We provide a convergence rate analysis for a variant of the domain decomposition method introduced by Gropp and Keyes for solving the algebraic equations that arise from finite element discretization of nonsymmetric and indefinite elliptic problems with Dirichlet boundary conditions in R^2. We show that the convergence rate of the preconditioned GMRES method is nearly optimal in the sense that the rate of convergence depends only logarithmically on the mesh size and the number of substructures, if the global coarse mesh is fine enough.

Convergence Rate Estimate for a Domain Decomposition Method

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1 Introduction

In the past several years, a well-developed theory has been established for the domain decomposition solution of symmetric positive definite elliptic equations. Despite their lack of a corresponding mathematical theory, the natural parallel computing possibilities of domain decomposition methods also attract a large number of applications for nonsymmetric or indefinite problems. Among them is the "tile algorithm" introduced by Gropp and Keyes in [12]. A multi-architecture parallel code based on this method for the linear systems of equations obtained from the discretization of second-order elliptic problems in R^2 is described in [13]. We study herein the convergence rate of a variant of this method more amenable to analysis than the original through establishing bounds for the preconditioned system. We show that these bounds depend only mildly on the mesh and subdomain sizes. Some related methods and analyses for this class of problems have appeared recently in [1], [3], [6] and [21].

By rewriting the algorithm of [12] in terms of subspaces and projections, we show that it is closely related to the well-known substructuring algorithm introduced in [2], which is designed for symmetric positive definite problems, and also the iterative substructuring algorithm in [4] and [18].

Though the algorithm we consider in this paper is based on a nonoverlapping decomposition of the domain, we employ in the convergence proof many ideas from the theory of the additive Schwarz method [5], [6], which is based on an overlapped subdomain decomposition. The basic technique involves a decomposition of the finite element space into subspaces and the application of related projections. A summary of these techniques for symmetric problems is given in [10].

This paper is organized as follows. In Section 2, we present a Dirichlet boundary value problem together with some assumptions which are basic to the analysis. A two-level triangulation of the domain, the corresponding finite element spaces, and the GMRES iterative method are introduced in Section 3. The convergence rate analysis of Section 4 is the central and most technical part of this contribution. We conclude in Section 5 with some preliminary numerical experiments and a discussion of possible relaxations of the hypotheses.

2 A Dirichlet boundary value problem

Let Ω be an open bounded polygon in R^2, with boundary $\partial\Omega$. Consider the homogeneous Dirichlet boundary value problem

$$\begin{cases} Lu = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

where L is an elliptic operator of the following form:

$$Lu(x) = -\sum_{i,j=1}^{2} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}) + 2\sum_{i=1}^{2} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x).$$

We assume that the matrix $\{a_{ij}(x)\}$ is symmetric and uniformly positive definite for $x \in \Omega$, $f \in L^2(\Omega)$, and that the equation has a unique solution in $H_0^1(\Omega)$.

Let $(\cdot,\cdot)$ denote the usual L^2 inner product. The weak form of equation (1) is: Find $u \in H_0^1(\Omega)$ such that

$$B(u,v) = (f,v), \qquad \forall v \in H_0^1(\Omega), \tag{2}$$

where the bilinear form $B(u,v)$ is defined as

$$B(u,v) = A(u,v) + S(u,v) + (\tilde{c}u, v).$$

Here

$$A(u,v) = \sum_{i,j=1}^{2} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dx,$$

$$S(u,v) = \sum_{i=1}^{2} \int_\Omega (b_i \frac{\partial u}{\partial x_i} v + \frac{\partial (b_i u)}{\partial x_i} v) dx$$

and $\tilde{c}(x) = c(x) - \sum_{i=1}^{2} \frac{\partial b_i}{\partial x_i}$. We note that for sufficiently negative $\tilde{c}$ the bilinear form is indefinite. The Helmholtz equation is an example of this kind.

Throughout this paper, c and C, with or without subscripts, denote generic, strictly positive constants, which may have different values in different places. They are independent of the mesh parameters, which will be introduced later.

We make the following basic boundedness assumptions for these bilinear forms.

(i) $A(\cdot,\cdot)$ is equivalent to the square of the $H_0^1(\Omega)$ norm; i.e., there exist two constants $c > 0$ and C such that

$$c\|u\|^2_{H_0^1(\Omega)} \le A(u,u) \le C\|u\|^2_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

We shall use $\|\cdot\|_A$ to denote the A-norm and exchange freely between the A-norm and H_0^1 norm.

(ii) $S(\cdot,\cdot)$ is bounded; i.e., there exists a constant C such that

$$|S(u,v)| \le C\|u\|_{H_0^1(\Omega)}\|v\|_{L^2(\Omega)}, \quad \forall u,v \in H_0^1(\Omega).$$

It is easy to verify that the bilinear form $S(\cdot,\cdot)$ satisfies

$$S(u,v) = -S(v,u), \quad \forall u,v \in H_0^1(\Omega).$$

(iii) $\|\tilde{c}\|_{L^\infty(\Omega)} \le C$.

As a consequence of assumptions (i), (ii) and (iii), the following bounds and regularity for the bilinear form $B(\cdot,\cdot)$ can be established.

(1) $|B(u,v)| \le C\|u\|_{H_0^1(\Omega)}\|v\|_{H_0^1(\Omega)}, \forall u,v \in H_0^1(\Omega)$.

(2) Gårding's inequality holds; i.e., there exist two constants $c > 0$ and $C > 0$ such that

$$B(u,u) \ge c\|u\|^2_{H_0^1(\Omega)} - C\|u\|^2_{L^2(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

(3) The solution w of the equation

$$B(\phi, w) \;=\; (g,\phi), \;\; \forall \phi \in H_0^1(\Omega)$$

satisfies $\|w\|_{H^{1+\gamma}(\Omega)} \le C\|g\|_{L^2(\Omega)}$, where $\gamma \in [\frac{1}{2}, 1]$; cf. [15].

We note that the upper bounds on the magnitudes of $B(\cdot,\cdot)$ and $S(\cdot,\cdot)$ are of different forms since one of the factors in each term of $S(\cdot,\cdot)$ is an order lower than those of the factors in $A(\cdot,\cdot)$. This enables us to control the skew-symmetric terms, and makes our analysis possible.

3 Triangulation, Galerkin approximation and GMRES

We solve equation (2) by a Galerkin conformal finite element method. For simplicity, only piecewise linear triangular elements are considered. The use of higher order elements is possible but is not discussed here. In this section, we first introduce a two-level triangulation of $\Omega \in R^2$ and the corresponding finite element spaces, then make the Galerkin approximation, and conclude with a summary of properties of the GMRES method salient to the convergence proof.

3.1 A two-level triangulation

We employ the same two-level triangulation previously described in, for example, [2], [4], [9], and [12] and repeated here for notational self-containedness only. In the first step, for a given polygonal region $\Omega \in R^2$, we define $T^H = \{\tau_i^H\}$ to be a shape regular finite element triangulation of Ω. $\{\tau_i^H\}$ is a set of nonoverlapping triangles with diameter of order H. By shape regular, we mean that there exist constants $c > 0$ and $C > 0$ such that each τ_i^H contains a ball of radius cH and is contained in a ball of radius CH. We shall refer to this as the H-level triangulation, or the coarse mesh, and the vertices of the triangles that lie inside Ω as the cross points, see the right figure of Figure 1.

We further divide each τ_i^H into smaller triangles, denoted as τ_j^h. We assume that each τ_j^h has a diameter of order h and $\{\tau_j^h\}$ comprises a shape regular finite element subdivision of Ω in the same sense as above. We call $T^h = \{\tau_i^h\}$ the fine mesh, or h-level triangulation, of Ω, see the left figure of Figure 1.

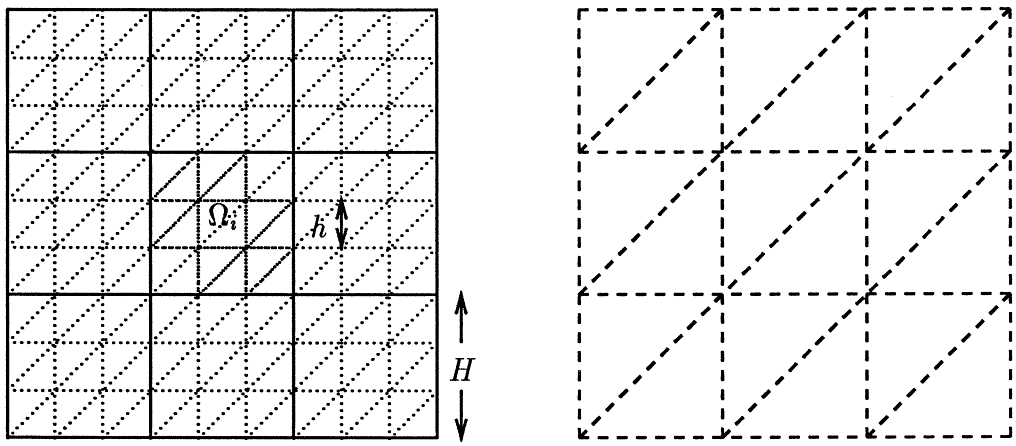

Figure 1: A sample two-level triangulation of Ω. The left figure shows the fine mesh and also the decomposition of the domain into Ω_i's. The right figure shows the coarse mesh.

Given an H-level triangulation, we can regard Ω as the union of a finite number of subregions, denoted as Ω_i, with diameters of order H. In practice, the Ω_i are usually chosen to be either quadrilaterals or triangles whose sides coincide with the mesh lines of T^H. We refer to each Ω_i as a substructure, or "tile".

We now define the piecewise linear finite element function spaces over the H-level and h-level subdivisions of Ω:

$$\begin{aligned}
V^h &= \{v^h \mid\in C(\overline{\Omega}),\ v^h|_{\tau_i^h} \text{ is linear on } \tau_i^h,\ \forall i,\ v^h = 0 \text{ on } \partial\Omega\} \\
V_C^h &= \{v^H \mid\in C(\overline{\Omega}),\ v^H|_{\tau_i^H} \text{ is linear on } \tau_i^H,\ \forall i,\ v^H = 0 \text{ on } \partial\Omega\} \\
V_I^h(\Omega_i) &= \{v^h \mid\in V^h,\ v^h = 0 \text{ on } \partial\Omega_i\} \\
V_I^h &= \oplus_i V_I^h(\Omega_i),
\end{aligned}$$

and we also define V_E^h, a subspace of V^h, to consist of all functions that are B-discrete piecewise harmonic in the interior of all Ω_i's and vanish at the cross points. v_E^h is B-discrete harmonic in Ω_i if

$$B(v_E^h, \phi^h) = 0, \ \forall \phi^h \in V_I^h(\Omega_i).$$

We denote by $\Gamma = \cup_i \partial\Omega_i - \partial\Omega$ the union of the internal interfaces of the substructures, and define the corresponding subspace $V_\Gamma^h \subset V^h$ to consist of all functions that are linear combinations of the h-level nodal basis functions with nodes on Γ and that vanish at all cross points. These functions are zero at all the nodes that are interior to any Ω_i. Throughout this paper, Γ_{ij} denotes the common edge of Ω_i and Ω_j, and the corresponding piecewise linear continuous function space is equipped with an inner product

$$\langle f, g\rangle_{\Gamma_{ij}} = (l_0^{1/2} f, g)_{L^2(\Gamma_{ij})},$$

where $f, g \in H_{00}^{1/2}(\Gamma_{ij})$, and the operator l_0 is defined by

$$(l_0 f, \phi)_{L^2(\Gamma_{ij})} = (f', \phi')_{L^2(\Gamma_{ij})}, \quad \forall \phi \in H_{00}^{1/2}(\Gamma_{ij}).$$

The prime denotes differentiation with respect to arc length along Γ_{ij}. The operator $l_0^{1/2}$ and the space $H_{00}^{1/2}(\Gamma_{ij})$ were also used in [2] and [8]. A description of the computation of the inner product in the discrete case can be found in [2].

For any function $u^h \in V^h$, $\tilde{u}^h$ denotes its restriction, or trace, on Γ. If $\tilde{u}^h, \tilde{v}^h \in H_{00}^{1/2}(\Gamma_{ij})$ for all Γ_{ij}, we define the inner product for functions on the interfaces as

$$\langle \tilde{u}^h, \tilde{v}^h \rangle = \sum \langle \tilde{u}^h, \tilde{v}^h \rangle_{\Gamma_{ij}}$$

and the corresponding norm as $|\cdot|_\Gamma = \langle \cdot, \cdot \rangle^{1/2}$.

3.2 Galerkin finite element approximation

The Galerkin approximation of equation (2) reads as follows: Find $\mathbf{u}^h \in V^h$ such that

$$B(\mathbf{u}^h, v^h) = (f, v^h), \quad \forall v^h \in V^h. \tag{3}$$

If the mesh size h is small enough, the existence and uniqueness of $\mathbf{u}^h$ have been extensively studied in the literature; see [17]. By using the nodal basis functions, equation (3) can be transformed into a linear system of algebraic equations, which is usually very large, sparse, and relatively ill-conditioned.

3.3 A brief discussion of the GMRES method

We are interested in using the generalized minimal residual (GMRES) method, introduced in [16] and mathematically equivalent to the generalized conjugate residual (GCR) method [11], to solve the following linear system of equations on R^n:

$$Px = b,$$

where b is given in R^n and the explicit matrix expression of P need not be known.

The method begin with an initial approximate solution $x_0 \in R^n$ and an initial residual $r_0 = b - Px_0$. At the m^{th} iteration, a correction vector z_m is computed in the Krylov subspace

$$\mathcal{K}_m(r_0) = \text{span}\{r_0, Pr_0, \cdots, P^{m-1}r_0\},$$

which minimizes the residual, $\min_{z \in \mathcal{K}_m(r_0)} \|b - P(x_0 + z)\|$. The m^{th} iterate is then $x_m = x_0 + z_m$. It can be shown that the solution is reached in no more than n iterations if exact arithmetic operations are performed.

According to the theory of [11], the rate of convergence of the GMRES method can be characterized (not necessarily tightly) by the ratio of the minimal eigenvalue of the symmetric part of the operator to the norm of the operator. These two quantities are defined as follows:

$$c_P = \inf_{x \neq 0} \frac{[x, Px]}{[x, x]} \quad \text{and} \quad C_P = \sup_{x \neq 0} \frac{\|Px\|}{\|x\|},$$

where $[\cdot, \cdot]$ is the inner product being used and $\| \cdot \|$ is the corresponding norm. From [11] we then have the following theorem.

Theorem 1 *If $c_P > 0$, then the GMRES method converges and at the m^{th} iteration, the residual is bounded as*

$$\|r_m\| \leq \left(1 - \frac{c_P^2}{C_P^2}\right)^{m/2} \|r_0\|.$$

In our application, the operator P is the preconditioned stiffness matrix of the discrete elliptic problem and b is premultiplied by the corresponding preconditioner. Our main contribution is obtaining bounds for c_P and C_P in terms of the mesh size parameters.

4 Algorithm and analysis

In this section, we introduce our substructuring-based domain decomposition algorithm and provide convergence rate estimates in Theorem 2. This algorithm consists of three types of subproblem solves at each iteration, namely, cross-point, interior and interface solves, and is thus generically similar to the substructuring algorithm introduced in [2].

We observe that the finite element space V^h can be represented as the sum of some of the subspaces previously introduced, i.e.,

$$V^h = V_I^h + V_C^h + V_E^h.$$

Thus, certain projections from the finite element space V^h to its subspaces can be introduced naturally in terms of the bilinear form $B(\cdot, \cdot)$.

Definition: For any $u^h \in V^h$, $P_I u^h \in V_I^h$ is the solution of the finite element equation

$$B(P_I u^h, v^h) = B(u^h, v^h), \quad \forall v^h \in V_I^h. \tag{4}$$

Similarly, $P_C u^h \in V_C^h$ is defined as the solution of

$$B(P_C u^h, v^h) = B(u^h, v^h), \quad \forall v^h \in V_C^h. \tag{5}$$

Before defining $P_E u^h \in V_E^h$, we first define the restriction of $P_E u^h$ to Γ as follows:

$$\langle \widetilde{P_E u^h}, \tilde{v}^h \rangle \;=\; B(u^h - P_I u^h, v^h), \;\; \forall v^h \in V_\Gamma^h. \tag{6}$$

Note that $\widetilde{P_E u^h}$ vanishes at all cross points. We then define $P_E u^h$ to be the piecewise B-discrete harmonic extension of its boundary value into the interior of each Ω_i; i.e.,

$$B(P_E u^h, v^h) \;=\; 0, \;\; \forall v^h \in V_I^h.$$

We notice that due to the nonsymmetry of $B(\cdot,\cdot)$, these projections are not in general orthogonal. Let us denote the sum of these projections by P:

$$P \;=\; P_I \;+\; P_C \;+\; P_E.$$

It is easy to see that the computation of the projections of an arbitrary function $v^h \in V^h$ involves only the solution of some standard finite element linear systems of algebraic equations defined on the Ω_i, the Γ_{ij}, and the coarse mesh. All of them contain only a relatively small number of degrees of freedom compared to the original large system, and most of these subproblems are independent of each other, in fact that leads to an obvious parallelization, as specifically noted below.

Let us denote the projection of the exact solution $\mathbf{u}^h$ of the Galerkin equation (3) as $\mathbf{b}$:

$$\mathbf{b} \;=\; P\mathbf{u}^h \;=\; P_I\mathbf{u}^h \;+\; P_C\mathbf{u}^h \;+\; P_E\mathbf{u}^h.$$

It is important to understand that $\mathbf{b} \in V^h$ can be computed without *a priori* knowledge of the solution $\mathbf{u}^h$ of (3). The computation is done by first solving the following equations,

$$\begin{aligned}
B(P_I\mathbf{u}^h, v^h) \;&=\; (f, v^h), \;\; \forall v^h \in V_I^h, \\
B(P_C\mathbf{u}^h, v^h) \;&=\; (f, v^h), \;\; \forall v^h \in V_C^h, \\
\langle \widetilde{P_E\mathbf{u}^h}, \tilde{v}^h \rangle \;&=\; (f - P_I\mathbf{u}^h, v^h), \;\; \forall v^h \in V_\Gamma^h,
\end{aligned}$$

$$\left\{ \begin{aligned} B(P_E\mathbf{u}^h, v^h) \;&=\; 0, \;\; \forall v^h \in V_I^h \\ P_E\mathbf{u}^h|_\Gamma \;&=\; \widetilde{P_E\mathbf{u}^h}, \end{aligned} \right.$$

and then taking the sum of the above solutions.

Using $\mathbf{b}$ as the right-hand side, we can define a new system of linear equations as

$$P\mathbf{u}^h \;=\; \mathbf{b} \tag{7}$$

Because of the special choice of the right-hand side, it is not difficult to verify that equation (7) has the same solution as the Galerkin equation (3) when the operator P is invertible.

In general, the preconditioned operator P is not symmetric even in the case that L is selfadjoint. A method based on GMRES has been introduced in [12] to solve a system of similar structure. Its convergence rate is the subject of the sequel. The algorithm is described by specifying how to perform the matrix vector multiply Pu^h.

Algorithm: For a given $u^h \in V^h$,

Step 1. Find $P_I u^h$ by solving the interior problems.

$$B(P_I u^h, v^h) = B(u^h, v^h), \quad \forall v^h \in V_I^h.$$

Note that V_I^h is a direct sum of $V_I^h(\Omega_i)$'s. The problems on each substructure are independent of each other and can be solved in parallel.

Step 2. Find $P_C u^h$ by solving a coarse mesh problem.

$$B(P_C u^h, v^h) = B(u^h, v^h), \quad \forall v^h \in V_C^h.$$

This is independent of step 1, and therefore can go in parallel with step 1.

Step 3. Find $\widetilde{P_E u^h}$ by solving the interface problems.

$$\langle \widetilde{P_E u^h}, \tilde{v}^h \rangle = B(u^h - P_I u^h, v^h), \quad \forall v^h \in V_\Gamma^h.$$

The problems on each Γ_{ij} are independent, and therefore can be solved in parallel, once the solutions from step 1 are available in both subdomains.

Step 4. Find $P_E u^h$ by solving the interior problems. For each Ω_i,

$$B(P_E u^h, v^h) = 0, \quad \forall v^h \in V_I^h(\Omega_i),$$

with the boundary condition $P_E u^h |_{\partial\Omega_i}$, which is available after step 3. The solutions on different subregions are independent and may be done in parallel.

Step 5. Find $Pu^h = P_C u^h + P_I u^h + P_E u^h$.

According to Theorem 1, the rate of convergence of the algorithm can be estimated in terms of certain spectral bounds for the operator P. The following main theorem shows that the operator P is nearly uniformly well-conditioned in the sense that the bounds depend only mildly on the mesh parameters H and h, if the coarse mesh size H is small enough.

Theorem 2 *There exist constants $H_0 > 0$, $c(H_0) > 0$ and $C(H_0)$ such that, if $\max\{H^\gamma, H(1+\ln(H/h)^3)\} \leq H_0$ holds, then*

(1) $\|Pu^h\|_A \leq C(H_0)\|u^h\|_A, \quad \forall u^h \in V^h.$

(2) $A(u^h, Pu^h) \geq c(H_0)/(1+\ln(H/h)^3) A(u^h, u^h), \quad \forall u^h \in V^h.$

The proof will be given at the end of this section.

Remarks: (a) The requirement that $H(1+\ln(H/h)^3) \le H_0$ arises from the existence of the nonsymmetric and indefinite terms. In the subsequent proofs it can be seen that the condition is unnecessary if L is symmetric positive definite, and the constants $c(H_0)$ and $C(H_0)$ are independent of the coarse mesh size.

(b) The constant H_0 effectively determines the maximum size of the substructures, or their minimum number. It is a problem-dependent constant. Generally speaking, H_0 decreases if we increase the coefficients of the skew-symmetric terms or make $\tilde{c}$ more negative. It also depends on the shape of the domain Ω; for example, if the domain is not convex, H_0 will depend on the sizes of the reentrant angles. This is reflected in the constant γ, which was introduced in the end of Section 2, cf. [15].

The second half of this section is devoted to the proof of Theorem 2. We begin by defining some Sobolev norms. The Sobolev space of order one-half on $\partial\Omega_i$ is denoted as $H^{1/2}(\partial\Omega_i)$; cf. [2], pp. 112. Because Γ, which can be regarded as the union of all Γ_{ij}s, is not a simple curve, we define the corresponding semi-norm as

$$|\tilde{v}(x)|^2_{H^{1/2}(\Gamma)} = \sum |\tilde{v}(x)|^2_{H^{1/2}(\partial\Omega_i)},$$

where $v(x) \in V_\Gamma^h$ vanishes on $\partial\Omega$, and the sum is taken over all i's.

To prove Theorem 2, we need to establish some technical lemmas. The first lemma was essentially proved in [2].

Lemma 1 *There exists an $H_0 > 0$ such that, if $H < H_0$, then for any $u^h \in V^h$, there exist $u_I^h \in V_I^h$, $u_C^h \in V_C^h$ and $u_E^h \in V_E^h$, such that,*

$$u^h = u_I^h + u_C^h + u_E^h,$$

and also

$$\|u_I^h\|^2_{H_0^1(\Omega)} + \|u_C^h\|^2_{H_0^1(\Omega)} + \|u_E^h\|^2_{H_0^1(\Omega)} \le C(1+\ln(H/h))\|u^h\|^2_{H_0^1(\Omega)},$$

where $C = C(H_0) > 0$ is a constant independent of H and h.

Proof: The decomposition can be constructed easily. For any given $u^h \in V^h$, let $w^h = u^h - u_C^h$, where u_C^h is the linear interpolant of u^h on the H-level nodes. We choose u_I^h to be the solution of the equation

$$B(u_I^h, \phi^h) = B(w^h, \phi^h), \quad \forall \phi^h \in V_I^h,$$

with a zero boundary condition on the interfaces, and define $u_E^h = u^h - u_C^h - u_I^h$. It is easy to verify that $u_E^h \in V_E^h$.

Using Lemma 3.4 in [2], we have

$$\|u_C^h\|^2_{H_0^1(\Omega)} \le C(1+\ln(H/h))\|u^h\|^2_{H_0^1(\Omega)}.$$

Since

$$B(u_I^h, u_I^h) \;=\; B(w^h, u_I^h),$$

by Gårding's inequality and Friedrich's inequalities,

$$(1 - CH^2)\|u_I^h\|^2_{H_0^1(\Omega)} \le C\|w^h\|_{H_0^1(\Omega)}\|u_I^h\|_{H_0^1(\Omega)}.$$

Thus, if H is small enough,

$$\|u_I^h\|_{H_0^1(\Omega)} \le C(H_0)\|w^h\|_{H_0^1(\Omega)} \le C(H_0)(1 + \ln(H/h))^{1/2}\|u^h\|_{H_0^1(\Omega)}.$$

The bound for $\|u_E^h\|^2_{H_0^1(\Omega)}$ can be obtained by using the triangle inequality. □

Lemma 2 *There exists a constant C, such that, for any B-discrete harmonic function $w^h \in V^h(\Omega_i)$, which vanishes at all cross points*

$$A(w^h, w^h) \le C\langle w^h|_{\partial\Omega_i}, w^h|_{\partial\Omega_i}\rangle.$$

Proof: Let us take a special B-discrete harmonic function w^h which vanishes on $\Gamma \setminus \Gamma_{ij}$, where Γ_{ij} is one of the edges comprising $\partial\Omega_i$. If we can prove the lemma for this type of function, we are done.

Let $w \in H_0^1(\Omega_i)$ be the solution of the problem

$$B(w, \phi) = 0, \;\; \forall\phi \in H_0^1(\Omega_i), \quad w = w^h \text{ on } \partial\Omega_i.$$

By a well-known *a priori* inequality; see, *e.g.,* [15], we have

$$|w|^2_{H^{1+\epsilon}(\Omega_i)} \le C|w^h|^2_{H_{00}^{1/2+\epsilon}(\Gamma_{ij})},$$

for a small positive $\epsilon > 0$. We also have, using the approximation property of finite elements, that

$$\begin{aligned} |w - w^h|^2_{H^1(\Omega_i)} &\le Ch^{2\epsilon}|w|^2_{H^{1+\epsilon}(\Omega_i)} \\ &\le Ch^{2\epsilon}|w^h|^2_{H_{00}^{1/2+\epsilon}(\Gamma ij)} \\ &\le C|w^h|^2_{H_{00}^{1/2}(\Gamma ij)}. \end{aligned}$$

The last step is due to an inverse property of the finite element function on Γ_{ij}. Consequently,

$$|w^h|^2_{H^1(\Omega_i)} \le C(|w - w^h|^2_{H^1(\Omega_i)} + |w|^2_{H^1(\Omega_i)}) \le C|w^h|^2_{H_{00}^{1/2}(\Gamma ij)}.$$

The proof is completed by noticing the equivalence of the $A(\cdot,\cdot)$ norm and the square of the H^1 norm, and also the fact that $|w^h|^2_{H_{00}^{1/2}(\Gamma ij)} \le C\langle w^h|_{\partial\Omega_i}, w^h|_{\partial\Omega_i}\rangle$. □

We have defined two norms for the interface functions. In the next lemma, we give the relations between these two norms. For the proof, we refer to [2].

Lemma 3 *There exist $c > 0$ and C such that, for any w^h defined on Γ that vanishes at all cross points,*

$$c|w^h|^2_{H^{1/2}(\Gamma)} \leq \langle w^h, w^h \rangle \leq C(1 + \ln(H/h)^2)|w^h|^2_{H^{1/2}(\Gamma)}.$$

The next lemma, which is essentially due to Schatz [17], gives the approximation estimates for the coarse mesh projection. A proof can be found in [6].

Lemma 4 *There exist constants $H_0 > 0$ and $C(H_0)$ such that if $H \leq H_0$, then*
(1)$\|P_C u^h - u^h\|_{H_0^1(\Omega)} \leq C\|u^h\|_{H_0^1(\Omega)}, \ \forall u^h \in V^h$,
(2)$\|P_C u^h - u^h\|_{L^2(\Omega)} \leq CH^\gamma \|P_C u^h - u^h\|_{H_0^1(\Omega)}, \ \forall u^h \in V^h$.

Remark: Let us recall that $\gamma \in [\frac{1}{2}, 1]$. If the polygonal domain Ω is convex, then H can replace H^γ in the second part of this lemma.

The following lemma is essential to our spectral lower bound estimate.

Lemma 5 *There exist constants $H_0 > 0$, $c(H_0) > 0$ such that, if $H \leq H_0$, then*

$$\|P_I u^h\|^2_{H_0^1(\Omega)} + \|P_C u^h\|^2_{H_0^1(\Omega)} + |\widetilde{P_E u^h}|^2_\Gamma \geq c/(1 + \ln(H/h)^3)\|u^h\|^2_{H_0^1(\Omega)},$$

for any $u^h \in V^h$.

Proof: By Lemma 4, we can easily obtain that

$$\|u^h\|^2_{L^2(\Omega)} \leq C(H^{2\gamma}\|u^h\|^2_{H_0^1(\Omega)} + \|P_C u^h\|^2_{L^2(\Omega)}). \tag{8}$$

Replacing the L^2 term in Gårding's inequality for u^h by (8), we have

$$(c - CH^{2\gamma})\|u^h\|^2_{H_0^1(\Omega)} \leq \|P_C u^h\|^2_{L^2(\Omega)} + B(u^h, u^h).$$

Let us assume that H is small enough so that

$$c - CH^{2\gamma} \geq c_1 > 0.$$

Hence,

$$c_1\|u^h\|^2_{H_0^1(\Omega)} \leq \|P_C u^h\|^2_{L^2(\Omega)} + B(u^h, u^h). \tag{9}$$

By using the decomposition Lemma 1, there exist $u_I^h \in V_I^h$, $u_C^h \in V_I^h$ and $u_E^h \in V_E^h$ such that

$$u^h = u_I^h + u_C^h + u_E^h.$$

Consequently, we have

$$B(u^h, u^h) = B(u^h, u_I^h) + B(u^h, u_C^h) + B(u^h, u_E^h). \tag{10}$$

We now bound the right-hand side of (10) term by term.

By using the definition of the projection, the first two terms can be bounded easily:

$$B(u^h, u_I^h) \;=\; B(P_I u^h, u_I^h) \;\le\; C\|P_I u^h\|_{H_0^1(\Omega)} \|u_I^h\|_{H_0^1(\Omega)} \qquad (11)$$

and

$$B(u^h, u_C^h) \;=\; B(P_C u^h, u_C^h) \;\le\; C\|P_C u^h\|_{H_0^1(\Omega)} \|u_C^h\|_{H_0^1(\Omega)}. \qquad (12)$$

However, the third term cannot be estimated as simply. We show that

$$B(u^h, u_E^h) \le C((1+\ln(H/h))|\widetilde{P_E u^h}|_\Gamma \|u_E^h\|_{H_0^1(\Omega)} + \|P_I u^h\|_{H_0^1(\Omega)} \|u_E^h\|_{H_0^1(\Omega)}). \quad (13)$$

From the definition and the fact that

$$B(u^h - P_I u^h, \phi^h) = 0, \;\; \forall \phi^h \in V_I^h,$$

we obtain that

$$\langle \widetilde{P_E u^h}, \tilde{u}_E^h \rangle \;=\; B(u^h - P_I u^h, u_E^h). \qquad (14)$$

Hence, from (14),

$$B(u^h - P_I u^h, u_E^h) \le |\langle \widetilde{P_E u^h}, \tilde{u}_E^h \rangle|, \qquad (15)$$

which can be bounded by

$$\begin{aligned} |\langle \widetilde{P_E u^h}, \tilde{u}_E^h \rangle| \;&\le\; |\widetilde{P_E u^h}|_\Gamma |\tilde{u}_E^h|_\Gamma \\ &\le\; C(1+\ln(H/h))|\widetilde{P_E u^h}|_\Gamma |\tilde{u}_E^h|_{H^{1/2}(\Gamma)}. \end{aligned}$$

By using the trace theorem, it can be further bounded by

$$C(1+\ln(H/h))|\widetilde{P_E u^h}|_\Gamma \|u_E^h\|_{H_0^1(\Omega)}.$$

The term $B(u^h, u_E^h)$ can then be bounded as

$$\begin{aligned} B(u^h, u_E^h) \;&=\; B(u^h - P_I u^h, u_E^h) + B(P_I u^h, u_E^h) \\ &\le\; C(1+\ln(H/h))|\widetilde{P_E u^h}|_\Gamma \|u_E^h\|_{H_0^1(\Omega)} + C\|P_I u^h\|_{H_0^1(\Omega)} \|u_E^h\|_{H_0^1(\Omega)}. \end{aligned}$$

Combining the above results and applying Schwarz's inequality to (10), we have

$$\begin{aligned} B(u^h, u^h) \;&\le\; C\sqrt{2\|P_I u^h\|^2_{H_0^1(\Omega)} + \|P_C u^h\|^2_{H_0^1(\Omega)} + (1+\ln(H/h))^2 |\widetilde{P_E u^h}|^2_\Gamma} \\ &\quad \times \sqrt{\|u_I^h\|^2_{H_0^1(\Omega)} + \|u_C^h\|^2_{H_0^1(\Omega)} + 2\|u_E^h\|^2_{H_0^1(\Omega)}}. \end{aligned} \qquad (16)$$

Thus, the proof can be accomplished by returning to (9), using the fact that

$$\|P_C u^h\|^2_{L^2(\Omega)} \le C\|P_C u^h\|_{H_0^1(\Omega)} \|u^h\|_{H_0^1(\Omega)},$$

Schwarz's inequality and also Lemma 1.□

We have mentioned before that the contribution from the skew-symmetric or zeroth-order terms are of lower order in H compared with the second order terms. The use of this fact is made through the following lemma. We call this the H-lemma.

Lemma 6 *There exists a constant C, independent of H and h, such that, for any $u^h \in V^h$,*

(1) $|(\tilde{c}u^h, P_E u^h)| \leq CH(A(u^h, u^h) + A(P_E u^h, P_E u^h))$.

(2) $|(\tilde{c}u^h, P_I u^h)| \leq CH(A(u^h, u^h) + A(P_I u^h, P_I u^h))$.

(3) $|(\tilde{c}P_I u^h, P_I u^h)| \leq CH^2 A(P_I u^h, P_I u^h)$.

(4) $|(\tilde{c}(u^h - P_C u^h), P_C u^h)| \leq CH(A(u^h, u^h) + A(P_C u^h, P_C u^h))$.

(5) $|S(P_E u^h, P_I u^h)| \leq CH(A(P_E u^h, P_E u^h) + A(P_I u^h, P_I u^h))$.

(6) $|S(u^h, P_I u^h)| \leq CH(A(u^h, u^h) + A(P_I u^h, P_I u^h))$.

(7) $|S(u^h - P_C u^h, P_C u^h)| \leq CH(A(u^h, u^h) + A(P_C u^h, P_C u^h))$.

Proof: The proofs for each statement are straightforward, so we shall just mention the ideas briefly. For the L^2 inner products, we bound by the L^2 norm of each component. One or both components can contribute an H factor as we move from L^2 to H_0^1. To do so, we apply Friedrich's inequality to $P_I u^h$, or apply Lemma 4 to $u^h - P_C u^h$, or make use of the approximation property of $P_E u^h$ (cf. Theorem 3.3 in [19]), together with the fact that $P_E u^h$ vanishes at all cross points; then use the inequality $ab \leq 1/2(a^2 + b^2)$.

As to the inequalities for $S(\cdot, \cdot)$, we first apply assumption (ii), using the L^2 norm for the component that can contribute an H factor, and then employ the same devices as for the L^2 inner products mentioned above. □.

Now, we are ready to prove Theorem 2. First, the upper bound part:

Since P is the sum of three projections, we estimate them one at a time. For P_I, by using its definition,

$$B(P_I u^h, P_I u^h) = B(u^h, P_I u^h).$$

Applying Gårding's inequality to the left-hand side and using the boundedness of $B(\cdot, \cdot)$ for the right-hand side, we obtain

$$c\|P_I u^h\|^2_{H_0^1(\Omega)} - C\|P_I u^h\|^2_{L^2(\Omega)} \leq \|u^h\|_{H_0^1(\Omega)} \|P_I u^h\|_{H_0^1(\Omega)}.$$

Noting that $P_I u^h \in \oplus_i H_0^1(\Omega_i)$, by using Friedrich's inequality in each subregion Ω_i, and taking the sum over all subregions, we have

$$\|P_I u^h\|^2_{L^2(\Omega)} \leq CH^2 \|P_I u^h\|^2_{H_0^1(\Omega)}.$$

Therefore, if H is small enough, as argued for (9), we obtain

$$\|P_I u^h\|_{H_0^1(\Omega)} \leq 1/c_1 \|u^h\|_{H_0^1(\Omega)}.$$

The bound for P_C can be obtained easily by using Lemma 4 and the triangle inequality,

$$\|P_C u^h\|_{H_0^1(\Omega)} \leq C\|u^h\|_{H_0^1(\Omega)}.$$

We now proceed to bound $P_E u^h$. By definition,

$$\langle \widetilde{P_E u^h}, \widetilde{P_E u^h} \rangle = B(u^h - P_I u^h, P_E u^h),$$

which can be bounded from above by

$$C\|u^h\|_{H_0^1(\Omega)}\|P_E u^h\|_{H_0^1(\Omega)}.$$

Applying Lemma 2, we have

$$\|P_E u^h\|^2_{H_0^1(\Omega)} \leq C\langle \widetilde{P_E u^h}, \widetilde{P_E u^h} \rangle \leq C\|u^h\|_{H_0^1(\Omega)}\|P_E u^h\|_{H_0^1(\Omega)},$$

i.e.,

$$\|P_E u^h\|_{H_0^1(\Omega)} \leq C\|u^h\|_{H_0^1(\Omega)}.$$

The upper bound for P can be obtained by combining the estimates for P_I, P_C and P_E.

Next, the lower bound part:

For any given $u^h \in V^h$, by using the definition of P, we have

$$A(u^h, Pu^h) = A(u^h, P_I u^h) + A(u^h, P_C u^h) + A(u^h, P_E u^h).$$

We estimate the right-hand side term by term. It is easy to see, using the H-lemma and letting H be small enough, that

$$\begin{aligned} A(u^h, P_I u^h) &= B(u^h, P_I u^h) - S(u^h, P_I u^h) - (\tilde{c}u^h, P_I u^h) \\ &= A(P_I u^h, P_I u^h) + (\tilde{c}(P_I u^h - u^h), P_I u^h) \\ &\quad - S(u^h, P_I u^h) \\ &\geq c(H_0)A(P_I u^h, P_I u^h) - CH\|u^h\|^2_{H_0^1(\Omega)} \end{aligned} \tag{17}$$

and similarly,

$$\begin{aligned} A(u^h, P_C u^h) &= A(P_C u^h, P_C u^h) + (\tilde{c}(P_C u^h - u^h), P_C u^h) \\ &\quad - S(u^h - P_C u^h, P_C u^h) \\ &\geq c(H_0)A(P_C u^h, P_C u^h) - CH\|u^h\|^2_{H_0^1(\Omega)} \end{aligned} \tag{18}$$

In our next step, using the H-lemma, we bound the term $A(u^h, P_E u^h)$ from below.

$$\begin{aligned}
\langle \widetilde{P_E u^h}, \widetilde{P_E u^h} \rangle &= B(u^h - P_I u^h, P_E u^h) \\
&= B(P_E u^h, u^h - P_I u^h) + 2S(u^h - P_I u^h, P_E u^h) \\
&= B(P_E u^h, u^h) + 2S(u^h - P_I u^h, P_E u^h) \\
&= B(u^h, P_E u^h) + 2S(-P_I u^h, P_E u^h) \\
&\leq C(A(u^h, P_E u^h) + \tfrac{H}{2} A(P_I u^h, P_I u^h) + \\
&\quad \tfrac{H}{2} A(P_E u^h, P_E u^h)) + CH \|u^h\|^2_{H_0^1(\Omega)}.
\end{aligned} \tag{19}$$

Using Lemma 2, we have

$$A(P_E u^h, P_E u^h) \leq C \langle \widetilde{P_E u^h}, \widetilde{P_E u^h} \rangle.$$

Together with (19), we obtain

$$\begin{aligned}
& A(u^h, P_E u^h) + \tfrac{H}{2} A(P_I u^h, P_I u^h) \\
& \geq c(1 - CH) \langle \widetilde{P_E u^h}, \widetilde{P_E u^h} \rangle - CH \|u^h\|^2_{H_0^1(\Omega)}.
\end{aligned} \tag{20}$$

Finally, by combining all the above estimates and assuming that H is small enough, we have

$$\begin{aligned}
A(u^h, P u^h) &\geq c(A(P_I u^h, P_I u^h) + A(P_C u^h, P_C u^h) \\
&\quad + \langle \widetilde{P_E u^h}, \widetilde{P_E u^h} \rangle) - CH \|u^h\|^2_{H_0^1(\Omega)} \\
&\geq (c/(1 + \ln(H/h)^3) - CH) \|u^h\|^2_{H_0^1(\Omega)} \\
&= (1 + \ln(H/h)^3)^{-1} (c - CH(1 + \ln(H/h)^3)) \|u^h\|^2_{H_0^1(\Omega)},
\end{aligned}$$

which completes the lower bound proof, if we require that $H(1 + \ln(H/h)^3)$ be small enough.□

5 A numerical check and concluding remarks

Motivated by empirically slowly growing iteration counts for the "tile algorithm" as $h \to 0$ with (H/h) fixed on a variety of boundary value problems (as in Table 3 of [12]), we have theoretically established bounds for a variant preconditioned system. These bounds depend on H and h only through the logarithm of their ratio, and are thus similar in form, though a logarithm or more weaker, to those of [2], [9] and [4] for other problems in the plane. In the union of this and these three previous papers, domain decomposed preconditioned Krylov iteration for combinations of the cases of selfadjoint or non-selfadjoint and overlapping or non-overlapping subdomains all receive coverage. We conclude with a number of remarks on the applicability of the analysis.

In the non-selfadjoint cases, H must be "sufficiently small" for the proofs to go through. "Sufficiently small" refers to the dominance of the second-order terms relative to the first- and zeroth-order terms. Roughly speaking, in

$$\Delta u + K_1 u_\alpha + K_0 u = f \ ,$$

where u_α represents a directional derivative, we must have $H|K_1|$ and $H^2|K_0|$ bounded by sufficiently small constants. The product $H|K_1|$ is a substructure Reynolds number. The restriction on the size of H is not an unexpected weakness in domain-decomposed preconditioning. Similar requirements of a "sufficiently fine" coarse grid are needed to make multigrid proofs go through in the presence of skew-symmetric perturbations to the elliptic terms; see, *e.g.*, [14].

When the bound on $H|K_1|$ is violated, full GMRES will still converge, but may become uneconomical, and restarted GMRES may theoretically fail. Thus, it may in general be necessary to accept a large cross point system in the step of the preconditioner involving P_C. In practice, in [12] the only restrictions on the size of the cross point system were related to overall storage, not to convergence failure.

Theorem 2 is not guaranteed to hold for the tile algorithm as formulated in [12] since the interface solve is defined therein by projecting the function itself to the interface instead of the B-discrete harmonic part, and by using as an interface operator simply the tangential part of L. Thus, the first set of subdomain solves (step 1 in the algorithm described herein) is skipped, and both the operators and right-hand sides of step 3 are different. As a result, Lemma 2 cannot be used in bounding the symmetric part of P from below in Theorem 2, and a factor of h appears in the final result. Of course, the tangential interface operator may have a better constant than the $l_0^{1/2}$-type operator in strongly convective problems, (see, *e.g.*, the comparisons in [7]), but the latter is better either as $h \to 0$ or as $|K_1| \to 0$. Numerical experiments with the self-adjoint example #5 from [12], in which the tangential operator is replaced with an FFT-implementable $l_0^{1/2}$, bear this expectation out.

We have implemented the present algorithm for homogeneous Dirichlet boundary conditions and tested it on the convection-diffusion problem (see also [20])

$$-\Delta u + \sigma \cdot \nabla u = f$$

on a uniformly gridded unit square, with $\sigma^t = (10, 10)$ and f so chosen that the solution is $e^{xy}\sin(\pi x)\sin(\pi y)$.

We report in the table below the iteration counts for three codes at three different levels of refinement, for a reduction in the initial residual (for a zero starting guess) of 10^{-5}. The first two columns give the fine and coarse mesh parameters, which are maintained at a constant ratio of 8 mesh cells per tile, as the mesh is refined. Note that the cell Reynolds numbers (essentially $h|\sigma|$) are modest in this problem so that strict Galerkin discretizations may be used without loss of monotonicity in the solution. The first column of results is for the code described in [12], the second for the present algorithm, and the third for an additive Schwarz algorithm for nonsymmetric problems, which also employs GMRES (see [4]). All three algorithms were implemented through a set of modules described in [12] and [13]. In the additive Schwarz algorithm, only one fine mesh width of overlap was employed. The three codes are not uniformly optimized, so no execution times are indicated. We point out that the iterations of the original tile algorithm are inherently cheaper than the present algorithm, since they involve half as many subdomain solves, and thus comparable times are achieved on the test problem. The additive Schwarz times are better than either on the finest grid.

h^{-1}	H^{-1}	Orig. Tile Alg.	Present	Add. Schw.
16	2	9	12	10
32	4	20	20	11
64	8	26	20	10

It is clear from the table that the asymptotic bound of Theorem 2 is achieved rather soon in h, and also that the $O(1)$ bound of additive Schwarz is achieved even sooner.

We have assumed throughout Dirichlet boundary conditions on Ω, a quasi-uniform h-triangulation, and exact subdomain solves in the preconditioner. In applications, it is of interest to relax all three. The extension to Robin or Neumann boundary conditions (as long as the degenerate Neumann case is avoided) involves a standard modification of the proof, in which boundary terms are retained in (2). Only the constants change in Theorem 2.

Locally adaptive refinement is a practical motivation for moving away from a quasi-uniform h-triangulation. The experiments with refinement in [12] show comparable iteration counts for globally and locally refined problems with the same effective h, although we have not so extended the main theorem of this paper.

Like adaptive refinement, the use of approximate subdomain solves in defining the preconditioner (a modification of the definition of P_I), introduces complications that we have not included in the proofs. Approximate factorizations (ILU and MILU) have proved to be fairly *un*satisfactory replacements for the exact P_I in [12]. It is expected that spectrally equivalent fast Poisson solvers will prove to be acceptable replacements, in the $h \to 0$ limit, and that the main theorem can be generalized to include them with suitable replacement of the interior problem in (4).

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