \alpha_{11} &= (v_l, \chi_l) \quad , \qquad \alpha_{12} = (v_l, \chi_r), \\ \alpha_{21} &= (v_r, \chi_l) \quad , \qquad \alpha_{22} = (v_r, \chi_r) \ , \end{aligned} \tag{27}$$

and the coefficients δ_l, δ_r by

$$\begin{aligned} \delta_l &= (v_l, \sigma), \\ \delta_r &= (v_r, \sigma), \end{aligned} \tag{28}$$

where σ is the solution to equation (15).

3.2. Analysis of the operators P_{AB}, P_{BA}.

In this subsection, we observe that each of the operators P_{AB} and P_{BA} is of rank one and give simple expressions for these operators as outer products.

Lemma 3.1 *In the notation of the preceding subsection,*

$$P_{AB} = \psi_{r_{|A}} \circ v_{r_{|B}}^T, \tag{29}$$

$$P_{BA} = \psi_{l_{|B}} \circ v_{l_{|A}}^T. \tag{30}$$

Proof. Combining expressions (11), (12) and (17), and observing that $x \leq t$ for any $x \in [a,b]$, $t \in [b,c]$, we have for any $\sigma \in L^2[b,c]$,

$$\begin{aligned} P_{AB}(\sigma)(x) &= \tilde{p}(x) \cdot \int_b^c G_1(x,t) \cdot \sigma(t)\, dt + \tilde{q}(x) \cdot \int_b^c G_0(x,t) \cdot \sigma(t)\, dt \\ &= \tilde{p}(x) \cdot \int_b^c u_r'(x) \cdot v_r(t) \cdot \sigma(t)\, dt + \tilde{q}(x) \cdot \int_b^c u_r(x) \cdot v_r(t) \cdot \sigma(t)\, dt \\ &= (\tilde{p}(x) \cdot u_r'(x) + \tilde{q}(x) \cdot u_r(x)) \cdot (v_{r_{|B}}, \sigma). \end{aligned} \tag{31}$$

The result (29) now follows from the definition of ψ_r in equation (21).

Similarly, combining expressions (11), (12) and (18), we observe that for any $\sigma \in L^2[b,c]$ and $x \in [b,c]$,

$$\begin{aligned} P_{BA}(\sigma)(x) &= \tilde{p}(x) \cdot \int_a^b G_1(x,t) \cdot \sigma(t)\, dt + \tilde{q}(x) \cdot \int_a^b G_0(x,t) \cdot \sigma(t)\, dt \\ &= \tilde{p}(x) \cdot \int_a^b u_l'(x) \cdot v_l(t) \cdot \sigma(t)\, dt + \tilde{q}(x) \cdot \int_a^b u_l(x) \cdot v_l(t) \cdot \sigma(t)\, dt \\ &= (\tilde{p}(x) \cdot u_l'(x) + \tilde{q}(x) \cdot u_l(x)) \cdot (v_{l_{|A}}, \sigma). \end{aligned} \tag{32}$$

The result (30) follows from the definition of ψ_l in equation (20). □

3.3. Recursive solution of the integral equation (15)

We now consider the original integral equation (15)

$$P\sigma = f$$

and two auxilliary equations

$$P_{AA}(\eta_A) = f_{|A} \tag{33}$$

$$P_{BB}(\eta_B) = f_{|B}\ . \tag{34}$$

The main result of this subsection is the following lemma, which constructs the solution σ of equation (15) from the solutions η_A, η_B of equations (33) and (34).

Lemma 3.2 *If, in the notation of Subsection 3.1, all three operators P, P_{AA}, P_{BB} are non-singular, then*

$$\sigma_{|A} = \eta_A - (\delta_r^B - \frac{\alpha_{21}^B}{\Delta} \cdot (\delta_l^A - \alpha_{12}^A \cdot \delta_r^B)) \cdot \phi_{r_A}, \tag{35}$$

$$\sigma_{|B} = \eta_B - (\delta_l^A - \frac{\alpha_{12}^A}{\Delta} \cdot (\delta_r^B - \alpha_{21}^B \cdot \delta_l^A)) \cdot \phi_{l_B}, \tag{36}$$

where the real numbers δ_l^A, δ_l^B, δ_r^A, δ_r^B, Δ are given by

$$\begin{aligned} \delta_l^A &= (v_{l_{|A}}, \eta_A), \\ \delta_l^B &= (v_{l_{|B}}, \eta_B), \\ \delta_r^A &= (v_{r_{|A}}, \eta_A), \\ \delta_r^B &= (v_{r_{|B}}, \eta_B), \end{aligned} \tag{37}$$

$$\Delta = 1 - \alpha_{21}^B \cdot \alpha_{12}^A. \tag{38}$$

Proof. Using definitions (14) - (19), the integral equation

$$P\sigma = f$$

can be rewritten in the form

$$P_{AA}(\sigma_{|A}) + P_{AB}(\sigma_{|B}) = f_{|A}, \tag{39}$$

$$P_{BA}(\sigma_{|A}) + P_{BB}(\sigma_{|B}) = f_{|B}. \tag{40}$$

The outer product expansions (29) and (30) for P_{AB} and P_{BA}, respectively, can then be used to obtain an explicit solution to the coupled equations (39) and (40) in terms of the functions η_A, η_B, ϕ_{l_A}, ϕ_{r_A}, ϕ_{l_B} and ϕ_{r_B}. Indeed, applying the operator P_{AA}^{-1} to equation (39) and the operator P_{BB}^{-1} to equation (40), we have

$$\sigma_{|A} + P_{AA}^{-1} \circ P_{AB}(\sigma_{|B}) = P_{AA}^{-1}(f_{|A}), \tag{41}$$

$$P_{BB}^{-1} \circ P_{BA}(\sigma_{|A}) + \sigma_{|B} = P_{BB}^{-1}(f_{|B}). \tag{42}$$

Substituting the outer product expansions (29) and (30) into (41) and (42) yields

$$\sigma_{|A} + P_{AA}^{-1} \circ \psi_{r_{|A}} \circ v_{r_{|B}}^T \circ \sigma_{|B} = \eta_A, \tag{43}$$

$$P_{BB}^{-1} \circ \psi_{l_{|B}} \circ v_{l_{|A}}^T \circ \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{44}$$

or

$$\sigma_{|A} + \phi_{r_A} \circ v_{r_{|B}}^T \circ \sigma_{|B} = \eta_A, \tag{45}$$

$$\phi_{l_B} \circ v_{l_{|A}}^T \circ \sigma_{|A} + \sigma_{|B} = \eta_B, \tag{46}$$

where we have used the definitions (24) for ϕ_{r_A} and ϕ_{l_B}. Now, multiplying (46) by $\phi_{r_A} \circ v_{r_{|B}}^T$ and subtracting it from (45), we obtain

$$(I - \phi_{r_A} \circ v_{r_{|B}}^T \circ \phi_{l_B} \circ v_{l_{|A}}^T)\, \sigma_{|A} = \eta_A - \phi_{r_A} \circ v_{r_{|B}}^T \circ \eta_B. \tag{47}$$

Similarly, multiplying (45) by $\phi_{l_B} \circ v_{l_{|A}}^T$ and subtracting it from (46), we get

$$(I - \phi_{l_B} \circ v_{l_{|A}}^T \circ \phi_{r_A} \circ v_{r_{|B}}^T)\, \sigma_{|B} = \eta_B - \phi^{l_B} \circ v_{l_{|A}}^T \circ \eta_A. \tag{48}$$

Due to (25) and (26), we can rewrite these equations in the form

$$(I - \alpha_{21}^B \cdot \phi_{r_A} \circ v_{l_{|A}}^T)\, \sigma_{|A} = \eta_A - \phi_{r_A} \circ v_{r_{|B}}^T \circ \eta_B, \tag{49}$$

$$(I - \alpha_{12}^A \cdot \phi_{l_B} \circ v_{r_{|A}}^T)\, \sigma_{|B} = \eta_B - \phi_{l_B} \circ v_{l_{|A}}^T \circ \eta_A. \tag{50}$$

By application of Lemma 2.1, we obtain

$$\sigma_{|A} = (I + \frac{\alpha_{21}^B}{\Delta} \cdot \phi_{r_A} \circ v_{l_{|A}}^T)\, (\eta_A - \phi_{r_A} \circ v_{r_{|B}}^T \circ \eta_B), \tag{51}$$

$$\sigma_{|B} = (I + \frac{\alpha_{12}^A}{\Delta} \cdot \phi_{l_B} \circ v_{r_{|B}}^T)\, (\eta_B - \phi_{l_B} \circ v_{l_{|A}}^T \circ \eta_A). \tag{52}$$

The results (35) and (36) now follow from simple algebraic manipulation using equations (25), (26) and (37). □

Remark 3.1. Suppose that b_1 and b_2 are a pair of real numbers such that $a < b_1 < b_2 < c$, and that the interval $[b_1, b_2]$ is denoted by C. We will denote by P_{CC} the restriction of the operator P to the interval C. Assuming that P_{CC} is invertible, we define the functions $\eta_C, \phi_{l_C}, \phi_{r_C} : C \to \mathbf{R}$ by

$$\eta_C = P_{CC}^{-1}(f_{|C}) \tag{53}$$

$$\phi_{l_C} = P_{CC}^{-1}(\psi_{l_{|C}})\,, \tag{54}$$

$$\phi_{r_C} = P_{CC}^{-1}(\psi_{r_{|C}})\,. \tag{55}$$

By applying the above lemma twice (once for the subinterval $[a, b_1]$ and once for $[a, b_2]$), we may easily observe that there exist two real numbers λ_1, λ_2 such that

$$\sigma(x) = \eta_C(x) + \lambda_1 \cdot \phi_{l_C}(x) + \lambda_2 \cdot \phi_{r_C}(x) \tag{56}$$

for all $x \in C$. The exact expressions for the coefficients λ_1, λ_2 are complicated, but irrelevant for the purposes of this paper. The existence of a relation of the form (56), however, will be critically important in Section 4.

3.4. Further Analytical Results

We now collect a number of identities which are necessary for the algorithm to be presented in Section 4. First, we apply Lemma 3.2 to the particular cases $f = \psi_l$, $f = \psi_r$ to obtain analytical expressions for the functions χ_l and χ_r defined in equations (22) and (23).

Corollary 3.1 *If, in the notation of Subsection 3.1, all three operators P, P_{AA}, P_{BB} are non-singular, then*

$$\chi_{l_{|A}} = \phi_{l_A} - \frac{(1-\alpha_{11}^A)\cdot\alpha_{21}^B}{\Delta}\cdot\phi_{r_A}, \tag{57}$$

$$\chi_{l_{|B}} = (1 - \frac{\alpha_{11}^A - \alpha_{12}^A\cdot\alpha_{21}^B}{\Delta})\cdot\phi_{l_B} = \frac{1-\alpha_{11}^A}{\Delta}\cdot\phi_{l_B}. \tag{58}$$

$$\chi_{r_{|A}} = (1 - \frac{\alpha_{22}^B - \alpha_{12}^A\cdot\alpha_{21}^B}{\Delta})\cdot\phi_{r_A}^2 = \frac{1-\alpha_{22}^B}{\Delta}\cdot\phi_{r_A}, \tag{59}$$

$$\chi_{r_{|B}} = \phi_{r_B} - \frac{(1-\alpha_{22}^B)\cdot\alpha_{12}^A}{\Delta}\cdot\phi_{l_B}\ , \tag{60}$$

where the coefficients α_{ij}^A and α_{ij}^B are given by equations (25) and (26).

We will also require analytical expressions for the inner products δ_l and δ_r defined in (28) in terms of the restricted inner products $\delta_l^A, \delta_l^B, \delta_r^A$ and δ_r^B defined in (37).

Corollary 3.2 *If, in the notation of Subsection 3.1, all three operators P, P_{AA}, P_{BB} are non-singular, then*

$$\begin{aligned}\delta_l &= (v_l, \sigma) = (v_{l_{|A}}, \sigma_{|A}) + (v_{l_{|B}}, \sigma_{|B}) \\ &= \frac{1-\alpha_{11}^B}{\Delta}\cdot\delta_l^A + \delta_l^B + \frac{(\alpha_{11}^B - 1)\cdot\alpha_{12}^A}{\Delta}\cdot\delta_r^B,\end{aligned} \tag{61}$$

$$\begin{aligned}\delta_r &= (v_r, \sigma) = (v_{r_{|A}}, \sigma_{|A}) + (v_{r_{|B}}, \sigma_{|B}) \\ &= \frac{1-\alpha_{22}^A}{\Delta}\cdot\delta_r^B + \delta_r^A + \frac{(\alpha_{22}^A - 1)\cdot\alpha_{21}^B}{\Delta}\cdot\delta_l^A\ .\end{aligned} \tag{62}$$

Proof. Multiplying equation (35) by $v_{l_{|A}}$ and $v_{r_{|A}}$, and equation (36) by $v_{l_{|B}}$ and $v_{r_{|B}}$, we obtain

$$(v_{l_{|A}}, \sigma_{|A}) = \frac{1}{\Delta} \cdot \delta_l^A - \frac{\alpha_{12}^A}{\Delta} \cdot \delta_r^B, \tag{63}$$

$$(v_{l_{|B}}, \sigma_{|B}) = \delta_l^B - \frac{\alpha_{11}^B}{\Delta} \cdot \delta_l^A + \frac{\alpha_{11}^B \cdot \alpha_{12}^A}{\Delta} \cdot \delta_r^B, \tag{64}$$

$$(v_{r_{|A}}, \sigma_{|A}) = \delta_r^A - \frac{\alpha_{22}^A}{\Delta} \cdot \delta_r^B + \frac{\alpha_{22}^A \cdot \alpha_{21}^B}{\Delta} \cdot \delta_l^A, \tag{65}$$

$$(v_{r_{|B}}, \sigma_{|B}) = \frac{1}{\Delta} \cdot \delta_r^B - \frac{\alpha_{21}^B}{\Delta} \cdot \delta_l^A. \tag{66}$$

Adding up the first pair of equations (63) and (64), we obtain the result (61). Adding up the second pair of equations (65) and (66), we obtain (62). □

A special case of Corollary 3.2 is obtained when $f = \psi^l$ or $f = \psi_r$. The proof follows easily from the definitions of χ_l and χ_r in (22) and (23).

Corollary 3.3 *If, in the notation of Subsection 3.1, all three operators P, P_{AA}, P_{BB} are non-singular, then*

$$\alpha_{11} = (v_l, \chi_l) = \frac{(1 - \alpha_{11}^B) \cdot (\alpha_{11}^A - \alpha_{12}^A \cdot \alpha_{21}^B)}{\Delta} + \alpha_{11}^B, \tag{67}$$

$$\alpha_{21} = (v_r, \chi_l) = \frac{\alpha_{21}^B \cdot (1 - \alpha_{22}^A) \cdot (1 - \alpha_{11}^A)}{\Delta} + \alpha_{21}^A. \tag{68}$$

$$\alpha_{12} = (v_l, \chi_r) = \frac{\alpha_{12}^A \cdot (1 - \alpha_{22}^B) \cdot (1 - \alpha_{11}^B)}{\Delta} + \alpha_{12}^B, \tag{69}$$

$$\alpha_{22} = (v_l, \chi_r) = \frac{(1 - \alpha_{22}^A) \cdot (\alpha_{22}^B - \alpha_{12}^A \cdot \alpha_{21}^B)}{\Delta} + \alpha_{22}^A. \tag{70}$$

Finally, combining Lemma 3.2 with the expressions (57)-(60), we have

Corollary 3.4 *Suppose that in the notation of Subsection 3.1, all three operators P, P_{AA}, P_{BB} are non-singular. Suppose further that the function $F : [a, c] \to \mathbf{R}$ is defined by the formula*

$$F(x) = \lambda_1 \cdot \chi_l(x) + \lambda_2 \cdot \chi_r(x) + \lambda_3 \cdot \sigma(x). \tag{71}$$

Then on the interval $[a, b]$,

$$F(x) = \mu_1 \cdot \phi_{l_A}(x) + \mu_2 \cdot \phi_{r_A}(x) + \mu_3 \cdot \eta_A(x). \tag{72}$$

with the coefficients μ_1, μ_2, μ_3 defined by the formulae

$$\begin{aligned} \mu_1 &= \lambda_1, \\ \mu_2 &= -\frac{(1 - \alpha_{11}^A) \cdot \alpha_{21}^B}{\Delta} \cdot \lambda_1 + \frac{1 - \alpha_{22}^B}{\Delta} \cdot \lambda_2 \\ &+ \left(\frac{\alpha_{21}^B}{\Delta} \cdot (\delta_l^A - \alpha_{12}^A \cdot \delta_r^B) - \delta_r^B\right) \cdot \lambda_3, \\ \mu_3 &= \lambda_3. \end{aligned} \tag{73}$$

Similarly, on the interval $[b,c]$,

$$F(x) = \nu_1 \cdot \phi_{l_B}(x) + \nu_2 \cdot \phi_{r_B}(x) + \nu_3 \cdot \eta_B(x), \tag{74}$$

with the coefficients ν_1, ν_2, ν_3 *defined by the formulae*

$$\begin{aligned} \nu_1 &= -\frac{(1-\alpha_{11}^A)}{\Delta} \cdot \lambda_1 + \frac{(1-\alpha_{22}^B)\cdot\alpha_{12}^A}{\Delta} \cdot \lambda_2 \\ &+ (\frac{\alpha_{12}^A}{\Delta} \cdot (\delta_r^B - \alpha_{21}^B \cdot \delta_l^A) - \delta_l^A) \cdot \lambda_3, \\ \nu_2 &= \lambda_2, \\ \nu_3 &= \lambda_3. \end{aligned} \tag{75}$$

IV. Description of the Algorithm

We turn now to the construction of the fast algorithm for the solution of the integral equation (15)

$$P\sigma = f \ ,$$

based on the apparatus developed in Section III. The main tool at our disposal is the ability to merge the solutions of restricted versions of the integral equation in adjacent subintervals (Lemma 3.2). As this suggests a recursive procedure, we begin by subdividing the whole interval $[a,c]$, on which the solution to (15) is sought, into a large number of subintervals. For the sake of simplicity, we assume that m is a positive integer and that $M = 2^m$ is the number of subintervals created. The boundary points of the subintervals are then given by a strictly increasing sequence of numbers

$$b_1, b_2, \ldots, b_M, b_{M+1} \tag{76}$$

with $b_1 = a$ and $b_{M+1} = c$. We define

$$B_i^m = [b_i, b_{i+1}] \qquad \text{for } i = 1, \ldots, M \tag{77}$$

and create a hierarchy of intervals B_i^k by recursively merging adjacent pairs. That is, for $k = m-1, \ldots, 1, 0$, we define

$$B_i^k = B_{2i-1}^{k+1} \bigcup B_{2i}^{k+1} \qquad \text{for } i = 1, \ldots, 2^k \ . \tag{78}$$

We will refer to each fixed k as a *level.* We will also refer to the two fine intervals B_{2i-1}^{k+1} and B_{2i}^{k+1} as *children* and to the larger interval B_i^k as a *parent.*

It is obvious that

$$B_i^k = [b_{1+(i-1)\cdot 2^{m-k}}, b_{1+i\cdot 2^{m-k}}] \tag{79}$$

and that for each level k,

$$[a,c] = \bigcup_{i=1}^{2^k} B_i^k \ . \tag{80}$$

4.1. Notation.

Generalizing the notation of Section III, we will denote by $P_{i,k}$ the restriction to the interval B_i^k of the integral operator P, so that

$$\begin{aligned} P_{i,k}(\sigma)(x) &= \sigma(x) + p(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-k}}}^{b_{1+i\cdot 2^{m-k}}} G_1(x,t) \cdot \sigma(t)\, dt \\ &+ q(x) \cdot \int_{b_{1+(i-1)\cdot 2^{m-k}}}^{b_{1+i\cdot 2^{m-k}}} G_0(x,t) \cdot \sigma(t)\, dt \end{aligned} \tag{81}$$

for any $\sigma \in L^2(B_i^k)$. For each B_i^k we will define the functions $\eta_{i,k}$, $\phi_{l_{i,k}}$, $\phi_{r_{i,k}} : B_i^k \to \mathbf{R}$ as the solutions of the equations

$$P_{i,k}(\eta_{i,k})) = f_{|B_i^k}, \tag{82}$$

$$P_{i,k}(\phi_{l_{i,k}}) = \psi_{l_{|B_i^k}}, \tag{83}$$

$$P_{i,k}(\phi_{r_{i,k}}) = \psi_{r_{|B_i^k}}, \tag{84}$$

provided that the operator $P_{i,k}$ is non-singular.

Remark 4.1. Suppose now that the operator $P_{i,k}$ is non-singular on the interval B_i^k. Then, by equation (56), there exist two numbers $\lambda_1^{i,k}, \lambda_2^{i,k} \in \mathbf{R}$ such that

$$\sigma(x) = \eta_{i,k}(x) + \lambda_1^{i,k} \cdot \phi_{l_{i,k}}(x) + \lambda_2^{i,k} \cdot \phi_{r_{i,k}}(x) \tag{85}$$

for all $x \in B_i^k$.

For each $k = 0, 1, \ldots, m$, and $i = 1, 2, \ldots, 2^k$, we define a 2×2 matrix $\alpha^{i,k}$, by

$$\begin{aligned} \alpha_{11}^{i,k} &= (v_{l_{|B_i^k}}, \phi_{l_{i,k}}), \\ \alpha_{21}^{i,k} &= (v_{r_{|B_i^k}}, \phi_{l_{i,k}}), \\ \alpha_{12}^{i,k} &= (v_{l_{|B_i^k}}, \phi_{r_{i,k}}), \\ \alpha_{22}^{i,k} &= (v_{r_{|B_i^k}}, \phi_{r_{i,k}}), \end{aligned} \tag{86}$$

and the vector $\delta^{i,k} = (\delta_l^{i,k}, \delta_r^{i,k})$ by

$$\begin{aligned} \delta_l^{i,k} &= (v_{l_{|B_i^k}}, \eta_{i,k}), \\ \delta_r^{i,k} &= (v_{r_{|B_i^k}}, \eta_{i,k}). \end{aligned} \tag{87}$$

4.1. Discretization of the Restricted Integral Equations.

Choosing an integer $p \geq 1$, we construct the p scaled Chebyshev nodes

$$\tau_i^j = \left(\frac{b_{i+1} - b_i}{2}\right) \cos\left[\frac{(2j-1)\pi}{2p}\right] + \left(\frac{b_{i+1} + b_i}{2}\right) \qquad j = 1, 2, \ldots, p \tag{88}$$

on each of the intervals B_i^m, $i = 1, 2, \ldots, M$. We then discretize the three integral equations (82), (83) and (84) via a Nyström algorithm based on $p-$point Chebyshev quadrature (see Appendix A). The resulting approximations to the functions $\eta_{i,k}$, $\phi_{l_{i,k}}$, $\phi_{r_{i,k}}$ at the nodes τ_i^j will be denoted by

$$\begin{aligned}
\tilde{\eta}_{i,k} &= (\tilde{\eta}_{i,k}^1, \tilde{\eta}_{i,k}^2, \ldots, \tilde{\eta}_{i,k}^p) , \\
\tilde{\phi}_{l_{i,k}} &= (\tilde{\phi}_{l_{i,k}}^1, \tilde{\phi}_{l_{i,k}}^2, \ldots, \tilde{\phi}_{l_{i,k}}^p) , \\
\tilde{\phi}_{r_{i,k}} &= (\tilde{\phi}_{r_{i,k}}^1, \tilde{\phi}_{r_{i,k}}^2, \ldots, \tilde{\phi}_{r_{i,k}}^p) ,
\end{aligned}$$

respectively.

Remark 4.2. It is well-known that the order of convergence of the approximations $\tilde{\eta}_{i,k}, \tilde{\phi}_{l_{i,k}}, \tilde{\phi}_{r_{i,k}}$ to the functions $\eta_{i,k}, \phi_{l_{i,k}}, \phi_{r_{i,k}}$ is p. Since all subsequent steps in the construction of an approximate solution $\tilde{\sigma}$ to the integral equation (15) are analytic, the convergence rate of the full algorithm depends entirely on the parameter p. For example, by using 16 scaled Chebyshev points on each subinterval at the finest level, one obtains a sixteenth order method.

4.2. Informal description of the algorithm.

We begin by directly solving the three integral equations (82), (83) and (84) on each subinterval B_i^m at the finest level, as discussed in the preceding subsection. Equation (85) then shows that σ restricted to B_i^m can be expressed as a linear combination of the three solutions $\eta_{i,m}$, $\phi_{l_{i,m}}$, $\phi_{r_{i,m}}$. Thus, it remains only to determine the two coefficients

$$\lambda_1^{i,m}, \lambda_2^{i,m}$$

for each of the M subintervals B_i^m. Fortunately, this can be done recursively. To see this, suppose that, at some coarse level $k \leq m-1$, we are given the coefficients $\lambda_1^{i,k}, \lambda_2^{i,k}$ for the subinterval B_i^k. Then Corollary 3.4 provides formulas for the calculation of the corresponding coefficients

$$\lambda_1^{2i-1,k+1}, \lambda_2^{2i-1,k+1} \qquad \text{and} \qquad \lambda_1^{2i,k+1}, \lambda_2^{2i,k+1}$$

for the two child intervals B_{2i-1}^{k+1} and B_{2i}^{k+1}, respectively. For initialization, observe that

$$\lambda_1^{1,0} = \lambda_2^{1,0} = 0 \tag{89}$$

(i.e. the solution of equation (82) on the whole interval $[a, c]$ is simply σ).

In order to use the formulas (73 and 75) of Corollary 3.4, however, we need the matrices $\alpha^{2i-1,k+1}, \alpha^{2i,k+1}$ and the vectors $\delta^{2i-1,k+1}, \delta^{2i,k+1}$. These quantities are also computed recursively but in the opposite direction, namely, from the finest level to the coarsest. They are certainly available at level m directly from the definition (86). For the interval B_i^k at any coarser level $k \leq m-1$, Corollaries 3.2 and 3.3 describe how $\alpha^{i,k}$ and $\delta^{i,k}$ are obtained from the matrices α and vectors δ of the two child intervals.

To summarize, the algorithm consists of three parts. First, a sufficiently fine subdivision $b_1, b_2, \ldots, b_{M+1}$ of the interval $[a, c]$ is chosen so that, on each of the intervals $B_{i,m}$, the functions $\eta_{i,m}, \phi_{l_{i,m}}$, and $\phi_{r_{i,m}}$ can be accurately represented by a low order Chebyshev expansion. On

each of the intervals $B_{i,m}$, the equations (82) - (84) are solved (approximately) by direct inversion of the linear system arising from a Nyström discretization. Second, the matrices $\alpha^{i,k}$ and vectors $\delta^{i,k}$ are computed in an upward sweep, beginning at the finest level m. Finally, the coefficients $\lambda_1^{i,k}$ and $\lambda_2^{i,k}$ are computed in a downward sweep, beginning at the coarsest level. The desired function σ is then recovered on each subinterval from equation (85).

The following is a more detailed description of the numerical procedure.

Algorithm

Comment [Define computational grid.]

Create $M = 2^m$ subintervals on $[a, c]$ by choosing a sequence of boundary points $b_1, b_2, \ldots, b_M, b_{M+1}$ with $b_1 = a$ and $b_{M+1} = c$. Choose the number p of Chebyshev nodes on each interval $B_i^m = [b_i, b_{i+1}]$ for $i = 1, \ldots, M$. Determine the locations of the scaled Chebyshev nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ on each interval B_i^m.

Step 1.

Comment [Construct the approximate solutions $\tilde{\eta}_{i,m}, \tilde{\phi}_{l_{i,m}}, \tilde{\phi}_{r_{i,m}}$ of equations (82) - (84) on each interval B_i^m.]

do i $= 1, 2, \ldots, M$

(1) Construct the three $p \times p$ linear systems on B_i^k obtained through a Nyström discretization of the corresponding integral equation (see Appendix A, Section 3).

(2) Solve the three $p \times p$ linear systems on B_i^k by Gaussian elimination, obtaining the values $\tilde{\eta}_{i,m}, \tilde{\phi}_{l_{i,m}}, \tilde{\phi}_{r_{i,m}}$.

enddo

Step 2.

Comment [Construct the coefficients $\alpha_{11}^{i,m}, \alpha_{12}^{i,m}, \alpha_{21}^{i,m}, \alpha_{22}^{i,m}, \delta_l^{i,m}, \delta_r^{i,m}$ on each interval B_i^m at the finest level.]

do i $= 1, 2, \ldots, M$

Evaluate the coefficients $\alpha_{11}^{i,m}, \alpha_{12}^{i,m}, \alpha_{21}^{i,m}, \alpha_{22}^{i,m}, \delta_l^{i,m}, \delta_r^{i,m}$ by applying the $k-$point Chebyshev quadrature formula (Appendix A, Section 2) to the inner product integrals (86), (87).

end do

Step 3 (Upward Sweep).

Comment [Construct the matrices $\alpha^{i,k}$ and the vectors $\delta^{i,k}$ for all intervals at all coarser levels $k = m-1, m-2, \ldots, 0$.]

do k= m-1, 0, -1

do i=1, 2^k

Use formulae (67) - (70) and (61) - (62) to compute the matrix $\alpha^{i,k}$ and the vector $\delta^{i,k}$ from the corresponding data in the two child intervals $(\alpha^{2i-1,k+1}, \alpha^{2i,k+1}, \delta^{2i-1,k+1}, \delta^{2i,k+1})$.

end do

end do

Step 4 (Downward Sweep).

Comment [Construct the coefficients $\lambda_1^{i,m}, \lambda_2^{i,m}$ for all intervals at the finest level.]

Set $\lambda_1^{0,1} = \lambda_2^{0,1} = 0$ (see equation (89) above).

do k=0,m-1
  do i=1, 2^k
    Use Corollary 3.4 to compute the coefficients $\lambda_1^{k+1,2\cdot i-1}, \lambda_2^{k+1,2\cdot i-1}, \lambda_1^{2i,k+1}, \lambda_2^{2i,k+1}$
    for the child intervals B_{2i}^{k+1} and B_{2i-1}^{k+1} from the coefficients $\lambda_1^{i,k}, \lambda_2^{i,k}$ of the parent interval B_i^k.
  end do
end do

Step 5.

Comment [Compute the solution σ of equation (15) at the nodes $\tau_i^1, \tau_i^2, \ldots, \tau_i^p$ for each interval B_i^m at the finest level.]

do i=1, M
  do j=1,p
    Determine the values of the solution σ of equation (15) at the node τ_i^j via formula (85).
  end do
end do

Step 6.

Comment [Compute the solution ϕ of equation (1) and its derivative ϕ' from the values of σ.]

Evaluate the integral (7), and its derivative by using composite Chebyshev quadrature
(see Remark 4.4 below).

Remark 4.3. Inspection of the above algorithm shows that the amount of work required is of the order $O(M \cdot p^3)$. Three $p \times p$ linear systems have to be solved for each of the M intervals B_i^m in Step 1, while Steps 2 - 5 require no more than $O(M \cdot p^2)$ operations. Since $N = M \cdot p$ is the total number of nodes in the discretization of the interval $[a, c]$, we can write the CPU time estimate in the form $O(N \cdot p^2)$. The cost of evaluating the solution ϕ of the differential equation (1) from the integral representation (7) is $O(N \log p)$ (see Remark 4.4 below).

Remark 4.4. The final step in the algorithm involves the evaluation of integrals of the form (7) at each of the Chebyshev nodes τ_i^j on each subinterval B_i^m, namely

$$\phi(\tau_i^j) = \int_a^c G_0(\tau_i^j, t) \cdot \sigma(t)\, dt \tag{90}$$

and

$$\phi'(\tau_i^j) = \int_a^c G_1(\tau_i^j, t) \cdot \sigma(t)\, dt\ . \tag{91}$$

If these integrals were calculated independently for each τ_i^j, the amount of work required would be of the order $O(N^2)$, and would dominate the construction of the function σ. In fact, this is unnecessary, for we may write

$$\begin{aligned}
\phi(\tau_i^j) &= u_l(\tau_i^j)\cdot\left[\int_a^{b_i} v_l(t)\cdot\sigma(t)\,dt + \int_{b_i}^{\tau_i^j} v_l(t)\cdot\sigma(t)\,dt\right] \\
&\quad + u_r(\tau_i^j)\cdot\left[+\int_{\tau_i^j}^{b_{i+1}} v_r(t)\cdot\sigma(t)\,dt + \int_{b_{i+1}}^{c} v_r(t)\cdot\sigma(t)\,dt\right], \qquad (92)\\
\phi'(\tau_i^j) &= u_l'(\tau_i^j)\cdot\left[\int_a^{b_i} v_l(t)\cdot\sigma(t)\,dt + \int_{b_i}^{\tau_i^j} v_l(t)\cdot\sigma(t)\,dt\right] \\
&\quad + u_r'(\tau_i^j)\cdot\left[+\int_{\tau_i^j}^{b_{i+1}} v_r(t)\cdot\sigma(t)\,dt + \int_{b_{i+1}}^{c} v_r(t)\cdot\sigma(t)\,dt\right], \qquad (93)\\
& \qquad (94)
\end{aligned}$$

where we have used the representations (11) and (12) and the fact that τ_i^j lies in the interval $B_i^m = [b_i, b_{i+1}]$. Step 6 can then be written in detail as follows:

Step 6 (a).

Comment [Precompute the integrals of $v_l\cdot\sigma$ and $v_r\cdot\sigma$ on each subinterval B_i^m by Chebyshev quadrature. These integrals will be denoted I_l and I_r, respectively.]

do i=1, M
$I_l(B_i^m) = \int_{b_i}^{b_{i+1}} v_l(t)\cdot\sigma(t)\,dt.$
$I_r(B_i^m) = \int_{b_i}^{b_{i+1}} v_r(t)\cdot\sigma(t)\,dt.$
end do

Step 6 (b).

Comment [March across interval from a to c, computing ϕ and ϕ' at each node in discretization. The varaibles $\mathcal{J}_l$ and $\mathcal{J}_r$ will be used to accumulate the integrals $\int_a^{b_i} v_l(t)\cdot\sigma(t)\,dt$ and $\int_{b_{i+1}}^{c} v_r(t)\cdot\sigma(t)\,dt$, respectively.

Set $\mathcal{J}_r = \sum_{i=2}^{M} I_r(B_i^m)$.
Set $\mathcal{J}_l = 0$.

do i=1, M
do j=1,p
For each τ_i^j, compute
$\phi(\tau_i^j) = u_l(\tau_i^j)\cdot\left[\mathcal{J}_l + \int_{b_i}^{\tau_i^j} v_l(t)\cdot\sigma(t)\,dt\right] + u_r(\tau_i^j)\cdot\left[\int_{\tau_i^j}^{b_{i+1}} v_r(t)\cdot\sigma(t)\,dt + \mathcal{J}_r\right]$
$\phi'(\tau_i^j) = u_l'(\tau_i^j)\cdot\left[\mathcal{J}_l + \int_{b_i}^{\tau_i^j} v_l(t)\cdot\sigma(t)\,dt\right] + u_r'(\tau_i^j)\cdot\left[\int_{\tau_i^j}^{b_{i+1}} v_r(t)\cdot\sigma(t)\,dt + \mathcal{J}_r\right]$
end do

$\mathcal{J}_l = \mathcal{J}_l + I_l(B_i^m)$
$\mathcal{J}_r = \mathcal{J}_r - I_r(B_{i+1}^m)$
end do

Thus, the amount of work required in Step 6(a) is $O(N)$. The integrals required on each subinterval in Step 6 (b) can be computed by spectral integration (see Appendix A, Section 2) using $O(p \log p)$ work. The total cost is therefore of the order $O(Mp \log p)$ or $O(N \log p)$.

V. Numerical Results

FORTRAN programs have been written implementing the algorithm described in the preceding section, for both real and complex valued functions. In this section, we discuss several details of our implementation, and demonstrate the peformance of the scheme with four numerical examples.

The following technical details of our implementation appear to be worth mentioning.

1. Originally, the algorithm was implemented for a fairly wide choice of background equations (6), and corresponding Green's functions (11). Our numerical experiments showed that the advantages of one background Green's function over another tend to be minor, unless the original equation (1) and the equation (6) from which the background Green's function is constructed can be chosen to be extremely close. Therefore, all subsequent implementations and all numerical experiments were performed with equation (6) of the form

$$\phi''(x) = 0. \tag{95}$$

2. The algorithm described in the preceding section requires that the number M of elementary subintervals on the interval $[a, c]$ be a power of 2. Cleary, this is not an essential limitation and it can be removed by simple bookkeeping changes. In the version of the algorithm used for numerical experiments, these changes were made.

3. The algorithm depends for its stability on the equations (82) - (84) having unique solutions for all subintervals B_i^k ($k = 0, 1, \ldots, M$, $i = 1, \ldots, 2^k$.) It is easy to construct examples for which this condition is violated, even though equation (15) has a unique solution. In such cases, a different subdivision of the interval $[a, c]$ can be attempted, such that none of the subintervals B_i^k of the new subdivision coincides with an interval of the original one. This procedure can be viewed as a form of pivoting, and it is easy to show that it is always possible to make it work. It has not been implemented at this point, and we have not so far encountered a need for it.

4. We have, however, implemented a crude scheme for detecting high condition numbers in the algorithm. These can occure in two places: in the solution of the linear systems on each of the finest level subintervals (Step 1), and while merging the solutions on two consecutive subintervals via formulae (61)-(62) and (67)- (70) (Step 3). In the first case, the condition number of the system being solved is estimated in the process of solution (we use a standard

LINPACK routine), and the largest of these is returned to the user. In the second case, the immediate reason for the ill-conditioning is the appearance of small values of the coefficient Δ in formulae (61)-(62) and (67)-(70). The smallest of these is also returned to the user. When an extremely large condition number is detected by the LINPACK routine, or an extremely small Δ is generated in the merging process, the resulting solution of the original ODE should be viewed as suspect. It is easy to show that when the differential operator is positive definite, this cannot happen. A more complete treatment of this subject requires further study.

5. In the upward sweep (Step 3), we evaluate the matrices $\alpha^{i,k}$ for all intervals $B_{i,k}$ and use these matrices to evaluate the vectors δ, the coefficients λ, and, finally, the solution σ of the integral equation (15). But the matrices $\alpha^{i,k}$ do not depend on the right-hand side f of equation (15), and it is easy to see that their evaluation accounts for more than three quarters of the work. Therefore, whenever the equation (15) has to be solved with multiple right-hand sides, we precompute the matrices $\alpha^{i,k}$ and store them, saving about 75% of the cost of the evaluation of subsequent solutions.

The algorithm of this paper has been applied to a variety of problems. Four experiments are described below, and their results are summarized in Tables 1-10. In each of these tables, the first column contains the total number N of nodes in the discretization of the interval $[a, c]$. The second column contains the relative L^2 error of the numerical solution as compared with the analytically obtained one, and the third column contains the maximum absolute error obtained at any node in the discretization. Columns four and five contain the same information for the derivative of the solution (i.e. its relative L^2 and L_∞ errors respectively). Finally, the last column contains the CPU time required to solve the problem. In all cases, the times given are for a SUN 3/60 computer using the 68881 floating point coprocessor.

Example 1. This example is taken from [10], where it is described as a reasonably difficult one due to the presence of rapidly growing solutions of the corresponding homogenous equation. The equation to be solved is

$$\phi'' + 400\phi = -400cos^2(\pi x) - 2\pi^2 cos(2\pi x) \tag{96}$$

with the boundary conditions

$$\phi(0) = \phi(1) = 0. \tag{97}$$

The algorithm has been applied to this problem with $p = 8, 16$ and 24, and the results of this experiment are presented in Tables 1-3.

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (sec.)
8	0.409×10^{-1}	0.316×10^{-1}	0.787×10^{-1}	0.972	0.400×10^{-1}
16	0.302×10^{-2}	0.186×10^{-2}	0.347×10^{-2}	0.371×10^{-1}	0.140×10^{0}
32	0.480×10^{-4}	0.426×10^{-4}	0.691×10^{-4}	0.891×10^{-3}	0.220×10^{0}
64	0.356×10^{-6}	0.513×10^{-6}	0.643×10^{-6}	0.106×10^{-4}	0.400×10^{0}
128	0.179×10^{-8}	0.369×10^{-8}	0.366×10^{-8}	0.782×10^{-7}	0.820×10^{0}
256	0.763×10^{-11}	0.198×10^{-10}	0.162×10^{-10}	0.424×10^{-9}	0.162×10^{1}
512	0.305×10^{-13}	0.918×10^{-13}	0.659×10^{-13}	0.196×10^{-11}	0.318×10^{1}
1024	0.136×10^{-14}	0.130×10^{-14}	0.915×10^{-15}	0.355×10^{-13}	0.638×10^{1}
2048	0.103×10^{-14}	0.171×10^{-14}	0.117×10^{-14}	0.532×10^{-13}	0.126×10^{2}

Table 1: Numerical results for Example 1, $p = 8$.

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (sec.)
16	0.954×10^{-5}	0.659×10^{-5}	0.959×10^{-5}	0.138×10^{-3}	0.200×10^{0}
32	0.457×10^{-8}	0.301×10^{-8}	0.545×10^{-8}	0.602×10^{-7}	0.340×10^{0}
64	0.401×10^{-12}	0.388×10^{-12}	0.581×10^{-12}	0.778×10^{-11}	0.700×10^{0}
128	0.658×10^{-15}	0.139×10^{-14}	0.106×10^{-14}	0.319×10^{-13}	0.134×10^{1}
256	0.626×10^{-15}	0.119×10^{-14}	0.106×10^{-14}	0.426×10^{-13}	0.262×10^{1}
512	0.635×10^{-15}	0.149×10^{-14}	0.934×10^{-15}	0.426×10^{-13}	0.520×10^{1}

Table 2: Numerical results for Example 1, $p = 16$.

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (sec.)
24	0.764×10^{-10}	0.524×10^{-10}	0.539×10^{-10}	0.804×10^{-9}	0.380×10^{0}
48	0.970×10^{-15}	0.155×10^{-14}	0.110×10^{-14}	0.319×10^{-13}	0.720×10^{0}
96	0.851×10^{-15}	0.175×10^{-14}	0.124×10^{-14}	0.319×10^{-13}	0.144×10^{1}

Table 3: Numerical results for Example 1, $p = 24$.

Example 2. The purpose of this example is to demonstrate the performance of the method when the coefficients p, q of the equation (6) are singular at the ends its interval of definition, while the particular solution being sought is smooth. We solve the Bessel equation

$$\phi''(x) + \frac{1}{x} \cdot \phi'(x) + \frac{x^2 - \nu^2}{x^2} = 0 \tag{98}$$

on the interval $[0, 600]$ with the boundary conditions

$$\phi(0) = 0, \tag{99}$$

$$\phi(600) = 1, \tag{100}$$

and $\nu = 100$. The difficulty of this problem is due to the fact that the two linearly independent solutions of equation (98) are $J_\nu(x)$ and $Y_\nu(x)$ (Bessel functions of the first and the second kinds, respectively). As is well known, $J_\nu(x)$ behaves in the vicinity of zero like x^ν, while $Y_\nu(x)$ behaves like $x^{-\nu}$; most methods have trouble finding the decaying solution. In addition, this is a fairly large-scale calculation, since the interval $[0, 600]$ contains almost 100 wavelengths of the solution to (98). The algorithm has been applied to this equation with $p = 16, 20$, and 24. The results are presented in Tables 4-6.

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (*sec.*)
192	0.945	0.103	0.101×10^{1}	0.172×10^{1}	0.194×10^{1}
384	0.651	0.901×10^{-1}	0.658	0.108×10^{1}	0.394×10^{1}
768	0.106×10^{-3}	0.151×10^{-4}	0.106×10^{-3}	0.177×10^{-3}	0.786×10^{1}
1536	0.284×10^{-8}	0.406×10^{-9}	0.285×10^{-8}	0.478×10^{-8}	0.156×10^{2}
3072	0.179×10^{-10}	0.265×10^{-11}	0.177×10^{-10}	0.310×10^{-10}	0.314×10^{2}
6144	0.100×10^{-10}	0.138×10^{-11}	0.101×10^{-10}	0.167×10^{-10}	0.625×10^{2}

Table 4: Numerical results for Example 2, $p = 16$.

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (*sec.*)
240	0.120×10^{1}	0.198	0.111×10^{1}	0.232×10^{1}	0.300×10^{1}
480	0.845×10^{-2}	0.118×10^{-2}	0.851×10^{-2}	0.140×10^{-1}	0.598×10^{1}
960	0.684×10^{-7}	0.979×10^{-8}	0.687×10^{-7}	0.114×10^{-6}	0.119×10^{2}
1920	0.205×10^{-11}	0.302×10^{-12}	0.202×10^{-11}	0.355×10^{-11}	0.239×10^{2}
3840	0.229×10^{-10}	0.325×10^{-11}	0.231×10^{-10}	0.382×10^{-10}	0.480×10^{2}

Table 5: Numerical results for Example 2, $p = 20$.

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (*sec.*)
288	0.889	0.113	0.942	0.155×10^{1}	0.430×10^{1}
576	0.765×10^{-4}	0.108×10^{-4}	0.770×10^{-4}	0.127×10^{-3}	0.866×10^{1}
1152	0.206×10^{-10}	0.295×10^{-11}	0.207×10^{-10}	0.346×10^{-10}	0.176×10^{2}
2304	0.356×10^{-11}	0.503×10^{-12}	0.356×10^{-11}	0.594×10^{-11}	0.343×10^{2}
4608	0.627×10^{-11}	0.856×10^{-12}	0.644×10^{-11}	0.982×10^{-11}	0.688×10^{2}

Table 6: Numerical results for Example 2, $p = 24$.

Remark 5.1. Problems like the preceding one are frequently encountered in the modeling of wave phenomena by means of separation of variables, and were the original motivation for this work.

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (sec.)
240	0.706	0.913	0.907	0.675×10^{6}	0.242×10^{1}
256	0.960×10^{-2}	0.468×10^{-1}	0.455×10^{-1}	0.338×10^{5}	0.256×10^{1}
272	0.701×10^{-4}	0.445×10^{-3}	0.476×10^{-3}	0.265×10^{3}	0.278×10^{1}
288	0.835×10^{-7}	0.635×10^{-6}	0.761×10^{-6}	0.404	0.288×10^{1}
304	0.198×10^{-10}	0.141×10^{-9}	0.200×10^{-9}	0.147×10^{-3}	0.308×10^{1}
320	0.378×10^{-11}	0.233×10^{-10}	0.254×10^{-10}	0.289×10^{-4}	0.320×10^{1}
336	0.610×10^{-11}	0.394×10^{-10}	0.321×10^{-10}	0.431×10^{-4}	0.338×10^{1}

Table 7: Numerical results for Example 3, $p = 16$.

Example 3. We solve a singular perturbation problem of the form

$$\epsilon \cdot \phi''(x) - \phi'(x) = 0, \tag{101}$$

$$\phi(-1) = 1, \tag{102}$$

$$\phi(1) = 2, \tag{103}$$

with $\epsilon = 10^{-6}$. The solution of this problem has an extremely sharp boundary layer near the right end of the interval $[-1, 1]$, causing severe numerical difficulties when most standard algorithms are used. In this case, we construct the intervals $B_i^m = [b_i, b_{i+1}]$ via the formula

$$\begin{aligned} b_1 &= -1\ , \\ b_i &= -1 + \sum_{j=1}^{i-1} \left(\frac{1}{2}\right)^{j-1} \quad \text{for } i = 2, \ldots, M\ , \\ b_{M+1} &= 1\ , \end{aligned} \tag{104}$$

so that they become progressively smaller near the right end of the interval $[-1, 1]$. The results of this experiment are presented in Table 7.

Example 4. Here, we solve a problem of the type which arises when dealing with a frequency domain equation for the vibrating string.

$$\phi''(x) + k^2 \cdot \phi(x) = 5 \cdot \sin(k \cdot x), \tag{105}$$

with Dirichlet boundary conditions

$$\phi(c) = \sin(k \cdot c), \tag{106}$$

$$\phi(d) = \sin(k \cdot d)\ . \tag{107}$$

These boundary conditions correspond to the solution $\phi = sin(k \cdot x)$. In Tables 8-10, we present the results obtained by with our algorithm for $c = -1, d = 1$, and $k = 630$, in order to demonstrate the performance of the method on large-scale oscillatory problems (the string is roughly 200 wavelengths long).

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (sec.)
800	$0.314\ 10^{1}$	$0.402\ 10^{1}$	$0.353\ 10^{1}$	$0.251\ 10^{4}$	$0.820\ 10^{1}$
1600	$0.782\ 10^{-4}$	$0.101\ 10^{-3}$	$0.887\ 10^{-4}$	$0.626\ 10^{-1}$	$0.165\ 10^{2}$
3200	$0.203\ 10^{-8}$	$0.281\ 10^{-8}$	$0.242\ 10^{-8}$	$0.173\ 10^{-5}$	$0.329\ 10^{2}$
6400	$0.153\ 10^{-8}$	$0.187\ 10^{-8}$	$0.158\ 10^{-8}$	$0.110\ 10^{-5}$	$0.664\ 10^{2}$

Table 8: Numerical results for Example 4, $p = 16$.

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (sec.)
600	$0.218\ 10^{1}$	$0.379\ 10^{1}$	$0.215\ 10^{1}$	$0.241\ 10^{4}$	$0.894\ 10^{1}$
1200	$0.839\ 10^{-4}$	$0.104\ 10^{-3}$	$0.918\ 10^{-4}$	$0.697\ 10^{-1}$	$0.179\ 10^{2}$
2400	$0.206\ 10^{-10}$	$0.361\ 10^{-10}$	$0.181\ 10^{-10}$	$0.153\ 10^{-7}$	$0.360\ 10^{2}$
4800	$0.411\ 10^{-10}$	$0.916\ 10^{-10}$	$0.353\ 10^{-10}$	$0.352\ 10^{-7}$	$0.721\ 10^{2}$

Table 9: Numerical results for Example 4, $p = 24$.

n	$E^2(\phi)$	$E^\infty(\phi)$	$E^2(\phi')$	$E^\infty(\phi')$	t (sec.)
800	$0.223\ 10^{-1}$	$0.400\ 10^{-1}$	$0.235\ 10^{-1}$	$0.283\ 10^{2}$	$0.164\ 10^{2}$
1600	$0.607\ 10^{-9}$	$0.837\ 10^{-9}$	$0.664\ 10^{-9}$	$0.522\ 10^{-6}$	$0.328\ 10^{2}$
3200	$0.194\ 10^{-10}$	$0.252\ 10^{-10}$	$0.168\ 10^{-10}$	$0.144\ 10^{-7}$	$0.675\ 10^{2}$
6400	$0.118\ 10^{-9}$	$0.172\ 10^{-9}$	$0.112\ 10^{-9}$	$0.935\ 10^{-7}$	$0.133\ 10^{3}$

Table 10: Numerical results for Example 4, $p = 32$.

The following observations can be made from Tables 1 - 10, and are corroborated by our more extensive experiments.

1. The practical convergence rate of the method is consistent with the theoretical one. For larger p, the exact numerical verification of the order of convergence tends to be difficult, since the precision of calculations is exhausted before the behavior of the scheme becomes asymptotic. However, this is often encountered when dealing with rapidly convergent algorithms.

2. For small-scale problems (such as in Example 1) and large p, the algorithm produces essentially exact results with a small number of nodes. For large-scale problems, double precision accuracy is achieved at approximately 20 nodes per wavelength with $p = 20$, at 12 nodes per wavelength with $p = 24$, and at 10 nodes per wavelength with $p = 32$. The optimal timings are achieved at p between 24 and 32 (provided that about 10 - 12 digits of accuracy are desired). There seems to be no reason for using the scheme with $p < 16$.

3. The scheme is completely indifferent to the extreme stiffness near the left end of the interval $[c, d]$ of equation (98) in the Example 2.

4. It is easy to use the algorithm in an adaptive manner, as demonstrated in Example 2, where we resolve a boundary layer of relative thickness 10^{-6}, without encountering any numerical

difficulties. However, a fully adaptive version of the scheme has not been implemented. The intervals B_i^m in Example 2 were provided by the calling program (as opposed to having been constructed by the algorithm itself).

5. The condition number of a Nyström discretization of a second kind integral equation is asymptotically bounded, and our results reflect this fact. The relatively poor accuracy (10 - 11 digits) obtained in Examples 2, 3 and 4 is due to the ill-conditioning of the original ODE, as opposed to that of the numerical scheme used.

VI. Generalizations and Conclusions

Generalizations: The results of this paper can be generalized in three obvious directions.

1. As described here, the algorithm is only applicable to boundary value problems for a single second order ODE. It can be generalized to systems of ODEs of arbitrary dimension. This generalization is quite straightforward, though technically somewhat involved. This work is currently in progress, and its results will be reported at a later date.
2. The algorithm of this paper can be used as a solver for the linearized problems which arise in applying Newton-type methods to non-linear boundary value problems of ordinary differential equations. This would involve converting the differential equation being solved into a non-linear second kind integral equation, with the subsequent application of an iterative (Newton) algorithm to the latter. In the context of non-linear problems, second kind integral equations retain their usual analytical and numerical advantages over the differential equation formulation. However, our numerical experience with such problems is extremely limited.
3. Attempts have been made to extend the results of this paper to elliptic partial differential equations. While this direction of research appears to be extremely attractive in principle, we have not been able to produce a workable algorithm of this type.

Conclusions: An algorithm has been presented for the solution of two-point boundary value problems of ordinary differential equations. The algorithm is based on reducing the differential equation to a second kind integral equation, with the subsequent solution of the latter via a Nyström type scheme. It has CPU time requirements proportional to $N \cdot p^2$, where N is the number of nodes in the discretization of the interval of definition of the equation, and p is the desired order of convergence of the scheme. The method does not involve the solution of linear systems with large condition numbers, permits the use of schemes with extremely high orders of convergence, and is quite insensitive to boundary layers or to end-point singularities in the coefficients of the differential equation.

Appendix A.
A High Order Scheme for the Solution of Equations (82) - (84)

In this appendix, we summarize a classical approach to the solution of integral equations via the Nyström algorithm based on Chebyshev quadrature. The facts used in this appendix are well-known, and can be found, for example, in [4,5,7].

A.1. Chebyshev approximation.

For any non-negative integer n, the Chebyshev polynomial T_n of degree n is defined by the formula

$$T_n(\cos\theta) = \cos(n\theta)\,. \tag{108}$$

Clearly, $|T_n(x)| \leq 1$ for $x \in [-1,1]$,

$$T_0(x) = 1, \qquad T_1(x) = x \tag{109}$$

and, using elementary trigonometric identities,

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad \text{for } n \geq 1\,. \tag{110}$$

It is easy to see that, for $n \geq 1$, the roots $t_n^1, t_n^2, \cdots, t_n^n$ of T_n are real, located on the interval $[-1,1]$, and given by the formula

$$t_n^i = \cos\frac{(2i-1)\pi}{2n}\,. \tag{111}$$

The Chebyshev polynomials constitute an orthonormal basis for $L^2[-1,1]$ with respect to the inner product

$$(f,g)_T = \int_{-1}^{1} f(t)\cdot g(t)\cdot(1-x^2)^{-\frac{1}{2}}\,dt, \tag{112}$$

Therefore, any function $f \in C^0[-1,1]$ can be represented by an expansion

$$f(x) = \sum_{i=0}^{\infty} a_i \cdot T_i(x), \tag{113}$$

with the coefficients a_i given by

$$a_i = (f, T_i)_T\,. \tag{114}$$

The popularity of Chebyshev expansions as a numerical tool is largely a consequence of the following two lemmas. The first demonstrates that the series converges rapidly for sufficiently smooth functions, while the second shows that numerical evaluation of the coefficients takes a particularly simple form. Proofs may be found in [7].

Lemma A.1. *Suppose that n and k are natural numbers, and that $f \in C^k[-1,1]$. Suppose further that the coefficients $a_0, a_1, \cdots, a_n$ are defined by formula (114). Then for any $x \in [-1,1]$,*

$$|f(x) - \sum_{i=0}^{n} a_i \cdot T_i(x)| = O(\frac{1}{n^{k-1}})\,. \tag{115}$$

In particular, if $f \in C^\infty$, then the expansion (113) converges to f superalgebraically.

A well-known property of the coefficients of the Chebyshev series is that they can be obtained from the cosine transform of the function $F(\theta) = f(\cos\theta)$. More precisely, a simple change of variables yields the result

$$\begin{aligned} a_0 &= (f, T_0)_T = \frac{1}{\pi}\int_0^\pi f(\cos\theta)d\theta \\ a_k &= (f, T_k)_T = \frac{2}{\pi}\int_0^\pi f(\cos\theta)\cos k\theta d\theta \qquad \text{for } k > 0. \end{aligned} \tag{116}$$

Lemma A.2. *Suppose that n and k are natural numbers, that $f \in C^k[-1,1]$, and that the coefficients a_i are defined by (114). Suppose further that the vector $\mathbf{f}_n = (f_n^1, f_n^2, \cdots, f_n^n)$ consists of the function values at the roots of $T_n(x)$, namely*

$$f_n^i = f(t_n^i), \qquad i = 1, 2, \ldots, N. \tag{117}$$

Let $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ be given by

$$\alpha = C_n(\mathbf{f}_n)\,, \tag{118}$$

where C_n denotes the discrete cosine transform of dimension n. Then

$$|a_i - \alpha_i| = O(\frac{1}{n^k})\ .$$

A.2. Chebyshev quadrature.

Lemmas A.1 and A.2 provide a tool for the construction of highly accurate interpolation schemes. The following two lemmas use Chebyshev expansions as an apparatus for the numerical evaluation of indefinite integrals.

Lemma A.3. *Suppose that $f : [-1,1] \to \mathbf{R}$ is given by the finite Chebyshev series*

$$f(x) = \sum_{i=0}^{n-1} \alpha_i \cdot T_i(x)\ . \tag{119}$$

Then the indefinite integral of f has a series expansion of the form

$$F_I(x) = \int_{-1}^{x} f(t)\ dt = \sum_{i=0}^{n} \beta_i \cdot T_i(x)\ . \tag{120}$$

The coefficients are given by

$$\begin{aligned} \beta_i &= \frac{1}{2\cdot i}\cdot(c_{i-1}\cdot\alpha_{i-1} - c_{i+1}\cdot\alpha_{i+1}) \qquad \textit{for } i \geq 1\ , \\ \beta_0 &= 2\cdot\sum_{i=1}^{n-1}(-1)^{i-1}\cdot\beta_i, \end{aligned} \tag{121}$$

where $c_0 = 2$, $c_i = 1$ *for* $i > 0$, *and* α_i *is assumed to be zero outside the range* $i = 0, 1, \ldots, n-1$.

Definition A.1 *Let* $\alpha = (\alpha_0, \alpha_1, \cdots, \alpha_{n-1})$ *and let* $\beta = (\beta_0, \beta_1, \cdots, \beta_n)$. *Then the linear mapping* $S^n : \mathbf{R}^n \to \mathbf{R}^{n+1}$ *defined by the formula*

$$S^n(\alpha) = \beta$$

is referred to as the spectral integration operator.

Lemma A.4. (Clenshaw and Curtis [2]) *Suppose that* $f \in C^k[-1, 1]$ *for some* $k > 1$, *and that the vector* $f^n = (f_n^1, f_n^2, \ldots, f_n^n) \in R^n$ *consists of the function values at the roots of* $T_n(x)$,

$$f_n^i = f(t_n^i), \qquad i = 1, 2, \ldots, N.$$

Suppose further that $F_l : [-1, 1] \to \mathbf{R}$ *is the indefinite integral*

$$F_l(x) = \int_{-1}^{x} f(t)\ dt$$

and that the vector $F^n = (F_n^1, F_n^2, \ldots, F_n^n) \in R^n$ *is defined by the formula*

$$F_n^i = F_l(t_n^i). \tag{122}$$

Then

$$\|F^n - C_n^{-1} \circ S^n \circ C_n\ (f^n)\|_\infty = O(\frac{1}{n^{k-1}})\ . \tag{123}$$

Furthermore, all elements of the matrix $C_n^{-1} \circ S^n \circ C_n$ *are strictly positive.*

The following theorem provides a trivial extension of the above lemma to the case of intervals of length other than 2.

Theorem A.1. *Suppose that* $[a, b] \subset \mathbf{R}$ *is an interval of non-zero length,* n, k *is a pair of natural numbers,* $f \in C^k[a, b]$, *and the finite sequence* $\tau_n^1, \tau_n^2, \cdots, \tau_n^n$ *is defined by the formula*

$$\tau_n^i = \left(\frac{b-a}{2}\right) \cos\left[\frac{(2i-1)\pi}{2n}\right] + \left(\frac{b+a}{2}\right)\ . \tag{124}$$

Suppose further that the function $F_l : [a, b] \to \mathbf{R}$ *is defined by the formula*

$$F_l(x) = \int_a^x f(t)\ dt. \tag{125}$$

Finally, suppose that the vectors $f^n = (f_n^1, f_n^2, \ldots, f_n^n)$ *and* $F^n = (F_n^1, F_n^2, \ldots, F_n^n)$ *are defined by*

$$f_n^i = f(\tau_n^i) \tag{126}$$

and

$$F_n^i = F_l(\tau_n^i). \tag{127}$$

Then

$$\|F^n - \frac{b-a}{2} \cdot C_n^{-1} \circ S^n \circ C_n\ (f^n)\| = O(\frac{1}{n^{k-1}})\,. \tag{128}$$

In the solution of the integral equation (8), we will also require the evaluation of indefinite integrals of the form

$$F_r(x) = \int_x^b f(t)\ dt.$$

For this purpose, we have the following theorem.

Theorem A.2. *Suppose that $[a,b] \subset \mathbf{R}$ is an interval of non-zero length, n, k is a pair of natural numbers, $f \in C^k[a,b]$, and the finite sequence $\tau_n^1, \tau_n^2, \cdots, \tau_n^n$ is defined by the formula*

$$\tau_n^i = \left(\frac{b-a}{2}\right) \cos\left[\frac{(2i-1)\pi}{2n}\right] + \left(\frac{b+a}{2}\right)\ . \tag{129}$$

Suppose further that the function $F_r : [a,b] \to \mathbf{R}$ is defined by the formula

$$F_r(x) = \int_x^b f(t)\ dt. \tag{130}$$

Finally, suppose that the vectors $f^n = (f_n^1, f_n^2, \ldots, f_n^n)$ and $F^n = (F_n^1, F_n^2, \ldots, F_n^n)$ are defined by

$$f_n^i = f(\tau_n^i) \tag{131}$$

and

$$F_n^i = F_r(\tau_n^i). \tag{132}$$

Then

$$\|F^n - \frac{b-a}{2} \cdot C_n^{-1} \circ \tilde{S}^n \circ C_n\ (f^n)\| = O(\frac{1}{n^{k-1}})\,, \tag{133}$$

where

$$\tilde{S}^n = T \circ S^n \circ T \tag{134}$$

and $T : \mathbf{R}^n \to \mathbf{R}^n$ is the transposition operator defined by

$$\begin{aligned} T_{i,n-i+1} &= 1, \\ T_{n-i+1,i} &= 1, \\ T_{i,j} &= 0 \qquad \textit{otherwise.} \end{aligned} \tag{135}$$

Definition A.2. *The matrix*

$$S_{a,b}^n = \frac{b-a}{2} \cdot C_n^{-1} \circ S^n \circ C_n \tag{136}$$

is known as the left spectral integration matrix of order n on the interval $[a,b]$. *The matrix*

$$\tilde{S}^n_{a,b} = \frac{b-a}{2} \cdot C_n^{-1} \circ \tilde{S}^n \circ C_n \tag{137}$$

is known as the right spectral integration matrix of order n on the interval $[a,b]$.

According to Theorems A.1 and A.2, $S^n_{a,b}$ converts the values of a function $f : [a,b] \rightarrow \mathbf{R}$ at the Chebyshev nodes on the interval $[a,b]$ into (approximate) values of the indefinite integral

$$\int_a^x f(t)\,dt$$

at these same nodes. Similarly, $\tilde{S}^n_{a,b}$ converts the values of f into (approximate) values of the indefinite integral

$$\int_x^b f(t)\,dt\ .$$

Furthermore, the order of convergence of the discrete approximations is equal to the number of continous derivatives f possesses. Finally, $S^n_{a,b}$ and $\tilde{S}^n_{a,b}$ can be applied to an arbitrary vector for a cost proportional to $n \cdot log(n)$, since it is a product of a diagonal matrix (S^n or $\tilde{S}^n$) and two cosine transforms. The latter fact, however, is irrelevant for the purposes of this paper.

A.3. The Nyström algorithm for the solution of second kind integral equations.

The Nyström algorithm associated with a family of n-point quadrature formulae $\eta^k = \{z^k_i, w_i\}$ replaces the integral equation

$$\phi(z) + \int_0^L K(z,t) \cdot \phi(t)dt = g(z) \tag{138}$$

with the system of linear algebraic equations

$$\phi_i + \sum_{j=1}^{n} w^i_j \cdot K(z_i, z_j) \cdot \phi_j = g(z_i) \qquad i = 1, \ldots, n. \tag{139}$$

We denote the matrix of the discretized system (139) by B_n, and view the solution $\phi_1, \ldots, \phi_n$ of (139) as an approximation to the solution ϕ of (138) at the nodes $z_1, \ldots, z_n$. If (139) has a unique solution, then for a wide class of quadrature formulae η^n and sufficiently large n, the system (138) also has a unique solution. Furthermore, under fairly broad assumptions, the convergence rate of the Nyström algorithm is the same as the convergence rate of the quadrature formula it is based on (see, for example, [1]).

A.4. A high order Nyström scheme for the solution of equation (8).

In this section, we describe a particular version of the Nyström scheme for the solution of equations of the form (8). The scheme to be described is based on Chebyshev quadrature and has a convergence rate of the order $k-1$, where k is the number of continous derivatives possessed by the solution σ of equation (8).

Given an interval $[a,c]$ and the Chebyshev nodes $\tau_1, \tau_2, \cdots, \tau_n$ on it, we will define the diagonal operators $\tilde{P}, \tilde{Q}, U^l, U^r, U^{l'}, U^{r'}, V^l, V^r : R^n \rightarrow R^n$ via the formulae

$$\begin{aligned} \tilde{P}_{i,i} &= \tilde{p}(\tau_i), \\ \tilde{Q}_{i,i} &= \tilde{q}(\tau_i), \\ U^l_{i,i} &= u_l(\tau_i), \\ U^r_{i,i} &= u_r(\tau_i), \\ U^{l'}_{i,i} &= u'_l(\tau_i), \\ U^{r'}_{i,i} &= u'_r(\tau_i), \\ V^l_{i,i} &= v_l(\tau_i), \\ V^r_{i,i} &= v_r(\tau_i). \end{aligned} \tag{140}$$

Finally, we will define the operator $A^n : R^n \rightarrow R^n$ by the formula

$$\begin{aligned} A^n = I &+ \tilde{P} \circ (U^{l'} \circ S^n_{[a,c]} \circ V^l + U^{r'} \circ \tilde{S}^n_{[a,c]} \circ V^r), \\ &+ Q \circ (U^l \circ S^n_{[a,c]} \circ V^l + U^r \circ \tilde{S}^n_{[a,c]} \circ V^r), \end{aligned} \tag{141}$$

and the vector $f^n \in R^n$ by the formula

$$f^n_i = f(\tau_i). \tag{142}$$

Observation A.1. The mapping A^n defined by the expression (141) can be viewed as an approximation to the operator $P_{i,k}$ defined by the expression (81). This is obvious from (11), (12), Theorem A.1 and Theorem A.2.

The following theorem provides an exact statement of the above observation. It is used in Section IV of this paper to construct highly accurate approximations to the solutions of equations (82), (83) and (84) on elementary subintervals B^m_i, and is the principal purpose of this Appendix.

Theorem A.2. Suppose that the equation (7) is chosen in such a manner that $v_l, v_r \in C^k[a,c]$ for some $k \geq 1$. Suppose further that the equation (82) (or (83) or (84)) has a unique solution σ such that $\sigma \in C^k[a,c]$. Then for all sufficiently large n, the equation

$$A^n(\sigma^n) = f^n \tag{143}$$

has a unique solution σ^n, and

$$\|\sigma^n_k - \sigma(\eta_k)\|_\infty = O(\frac{1}{n^{k-1}}) \, . \tag{144}$$

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