A scheme is presented for the solution of inverse scattering problems for the one-dimensional Helmholtz equation. The scheme is based on a combination of the standard Riccati equation for the impedance function with a new trace formula for the derivative of the potential, and can be viewed as a frequency domain version of the layer-stripping approach. The principal advantage of our procedure is that if the scatterer to be reconstructed has $m \geq 1$ continous derivatives, the accuracy of the reconstruction is proportional to $1/a^m$, where a is the highest frequency for which scattering data are available. Thus, a smooth scatterer is reconstructed very accurately from a limited amount of available data.

The scheme has the asymptotic cost $O(n^2)$, where n is the number of features to be recovered (as do classical layer-stripping algorithms), and is stable with respect to perturbations of the scattering data. The performance of the algorithm is illustrated with several numerical examples.

On the Inverse Scattering Problem for the Helmholtz Equation in One Dimension

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1 Introduction

During the last several decades, the inverse scattering problems for the Helmholtz equation have enjoyed a remarkable degree of popularity, both in pure and applied contexts (see, e.g., [1], [2]). A number of algorithms has been proposed for the numerical treatment of these problems, in such environments as medical diagnostics, non-destructive industrial testing, anti-submarine warfare, oil exploration, etc. In the design of such a scheme, three major problems have to be overcome.

1. The problem is highly non-linear, even in its purely mathematical form. In the one-dimensional case, the problem can be reduced to a linear one, but the procedure is not stable numerically.

2. Once a mathematically valid inversion scheme is constructed, it might or might not be stable numerically. In fact, no numerically robust schemes seem to exist at this time, except in one dimension.

3. The cost of applying the scheme on the computer tends to be extremely high, except in the one-dimensional case.

The existing attempts to solve inverse scattering problems for the Helmholtz equation can be roughly subdivided into four groups.

1. Linearized inversion schemes, attempting to approximate the inverse scattering problem by the problem of inverting an appropriately chosen linear operator (see, for example, [2]).

2. Methods based on the non-linear optimization techniques, attempting to recover the parameters of the problem iteratively, by solving a sequence of forward scattering problems (see, for example, [3], [4], [5]).

3. Gel'fand-Levitan and related techniques, converting the Helmholtz equation into the Schrödinger equation, the inverse problem for the latter being reducible to the solution of a linear Volterra integral equation (see, for example, [1], [6]).

4. Techniques based on the so-called trace formulae, connecting the high-frequency behavior of the solutions of the Helmholtz equation with the local values of the parameters to be recovered (see, for example, [7], [8], [9]).

The approach of this paper falls into the category 4 above, and is different from the preceding work in the choice of the trace formula (see Theorem 4.4 in Section 4.3). The new trace formula leads to an algorithm with superior convergence properties for smooth scatterers (see Section 5 below), and the resulting numerical procedure is extremely stable and efficient.

The plan of the paper is as follows. Section 2 contains the exact formulation of the problem to be addressed, together with the relevant notation. In Section 3, we summarize the background facts to be used in the paper. Section 4 is devoted to the development of the mathematical apparatus used to construct the algorithm, and in Section 5 the scheme itself is presented. In Section 6 we present several numerical examples demonstrating the actual performance of the procedure. Finally, in Section 7 we discuss the possible generalizations of the approach to higher dimensions and to systems of differential equations.

2 Formulation of the Problem

Following the standard practice, we will be considering the one dimensional scalar Helmholtz equation

$$\phi''(x,k) + k^2(1+q(x))\phi(x,k) = 0. \tag{1}$$

Unless specified otherwise, we will be assuming that $q \in c_0^2([0,1])$, i.e., that q is twice continuously differentiable everywhere, and that $q(x) = 0$ for all $x \notin [0,1]$. Defining the function $n : R \to R$ by the formula

$$n(x) = \sqrt{1+q(x)}, \tag{2}$$

we will denote by n_0, n_1 the minimum and maximum of n respectively, and assume that $0 < n_0$ so that

$$n_0 \le n(x) = \sqrt{1+q(x)} \le n_1. \tag{3}$$

For any complex k, we consider solutions of the Helmholtz equation $\phi_+(x,k)$ and $\phi_-(x,k)$ which have the form

$$\phi_+(x,k) = \phi_{inc+}(x,k) + \phi_{scat+}(x,k), \tag{4}$$
$$\phi_-(x,k) = \phi_{inc-}(x,k) + \phi_{scat-}(x,k) \tag{5}$$

with

$$\phi_{inc+}(x,k) = e^{ikx}, \tag{6}$$
$$\phi_{inc-}(x,k) = e^{-ikx} \tag{7}$$

and $\phi_{scat+}, \phi_{scat-}$ both satisfying the outgoing radiation boundary conditions

$$\phi'_{scat}(0,k) + ik\phi_{scat}(0,k) = 0, \tag{8}$$
$$\phi'_{scat}(1,k) - ik\phi_{scat}(1,k) = 0. \tag{9}$$

Normally, ϕ_{inc+} and ϕ_{inc-} are referred to as right-going and left-going incident fields respectively, and ϕ_{scat+} and ϕ_{scat-} are called scattered fields corresponding to the excitations ϕ_{inc+} and ϕ_{inc-}. The sum of an incident field and its corresponding scattered field is called the total field.

Remark 2.1. Throughout this paper, given a function $f(x,k)$, we will take the liberty to denote $\frac{\partial f}{\partial x}$ by $f'(x,k)$, so that the derivatives in the formulae (15), (16) are with respect to x.

As is well-known, for any complex k, the scattered fields $\phi_{scat+}(x,k)$ and $\phi_{scat-}(x,k)$ satisfy the nonhomogeneous Helmholtz equations

$$\phi''_{scat+}(x,k) + k^2(1+q(x))\phi_{scat+}(x,k) = -k^2 q(x)e^{ikx}, \tag{10}$$
$$\phi''_{scat-}(x,k) + k^2(1+q(x))\phi_{scat-}(x,k) = -k^2 q(x)e^{-ikx}. \tag{11}$$

Since $q(x) = 0$ for all $x \notin (0,1)$, it is easy to see that for any $k \in C$ there exist two complex numbers $\mu_+(k), \mu_-(k)$, identified as the reflection coefficients, such that

$$\phi_{scat}(x,k) = \mu_+(k)\cdot e^{-ikx},\ for\ all\ x \leq 0, \tag{12}$$
$$\phi_{scat}(x,k) = \mu_-(k)\cdot e^{ikx}.\ for\ all\ x \geq 1, \tag{13}$$

due to (10), (8) and (11), (9) respectively.

Denote by C^+ the upper half of the complex plane so that

$$C^+ = \{k \in C | Im(k) \geq 0\}. \tag{14}$$

For any $k \in C^+$, the impedance functions $p_+(x,k), p_-(x,k)$ associated with $\phi_+(x,k), \phi_-(x,k)$, respectively, are defined by the formulae

$$p_+(x,k) = \frac{\phi'_+(x,k)}{ik\phi_+(x,k)}, \tag{15}$$
$$p_-(x,k) = \frac{\phi'_-(x,k)}{-ik\phi_-(x,k)}. \tag{16}$$

Remark 2.2. For x outside the scatterer, it is easy to obtain explicit expressions for p_+, p_- in terms of reflection coefficients μ_+, μ_-. Indeed, combining (4) with (12), (5) with (13), we have

$$\phi_+(x,k) = e^{ikx} + \mu_+(k)e^{-ikx},\ for\ all\ x \leq 0, \tag{17}$$
$$\phi_-(x,k) = e^{-ikx} + \mu_-(k)e^{ikx},\ for\ all\ x \geq 1, \tag{18}$$

which can be reformulated as

$$\phi_+(x,k) = e^{ikx} + b_+(k)e^{-ikx+\alpha_+(k)},\ for\ all\ x \leq 0, \tag{19}$$
$$\phi_-(x,k) = e^{-ikx} + b_-(k)e^{ikx+\alpha_-(k)},\ for\ all\ x \geq 1. \tag{20}$$

with $\alpha_+(k), \alpha_-(k)$ real numbers and $b_+(k) \geq 0, b_-(k) \geq 0$, for any $k \in C$. Consequently,

$$p_+(x,k) = \frac{1 - b_+^2(k) + i2b_+(k)\sin(kx - \alpha_+(k))}{1 + b_+^2(k) + 2b_+(k)\cos(kx - \alpha_+(k))} \tag{21}$$

for all $x \leq 0$, and

$$p_-(x,k) = \frac{1 - b_-^2(k) + i2b_-(k)\sin(kx - \alpha_-(k))}{1 + b_-^2(k) + 2b_-(k)\cos(kx - \alpha_-(k))} \tag{22}$$

for all $x \geq 1$.

For any complex number k, the boundary value problems for ϕ_+, ϕ_- can be reformulated as initial value problems. More specifically, formulae (4), (5), (12) and (13) imply that there exist such complex constants α, β, depending only on k, that

$$\phi_+(x,k) = \alpha \cdot e^{ikx}, \ \textit{for all } x \geq 1, \tag{23}$$

$$\phi_-(x,k) = \beta \cdot e^{-ikx}, \ \textit{for all } x \leq 0. \tag{24}$$

Furthermore, α, β are nonzero because, e.g., if $\beta = 0$, then $\phi_-(0,k) = \phi'_-(0,k) = 0$, according to uniqueness theorem on initial value problems, $\phi_-(x,k) = 0$ for all $x \in R$, i.e.,

$$\phi_{scat-}(x,k) = -\phi_{inc-}(x,k) = -e^{-ikx}, \tag{25}$$

contradicting to (13). Clearly, formulae (23), (24) can be used as initial conditions for equation (1) to (uniquely) determine the total fields ϕ_+, ϕ_-.

Remark 2.3. While the existence and uniqueness of the functions $\phi_+(x,k), \phi_-(x,k)$ are quite obvious for any complex k, the functions $p_+(x,k), p_-(x,k)$ are only well-defined when $Im(k) \geq 0$, and the proof of this fact is somewhat involved (see lemmas in Section 4.1 below).

Remark 2.4. It is easy to see that the impedance functions p_+, p_- are independent of the nonzero coefficients α, β in (23), (24). Therefore, for simplicity, the initial conditions (23), (24) are reformulated as

$$\phi_+(x,k) = e^{ikx}, \ \textit{for all } x \geq 1, \tag{26}$$

$$\phi_-(x,k) = e^{-ikx}. \ \textit{for al } x \leq 0 \tag{27}$$

The functions ϕ_+, ϕ_- as solutions of equation (1) subject to boundary conditions (26), (27) differ from those subject to boundary conditions (23), (24) by constants.

The classical inverse scattering problem for the equation (1) is as follows:
Problem 1. Given the impedance function $p_+(0,k)$ for all $k \in R$, reconstruct the potential q for all $x \in [0,1]$.

It is well-known that this problem has a unique solution (and in the class of functions q much broader than $c_0^2([0,1])$, and several constructive schemes for that purpose have been proposed, most notably the Gelfand-Levitan and related methods. However, in applications the impedance function $p_+(0,k)$ is measured with a finite accuracy and at a finite number of (usually equispaced) values of the wavenumber k. Therefore, the following problem is more relevant in numerical applications

Problem 2. Suppose that the impedance function $p_+(0,k)$ is given at a finite number of frequencies $k_j, j = 1, 2, \ldots, N$ defined by the formulae $k_j = j \cdot h$, with h a positive constant. Suppose further that the values $p_+(0,k_j)$ are given with the relative accuracy ϵ. Reconstruct the potential q in the interval $[0,1]$ with the error that rapidly decreases with increasing N and decreasing h.

The present paper is devoted to the construction of an algorithm for the solution of the Problem 2.

Observation 2.1. The value of impedance function p_+ at $x = x_0$, $x_0 \leq 0$ can be obtained from $\phi_+(x_0, k)$ in the following manner. Assuming that at $x \leq 0$, the total field $\phi_+(x, k)$ is given by (17), from which $\mu_+(k)$ can be obtained

$$\mu_+(k) = \left(\phi_+(x_0, k) - e^{ikx_0}\right) e^{ikx_0}, \tag{28}$$

the value of the impedance function p_+ at $x = x_0$ is then

$$p_+(x_0, k) = \frac{\phi'_+(x_0, k)}{ik\phi_+(x_0, k)} = \frac{1 - \mu_+(k)e^{-2ikx_0}}{1 + \mu_+(k)e^{-2ikx_0}} \tag{29}$$

$$= 2\frac{e^{ikx_0}}{\phi_+(x_0, k)} - 1. \tag{30}$$

Similarly, for any $x_1 \geq 1$,

$$p_-(x_1, k) = \frac{1 - \mu_-(k)e^{2ikx_1}}{1 + \mu_-(k)e^{2ikx_1}} = 2\frac{e^{-ikx_1}}{\phi_-(x_1, k)} - 1. \tag{31}$$

3 Mathematical Preliminaries

In this section, we summarize several well-known mathematical facts to be used in the rest of the paper. These facts are given without proofs, since Lemmas 3.1–3.6 are found in standard textbooks (see, for example, [11], [10], Michelin) and Lemmas 3.7–3.10 are easy to verify directly.

Lemma 3.1 *Suppose that A is a linear mapping $C[0,1] \to C[0,1]$ and that $||A|| \leq \rho$, with ρ a real number such that $\rho < 1$. Then for any $g \in L^2[0,1]$, the equation*

$$\phi = A\phi + g \tag{32}$$

has a unique solution, which is the sum of the series (known as Neumann's series)

$$\phi = \sum_{j=0}^{\infty} A^j g. \tag{33}$$

Furthermore,

$$||\phi - \sum_{j=0}^{n} A^j g|| \leq \frac{\rho^{n+1}}{1-\rho}||g||. \tag{34}$$

Lemma 3.2 *Suppose that $f \in c_0^m([0,D])$ (i.e., f has m continuous derivatives and $f(x) = 0$ for all $x \notin (0,D)$), and that $f^{(m)}$ is absolutely continuous. Suppose further that $g \in c^{m+1}(R)$, $g^{(m+1)}$ is absolutely continuous and there exist real number $a > 0$ such that $g'(x) \geq a$ for all $x \in R$. Then there exists a real $c > 0$ such that*

$$\left| \int_0^D f(x) e^{ik(x+g(x))} dx \right| < \frac{c}{|k|^{m+1}} \tag{35}$$

for all complex k such that $Im(k) \geq 0$.

Lemma 3.3 *Suppose that $f \in c^l(R)$ with l a nonnegative integer. Suppose further that $f^{(j)}(0) = 0$ for $0 \leq j \leq l$, $f^{(l)}$ is absolutely continuous. Then there exists a positive number c such that*

$$\int_0^x f(t) e^{ik(x-t)} dt = -\sum_{j=1}^{l} \left(\frac{1}{2ik}\right)^j f^{j-1}(x) + \left(\frac{1}{2ik}\right)^{l+1} (f^l(x) + b(x,k)) \tag{36}$$

with $b : R \times C^+ \to C$ an absolutely continuous function of $x \in [0,1]$ such that

$$|b(x,k)| \leq c. \tag{37}$$

for all $x \in [0,1]$, $k \in C^+$. Furthermore, if $f(x) = 0$ for all $x \geq D$ with D a positive number, then

$$|b(x,k)| \leq c. \tag{38}$$

for all $(x,k) \in R \times C^+$

Lemma 3.4 *Suppose that $a : [0,1] \to R$ and $b : [0,1] \to C$ are two continuous functions, and that $a(x) > 0$, for all $x \in [0,1]$. Then for any two solutions u and v of the second order ODE*

$$(a(x)\phi'(x))' + b(x)\phi(x) = 0, \tag{39}$$

there exists a constant c such that

$$a(x)(u(x)v'(x) - v(x)u'(x)) = c \tag{40}$$

for all $x \in [0,1]$. Furthermore, $c \neq 0$ if and only if u and v are linearly independent. (The expression $W(u,v) = u(x)v'(x) - v(x)u'(x)$ is referred to as the Wronskian of the pair u, v).

Lemma 3.5 *Suppose that $G_k : [0,1] \times [0,1] \to C$ is the Green's function of the boundary value problem*

$$\psi''(x,k) + k^2\psi(x,k) = 0, \tag{41}$$

$$\psi'(0,k) + ik\psi(0,k) = 0, \tag{42}$$

$$\psi'(1,k) - ik\psi(1,k) = 0. \tag{43}$$

for any complex $k \neq 0$. Then the boundary value problem

$$\psi''(x,k) + (k^2 + \eta(x))\psi(x,k) = f(x,k) \tag{44}$$
$$\psi'(0,k) + ik\psi(0,k) = 0, \tag{45}$$
$$\psi'(1,k) - ik\psi(1,k) = 0. \tag{46}$$

is equivalent to a second kind integral equation

$$\psi(x,k) = -\int_0^1 G_k(x,t)\eta(t)\psi(t,k)dt + g(x,k) \tag{47}$$

with $f, g : [0,1] \times C \to C$ and g defined by the formula

$$g(x,k) = \int_0^1 G_k(x,t)f(t,k)dt. \tag{48}$$

Lemma 3.6 *For any complex $k \neq 0$, the Helmholtz equation*

$$\psi''(x,k) + k^2\psi(x,k) = 0 \tag{49}$$

with the outgoing radiation conditions (8) (9) has the Green's function

$$G_k(x,t) = \frac{1}{2ik}\begin{cases} e^{ik(t-x)}, & x \leq t, \\ e^{ik(x-t)}, & x \geq t. \end{cases} \tag{50}$$

Lemma 3.7 *Suppose that $q : R \to R$ is a c^2–function such that $q > -1$ for all $x \in R$. Suppose further that the functions $n, x, S, \eta, g : R \to R$ are defined by the formulae*

$$n(x) = \sqrt{1 + q(x)}, \tag{51}$$

$$t(x) = \int_0^x n(\tau)d\tau, \tag{52}$$

$$S(t) = e^{-\int_0^t \frac{n'(x(\tau))}{2(n(x(\tau)))^2}d\tau} \tag{53}$$

$$\eta(t) = \frac{S''(t)}{S(t)} - \frac{n'(x)}{2(n(x))^2}, \tag{54}$$

$$g(t) = \frac{f(x)}{S(t)}. \tag{55}$$

Finally, suppose that the function $\phi : R \times C \to C$ satisfies the equation

$$\phi''(x,k) + k^2(1 + q(x)) \cdot \phi(x,k) = f(x), \tag{56}$$

and the function $\psi : R \times C \to C$ is defined by the formula

$$\psi(t,k) = \phi(x(t),k)/S(t). \tag{57}$$

Then the function ψ satisfies the Schrödinger equation

$$\psi''(t,k) + (k^2 + \eta(t)) \cdot \psi(t,k) = g(t). \tag{58}$$

at all $t \in R$.

Remark 3.2. Lemma 3.7 provides a connection between the solutions of the Helmholtz equation (56) and those of the appropriately chosen Schrödinger equation (58). This connection will be used in the following section as an analytical tool. However, it is not very useful in numerical computations since the expression (53) can yield unmanageably large values of S even for perfectly well-behaved q.

Corollary 3.1 *Suppose that under the conditions of the preceding lemma that $q(x) = 0$ for all $x \notin (0,1)$. Suppose further that the functions $\psi_+, \psi_- : R \times C \to C$ are defined by the formulae*

$$\psi_+(t,k) = \phi_+(x(t),k)/S(t), \tag{59}$$
$$\psi_-(t,k) = \phi_-(x(t),k)/S(t). \tag{60}$$

Then ψ_+, ψ_- satisfy the ODEs

$$\psi_+''(t,k) + (k^2 + \eta(t)) \cdot \psi_+(t,k) = 0, \tag{61}$$
$$\psi_-''(t,k) + (k^2 + \eta(t)) \cdot \psi_-(t,k) = 0 \tag{62}$$

subject to the boundary conditions

$$\psi_+(t,k) = \xi(k) \cdot e^{ik(t-T_1)} \tag{63}$$

for all $t \geq T_1$, and

$$\psi_-(t,k) = e^{-ikt} \tag{64}$$

for all $t \leq 0$ with $T_1 > 0$, $\xi(k) \neq 0$ defined by the formulae

$$T_1 = t(1) = \int_0^1 n(\tau)d\tau, \tag{65}$$
$$\xi(k) = S(T_1)e^{ik}. \tag{66}$$

Furthermore,

$$p_+(x,k) = n(x)\frac{\psi_+'(t,k)}{ik\psi_+(t,k)} - \frac{n'(x)}{2ikn(x)}, \tag{67}$$
$$p_-(x,k) = n(x)\frac{\psi_-'(t,k)}{-ik\psi_-(t,k)} + \frac{n'(x)}{2ikn(x)}. \tag{68}$$

Observation 3.1. Suppose that $q(x) = 0$ for all $x \notin (0,1)$. Then according to Lemma 3.7 and Corollary 3.1,

$$t = x, \tag{69}$$
$$S(t) = 1, \tag{70}$$

and consequently

$$\phi_+(x,k) = \psi_+(t,k) \tag{71}$$

for all $x \leq 0$. Now, suppose the function ψ_+ is defined by formulae (59), (17). Defining the scattered field $\psi_{scat+} : R \times C \to C$ by the formula

$$\psi_+(t,k) = e^{ikt} + \psi_{scat+}(t,k), \tag{72}$$

we immediately see that

$$\psi_{scat+}(t,k) = \mu_+(k) \cdot e^{-ikt} \tag{73}$$

for all $x \leq 0$ due to (71), (17), (72). Finally, combining (72) with (61), we observe that ψ_{scat+} satisfies the Schrödinger equation

$$\psi''_{scat+}(t,k) + (k^2 + \eta(t))\psi_{scat+}(t,k) = \frac{-k^2 q(x(t)) e^{ikx(t)}}{S(t)} \tag{74}$$

subject to outgoing radiation conditions (8), (9) (the latter due to (73), (63)).

Lemma 3.8 *Suppose that under the conditions of the preceding lemma,*

$$\begin{aligned} \phi_+(x_0,k_0) &\neq 0, &(75)\\ \phi_-(x_0,k_0) &\neq 0 &(76) \end{aligned}$$

at some point $(x_0,k_0) \in R \times C$. Then there exists a neighborhood D of (x_0,k_0) such that the impedance functions p_+, p_- satisfy the Riccati equations

$$\begin{aligned} p'_+(x,k) &= -ik(p_+^2(x,k) - (1+q(x)), &(77)\\ p'_-(x,k) &= ik(p_-^2(x,k) - (1+q(x)) &(78) \end{aligned}$$

for all $(x,k) \in D$.

Observation 3.2. Combining formulae (23), (24), we easily observe that

$$\begin{aligned} p_+(x,k) &= 1, \ for\ all\ x \geq 1, &(79)\\ p_-(x,k) &= 1, \ for\ all\ x \leq 0, &(80) \end{aligned}$$

for all complex $k \neq 0$.

Lemma 3.9 *(Gronwall's inequality) Suppose that $u, v, w : [0,a] \to R$ are three continuous and nonnegative functions, satisfying the inequality*

$$w(x) \leq u(x) + \int_0^x v(t)w(t)dt \tag{81}$$

for all $x \in [0,a]$. Then

$$w(x) \leq u(x) + \int_0^x u(t)v(t)e^{\int_t^x v(\tau)d\tau} dt \tag{82}$$

for all $x \in [0, a]$.

The following lemma is a special case of the general theorem about continuous dependence on initial conditions and parameters of solutions of ODEs (see, for example, [10]).

Lemma 3.10 *Suppose that* $a : C \to C$ *is an entire function and that* $A : R \times C \to C^{n\times n}$ *is an* $n \times n$*-matrix whose entries* $a_{i,j}(x, z), i, j = 1, \ldots, n$ *are continuous functions of* x *and entire functions of* z *for all* $x \in R$. *Then for any* $z \in C$, *the differential equation*

$$y'(x, z) = A(x, z) \cdot y(x, z) \tag{83}$$

subject to the initial condition

$$y(0) = c(z) \tag{84}$$

has an unique solution $y(x, z)$ *for all* $x \in R$. *Moreover,* $y(x, z)$ *is an entire function of* z.

4 Impedance Functions and Their Properties

In this section, we investigate analytical properties of the impedance functions p_+, p_-. Our principal purpose here is to formulate exactly and prove the following three facts.

(1) For any $x \in R$, the impedance functions $p_+(x, k), p_-(x, k)$ are analytic functions of k in the upper half plane C^+. Furthermore,

$$p_+(x, k) = \sqrt{1 + q(x)} - \frac{q'(x)}{4(1 + q(x))} \cdot \frac{1}{ik} + O(k^{-2}), \tag{85}$$

$$p_-(x, k) = \sqrt{1 + q(x)} + \frac{q'(x)}{4(1 + q(x))} \cdot \frac{1}{ik} + O(k^{-2}), \tag{86}$$

for all $x \in R$, $k \in C^+$ (see Theorem 4.1 below).

(2) For large real k, the difference between $\overline{p_+}$ and p_- is extremely small (it decays like k^{-m}, where m is the smoothness of the scatterer, see Theorem 4.3 below). The expressions (85), (86) are the first two terms in WKB expansions of the functions p_+, p_-, respectively.

(3) For any $a > 0$, and all $x \in R$, we have the so-called trace formula

$$q'(x) = \frac{2}{\pi}(1 + q(x)) \int_{-a}^{a} (p_+(x, k) - p_-(x, k))dk + O(a^{-(m-1)}), \tag{87}$$

with m the smoothness of the scatterer (see see Theorem 4.4 below).

As often happens, the statements (1)–(3) above have extremely simple formulations, and a transparent physical interpretation. However, their proofs are technical and do not follow any simple physical intuition.

4.1 Boundedness

The following five lemmas establish the basic properties of the impedance functions p_+, p_- introduced in Section 1. Lemma 4.1 is a technical one, describing the behavior of ϕ_+, ϕ_- in the vicinity of $k = 0$ in the complex plane. Lemma 4.2 describes the properties of the impedance functions p_+, p_- near $k = 0$, Lemma 4.3 demonstrates the well-definedness of the impedance functions for real k, and Lemmas 4.4 and 4.5 provide upper and lower bounds for the impedance functions.

Lemma 4.1 *Suppose that $q \in c([0,1])$ and $A > 0$ is a real number. Then there exist three positive numbers δ, α and β such that*

1. $|\phi_+(x,k) - 1| \le \alpha|k|,$ (88)
2. $|\phi_-(x,k) - 1| \le \alpha|k|,$ (89)
3. $|\phi'_+(x,k) - ik| \le \beta|k|^2,$ (90)
4. $|\phi'_-(x,k) + ik| \le \beta|k|^2,$ (91)
5. $\phi_+(x,k) \ne 0,$ (92)
6. $\phi_-(x,k) \ne 0,$ (93)

for all real $x \in [-A, A]$ and complex k such that $|k| < \delta$.

Proof. Since the proofs of this lemma for ϕ_+, ϕ'_+ and for ϕ_-, ϕ'_- are identical, we only prove it in the case of ϕ_-, ϕ'_-. Defining two auxiliary functions $\phi_1, \psi : R \times C \to C$ by the formulae

$$\phi_1(x,k) = \phi_-(x,k) - 1, \tag{94}$$

$$\psi(x,k) = \phi'_-(x,k) + ik, \tag{95}$$

and combining (94), (95) with equation (1) and the initial condition (27), we observe that the functions ϕ_1, ψ satisfy the linear first order ODEs

$$\phi'_1(x,k) = \psi(x,k) + ik, \tag{96}$$

$$\psi'(x,k) = -k^2(1+q(x))(1+\phi_1(x,k)) \tag{97}$$

subject to the initial conditions

$$\phi_1(0,k) = 0, \tag{98}$$

$$\psi(0,k) = 0. \tag{99}$$

We start with showing that there exist continuous functions M, $N : R^+ \times R^+ \to R^+$ such that, for any $s \in R^+$, $M(s,t)$, $N(s,t)$ are monotonically increasing functions of t for all $t \in R^+$ and

$$|\phi_1(x,k)| \le M(A,|k|)|k|, \tag{100}$$

$$|\psi(x,k)| \le N(A,|k|)|k|^2. \tag{101}$$

First, we prove the estimate (101). Integrating (96) from 0 to x, we have

$$\phi_1(x,k) = \int_0^x (ik + \psi(t,k))dt, \tag{102}$$

and substituting (102) into (97) and integrating the result of the substitution, obtain

$$\psi(x,k) = -k^2 \int_0^x (1+q(t)) \left(1 + \int_0^t (ik + \psi(\tau,k))d\tau\right) dt. \tag{103}$$

Denoting $|\psi(x,k)|$ by $a(x,k)$ and observing that $1+q(x) \le n_1^2$ (see (3) in Section 2), we obtain

$$\begin{aligned} a(x,k) &\le |k|^2 n_1^2 \left(|x| + \frac{1}{2}x^2|k| + \int_0^x \int_0^t a(\tau,k)d\tau dt\right) \\ &\le |k|^2 n_1^2 \left(|x| + \frac{1}{2}x^2|k|\right) + |k|^2 n_1^2 \int_0^x (x-t)a(t,k)dt \end{aligned} \tag{104}$$

for any $x \in R$. Gronwall's inequality (see Lemma 3.9) implies that for any $x \in [0,A]$,

$$\begin{aligned} a(x,k) &\le |k|^2 n_1^2 \left(|x| + \frac{1}{2}x^2|k| + \int_0^x |t| + \frac{1}{2}t^2|k|(x-t)e^{\frac{1}{2}(x-t)^2}dt\right) \\ &\le N(A,|k|)|k|^2. \end{aligned} \tag{105}$$

It is easy to see that (105) is also valid for any $x \in [-A,0]$, and we obtain the estimate (101) with $N(A,k)$ defined by the formula

$$N(A,|k|) = \sup_{-A<x<A} n_1^2 \left(|x| + \frac{1}{2}x^2|k| + \int_0^x |t| + \frac{1}{2}t^2|k|(x-t)e^{\frac{1}{2}(x-t)^2}dt\right). \tag{106}$$

We now turn our attention to the estimate (100). Substituting (101) into (102), we obtain

$$\begin{aligned} |\phi_1(x,k)| &\le |x| \left(|k| + N(A,|k|)|k|^2\right) \\ &\le M(A,|k|)|k|, \end{aligned} \tag{107}$$

with

$$M(A,|k|) = A(1 + |k|N(A,|k|)), \tag{108}$$

for all real $x \in [-A,A]$ and complex k, which proves (100).

Now, the estimates (89) and (91) easily follow from (100) and (101). Indeed, since $M(A,t)$ is a continuous, monotonically increasing function of t, there exists a real δ such that

$$M(A,\delta) \cdot \delta < 1. \tag{109}$$

Denoting $M(A,\delta)$ by α, $N(A,\delta)$ by β and observing that $M(A,|k|), N(A,|k|)$ are monotonically increasing functions of $|k|$, we have

$$|\phi_1(x,k)| \le M(A,|k|)|k| \le M(A,\delta)|k| = \alpha|k|, \tag{110}$$

$$|\psi(x,k)| \le N(A,|k|)|k| \le N(A,\delta)|k| = \beta|k|^2, \tag{111}$$

from which (89), (91) follow immediately.

Finally, (93) is a direct consequence of (110) and (109). □

Lemma 4.2 *Suppose that $q \in c_0^2([0,1])$ and $A > 0$ is a real number. Then there exists $\delta > 0$ such that the impedance functions p_+, p_- are continuous functions of (x,k) for all real $(x,k) \in D$ with*

$$D = \{(x,k) | x \in [-A, A], k \in C, k \neq 0, |k| \leq \delta\} \tag{112}$$

Furthermore,

$$\lim_{k \to 0} p_+(x,k) = 1, \tag{113}$$
$$\lim_{k \to 0} p_-(x,k) = 1. \tag{114}$$

Proof. Due to Lemma 4.1, there exists a positive number δ such that $\phi_+(x,k) \neq 0$, $\phi_-(x,k) \neq 0$ for all real $(x,k) \in D$. Therefore, the functions p_+, p_- are well-defined in D, and their continuity follows from the continuity of ϕ_+, ϕ'_+, ϕ_-, ϕ'_- and the formulae (15), (16). Finally, (113), (114) are direct consequences of Formulae (88)– (91). □

Remark 4.1. While the impedance functions p_+, p_- are continuous in the vicinity of $k = 0$ in the complex plane, formulae (15), (16) fail to define p_+, p_- at $k = 0$. We now can define $p_+(x,0) = p_-(x,0) = 1$ for all $x \in R$ due to Lemma 4.2.

Lemma 4.3 *For any real $k \neq 0$ and all $x \in R$*

$$\phi_+(x,k) \neq 0, \tag{115}$$
$$\phi'_+(x,k) \neq 0, \tag{116}$$
$$\phi_-(x,k) \neq 0, \tag{117}$$
$$\phi'_-(x,k) \neq 0. \tag{118}$$

Proof. Again, since the proofs of this lemma for ϕ_+, ϕ'_+ and for ϕ_-, ϕ'_- are identical, we only prove (117) and (118). Denoting the real part of ϕ_- by u and the imaginary part by v, so that

$$\phi_-(x,k) = u(x,k) + iv(x,k), \tag{119}$$
$$\phi'_-(x,k) = u'(x,k) + iv'(x,k), \tag{120}$$

we observe that each of the functions u, v satisfies equation (1) (since the coefficients of the equation are real). Combining the initial condition (27) with (119), we immediately see that

$$u(x,k) = \cos(kx), \tag{121}$$
$$v(x,k) = \sin(kx) \tag{122}$$

for all $x \leq 0$ and $k \neq 0$. Therefore, the Wronskian of the pair u, v is

$$W(u,v) = k, \tag{123}$$

for any $x \in R$ (see Lemma 3.4), and $u(x,k)$, $v(x,k)$ can not be both zero, nor can $u'(x,k)$, $v'(x,k)$, for any $x \in R$ and $k \neq 0$. Now, formulae (117) and (118) immediately follow from (119) and (120) □

We have shown that the impedance functions p_+, p_- are well-defined for all real k (see Lemmas 4.2, 4.3 and Remark 4.1). Now, we turn our attention to the well-definedness of the impedance functions on the upper half of the k-plane. First we provide the lower bounds for p_+, p_-.

Lemma 4.4 *For all $x \in R$ and any k such that $Im(k) > 0$,*

$$
\begin{aligned}
Re(p_+(x,k)) &\geq n_0 \sin(\arg(k)), && (124)\\
Re(p_-(x,k)) &\geq n_0 \sin(\arg(k)) && (125)
\end{aligned}
$$

with $0 < n_0 \leq 1$ the minimum of $n(x)$ (see (3) in Section 2), and $\arg(k)$ the argument of the complex wave number k.

Proof. Since the proof of (124) and that of (125) are identical, we only provide the latter. Observing that

$$
Re(p_-(x,k)) = p_-(x,k) = 1 > n_0^2 \sin(\arg(k)), \tag{126}
$$

for any $Im(k) > 0$ and all $x \leq 0$ (see (80) in Section 3), we will prove (125) by showing that

$$
\frac{\partial}{\partial x}\left(Re(p_-(x,k))\right) \geq 0 \tag{127}
$$

for any $x > 0$ such that

$$
0 \leq Re(p_-(x,k)) \leq n_0 \sin(\arg(k)) \tag{128}
$$

(obviously, $0 < arg(k) < \pi$ for any k such that $Im(k) > 0$).

We will denote by a, b, u, v the real and imaginary parts of k and p_- respectively, so that

$$
\begin{aligned}
k &= a + ib, && (129)\\
p_-(x,k) &= u(x,k) + iv(x,k), && (130)
\end{aligned}
$$

with $b > 0$. Now, we can rewrite the Riccati equation (78) for p_- in the form

$$
\begin{aligned}
u' &= b(v^2 - u^2 + n^2) - 2auv, && (131)\\
v' &= -a(v^2 - u^2 + n^2) - 2buv. && (132)
\end{aligned}
$$

We observe that $\frac{\partial}{\partial x}u(x,k)$ is a function of u, v given by the formula

$$
\frac{\partial}{\partial x}u(x,k) = f(u,v) = b(v^2 - u^2 + n^2) - 2auv. \tag{133}
$$

Denoting the interval $[0, n_0 \sin(\arg(k))]$ by I, and defining the region $D \subset R \times R$ via the formula

$$D = \{(u,v) | u \in I, v \in R\}, \tag{134}$$

we observe that

$$\min_{(u,v)\in D} f(u,v) = b(n^2 - n_0^2) \geq 0 \tag{135}$$

which proves (127) given (128). Now, (125) follows immediately form (126), (127) and (128). □

As a direct consequence of Lemma 4.4, the following lemma establishes the upper bounds of the impedance functions in the upper half-plane.

Lemma 4.5 *For any k such that $Im(k) > 0$ and all $x \in R$,*

$$|(p_+(x,k))| \leq \frac{n_1}{\sin(\arg(k))}, \tag{136}$$

$$|(p_-(x,k))| \leq \frac{n_1}{\sin(\arg(k))}, \tag{137}$$

with $n_1 > 0$ the maximum of $n(x)$ (see (3) in Section 2).

Proof. Again, we only give the proof of (137) since the proof for (136) is identical. According to Lemma 4.4, the function

$$r(x,k) = 1/p_-(x,k) \tag{138}$$

is well-defined for any $Im(k) > 0$. Combining (138) with the equation (78) and the boundary condition (80) for p_-, we observe that $r(x,k)$ obeys the Riccati equation

$$r'(x,k) = ikn^2(x)\left(r^2(x,k) - \frac{1}{n^2(x)}\right), \tag{139}$$

subject to the initial condition $r(0,k) = 1$. Reproducing the proof of Lemma 4.4 almost verbatim, we obtain a lower bound for the real part of r

$$Re(r(x,k)) \geq \frac{\sin(\arg(k))}{n_1}. \tag{140}$$

Now, the upper bound

$$|(p_-(x,k))| \leq Re(r(x,k))^{-1} \leq \frac{n_1}{\sin(\arg(k))} \tag{141}$$

is readily obtained by combining (138) with (140). □

Corollary 4.1 *For all $x \in R$ and k such that $Im(k) > 0$,*

$$\phi_+(x,k) \neq 0, \tag{142}$$

$$\phi'_+(x,k) \neq 0, \tag{143}$$

$$\phi_-(x,k) \neq 0, \tag{144}$$

$$\phi'_-(x,k) \neq 0. \tag{145}$$

Proof. We prove this corollary by contradiction. First, we observe that

$$\phi_+(x,k) = 0 \tag{146}$$

implies

$$\phi'_+(x,k) = 0 \tag{147}$$

and vice versa, since both $\phi_+(x,k)$ and $\phi'_+(x,k)$ are continuous functions of x, and their ratio

$$ik \cdot p_+(x,k) = \frac{\phi'_+(x,k)}{\phi_+(x,k)} \tag{148}$$

is bounded from both above and below due to Lemmas 4.4, 4.5.

Suppose now that for some $x_0 \in R$, $Im(k_0) > 0$,

$$\phi_+(x_0,k_0) = \phi'_+(x_0,k_0) = 0. \tag{149}$$

Then the pair of functions

$$\phi(x) = \phi_+(x,k_0), \tag{150}$$

$$\psi(x) = \phi'_+(x,k_0) \tag{151}$$

satisfies the system of ODEs

$$\phi'(x) = \psi(x), \tag{152}$$

$$\psi'(x) = -k_0^2(1+q(x))\phi(x), \tag{153}$$

subject to the initial conditions

$$\phi(x_0) = \psi(x_0) = 0. \tag{154}$$

However, the initial value problem (152), (153), (154) has a unique solution

$$\phi(x) = \psi(x) = 0 \tag{155}$$

for all $x \le x_0$, which contradicts the condition (26), proving (142), (143).

The proof of (144), (145) is identical. □

Observation 4.1. Due to Lemma 3.8, it is easy to see that

$$\overline{p_+(x,k)} = p_+(x,-\overline{k}), \tag{156}$$

$$\overline{p_-(x,k)} = p_-(x,-\overline{k}), \tag{157}$$

for all $x \in R$ and $k \in C^+$. For real k, equalities (156), (157) assume the form

$$\overline{p_+(x,k)} = p_+(x,-k), \tag{158}$$

$$\overline{p_-(x,k)} = p_-(x,-k). \tag{159}$$

Indeed, combining the complex conjugate of (77) with that of (79), we obtain the ODE

$$\left(\overline{p_+(x,k)}\right)' = -i(-\overline{k})(\overline{p_+(x,k)})^2 - (1+q(x)) \tag{160}$$

subject to initial condition

$$\overline{p_+(0,k)} = 1. \tag{161}$$

Now, replacing k by $-\overline{k}$ in (77) and (79), we have

$$p_+'(x,-\overline{k}) = -i(-\overline{k})(p_+(x,-\overline{k})^2 - (1+q(x)), \tag{162}$$

and

$$p_+\left(0,-\overline{k}\right) = 1. \tag{163}$$

We notice that $\overline{p_+(x,k)}$, $p_+(x,-\overline{k})$ satisfy identical differential equations (160), (162) with identical boundary conditions (161), (163), from which (156) follows. A similar calculation proves (157).

4.2 Smoothness and Asymptotics

The following two technical lemmas describe the asymptotic behavior of the functions ψ_+, ψ_- (see Corollary 3.1 in Section 3), ϕ_+ and ϕ_- at large frequencies. They will be used in proofs of Theorems 4.1, 4.3, describing the high-frequency asymptotics of the impedance functions p_+, p_-. Theorems 4.1, 4.3 are in turn used in the following section to derive the trace formulae (278), (282), which are the principal analytical tool of this paper.

Lemma 4.6 *Suppose that for any $a \geq 0$, the region $K(a) \subset C$ is defined by the formulae*

$$K(a) = \{k | k \in C, Im(k) \geq 0, |k| \geq a\}. \tag{164}$$

Suppose further that $q \in c_0^2([0,1])$, $q(x) > -1$ for all $x \in R$, and the second derivative of q is absolutely continuous. Then there exist real numbers $A > 0$, $c > 0$ such that

$$\psi_+(t,k) = \xi(k)e^{ik(t-T_1)}\left(1 + \frac{1}{2ik}\int_t^1 \eta(\tau)d\tau + \epsilon_+(t,k)\right), \tag{165}$$

$$\psi_+'(t,k) = ik\xi(k)e^{ik(t-T_1)}\left(1 + \frac{1}{2ik}\int_t^1 \eta(\tau)d\tau + \delta_+(t,k)\right), \tag{166}$$

$$\psi_-(t,k) = e^{-ikt}\left(1 + \frac{1}{2ik}\int_0^t \eta(\tau)d\tau + \epsilon_-(t,k)\right), \tag{167}$$

$$\psi_-'(t,k) = -ike^{-ikt}\left(1 + \frac{1}{2ik}\int_0^t \eta(\tau)d\tau + \delta_-(t,k)\right), \tag{168}$$

with $\xi(k) : C \to C$, $T_1 > 0$ defined by (66), (65) (see Corollary 3.1 in Section 3), and $\epsilon_+, \epsilon_-, \delta_+, \delta_- : R \times K(A) \to C$ continuous functions such that

$$|\epsilon_+(t,k)| \leq c \cdot k^{-2}, \tag{169}$$

$$|\delta_+(t,k)| \leq c \cdot k^{-2}, \tag{170}$$

$$|\epsilon_-(t,k)| \leq c \cdot k^{-2}, \tag{171}$$

$$|\delta_-(t,k)| \leq c \cdot k^{-2}. \tag{172}$$

for all $(t,k) \in R \times K(A)$.

Proof. Since the proofs of this lemma for ψ_+, ψ'_+ and for ψ_-, ψ'_- are identical, we only prove it in the case of ψ_-, ψ'_-. Introducing two auxiliary functions $m, n : R \times C \to C$ by the formulae

$$m(t,k) = e^{ikt}\psi_-(t,k), \tag{173}$$

$$n(t,k) = -\frac{1}{ik}e^{ikt}\psi'_-(t,k) \tag{174}$$

and combining (62), (64) with (173), (174), we observe that m satisfies the equation

$$m''(t,k) - 2ikm'(t,k) = -\eta(t)m(t,k) \tag{175}$$

($\eta \in c_0([0,T_1])$ is absolutely continuous, see (54) for the definition of η) subject to the initial conditions

$$m(0,k) = 1, \tag{176}$$

$$m'(0,k) = 0. \tag{177}$$

Multiplying (175) by e^{-2ikt} and integrating the result from 0 to t, we have

$$m'(t,k) = -\int_0^t \eta(\tau)e^{2ik(t-\tau)}m(\tau,k)d\tau. \tag{178}$$

Integrating (178) from 0 to t, we obtain the second kind Volterra integral equation for m

$$m = F_k(m) + 1 \tag{179}$$

with the mapping $F_k : c(R) \to c(R)$ defined by

$$F_k(f)(t) = \frac{1}{2ik}\int_0^t \eta(\tau)(1 - e^{2ik(t-\tau)})f(\tau)d\tau. \tag{180}$$

Combining (178) with (173), (174), we observe that

$$n(t,k) = m(t,k) - \frac{1}{2ik}\int_0^t \eta(\tau)e^{2ik(t-\tau)}m(\tau,k)d\tau. \tag{181}$$

Since $\eta \in c_0([0,T_1])$, the function $\eta(\tau)(1 - e^{2ik(t-\tau)})$ is bounded for all real t, τ and $k \in K(0)$. Therefore, there exists a real number $c_1 > 0$ such that

$$||F_k|| \le \frac{c_1}{|k|}, \tag{182}$$

and hence there exists a real number $A > 0$ such that

$$||F_k|| \le 1 \tag{183}$$

for all $k \in K(A)$. Now, according to Lemma 3.1, for all $(t,k) \in R \times K(A)$, the unique solution of (179) can be approximated by the Neumann's series truncated at the second term

$$\begin{aligned} m(t,k) &= 1 + \frac{1}{2ik}\int_0^t \eta(\tau)(1 - e^{2ik(t-\tau)})d\tau + \alpha(t,k) \\ &= 1 + \frac{1}{2ik}\int_0^t \eta(\tau)d\tau + \beta(t,k) + \alpha(t,k) \end{aligned} \tag{184}$$

with $\alpha,\ \beta : R \times K(A) \to C$ such that

$$\|\alpha\| \leq \frac{2c_1^2}{|k|^2} \tag{185}$$

(see Lemma 3.1), and

$$\beta(t,k) = -\frac{1}{2ik}\int_0^t \eta(\tau)e^{2ik(t-\tau)}d\tau. \tag{186}$$

Since q'' is absolutely continuous and $q(x) = 0$ for all $x \leq 0$, we observe that η is absolutely continuous and $\eta(x) = 0$ for all $x \leq 0$ (see (54) in Lemma 3.7). According to Lemma 3.3, there exists $c_2 > 0$ such that

$$|\beta(t,k)| \leq \frac{c_2^2}{|k|^2}. \tag{187}$$

for all $x \in [0,1]$, $k \in C^+$. Now, combining (184) with (185) and (187), we observe that there exists $c_3 > 0$ such that

$$\left| m(t,k) - \left(1 + \frac{1}{2ik}\int_0^t \eta(\tau)d\tau\right)\right| \leq \frac{c_3}{|k|^2}. \tag{188}$$

for all $(t,k) \in R \times K(A)$. Similarly, there exists $c_4 > 0$ such that

$$\left| n(t,k) - \left(1 + \frac{1}{2ik}\int_0^t \eta(\tau)d\tau\right)\right| \leq \frac{c_4}{|k|^2} \tag{189}$$

due to (181), (188).

Now, (167), (171) follow immediately from (188), (176), and (168), (172) are a direct consequence of (189), (177). □

Lemma 4.7 *Suppose that $q \in c_0^\gamma([0,1])$, $\gamma \geq 2$, $q^{(\gamma)}$ is absolutely continuous and $q(x) > -1$ for all $x \in R$. Then for any integer $1 \leq l \leq \gamma$, the l-th interate $m_l : R \times C^+ \to C$ defined by the formulae*

$$m_0(t,k) = 0, \tag{190}$$

$$m_l(t,k) = 1 + F_k(m_{l-1})(t,k) \tag{191}$$

$$= 1 + \frac{1}{2ik}\int_0^t \eta(\tau)(1 - e^{2ik(t-\tau)})m_{l-1}(\tau,k)d\tau \tag{192}$$

(see (179), (180)) assumes the form

$$m_l(t,k) = 1 + \sum_{j=1}^{\gamma-1} \left(\frac{1}{2ik}\right)^j a_j(t) + \left(\frac{1}{2ik}\right)^\gamma a_\gamma(t,k) \tag{193}$$

with $a_j : R \to R,\ j = 1, \ldots, \gamma - 1,\ a_\gamma : R \times C^+ \to C$ *such that*

$$\frac{d^{\gamma-j} a_j(t)}{dx^{\gamma-j}} \tag{194}$$

are bounded and absolutely continuous for all $x \in R,\ j = 1, \ldots, \gamma - 1,$ *and*

$$a_\gamma(t,k) \tag{195}$$

is bounded and absolutely continuous function of t *for all* $(t,k) \in R \times C^+$.

Proof. We prove this lemma by induction. For $l = 1$, formulae (190), (192) yield

$$m_1(t,k) = 1 \tag{196}$$

for all $(t,k) \in R \times C^+$, which is already in the form (193) satisfying conditions (194), (195).

For $l \geq 1$, assuming that $m_l(t,k)$ is in the form (193) satisfying conditions (194), (195), we obtain m_{l+1} using (192):

$$\begin{aligned} m_{l+1}(t,k) &= 1 + \frac{1}{2ik} \int_0^t \eta(\tau)(1 - e^{2ik(t-\tau)}) m_l(\tau,k) d\tau \\ &= 1 + I_1(t,k) + I_2(t,k) + I_3(t,k) + I_4(t,k) \end{aligned} \tag{197}$$

with $I_j : R \times C^+ \to C,\ 1 \leq j \leq 4$ defined by the formulae

$$I_1(t,k) = \frac{1}{2ik} \int_0^t \eta(\tau) d\tau + \sum_{j=2}^{\gamma-1} \left(\frac{1}{2ik}\right)^j \int_0^t \eta(\tau) a_{j-1}(\tau) d\tau, \tag{198}$$

$$I_2(t,k) = -\frac{1}{2ik} \int_0^t \eta(\tau)(1 - e^{2ik(t-\tau)}) d\tau, \tag{199}$$

$$I_3(t,k) = -\sum_{s=2}^{\gamma-1} \left(\frac{1}{2ik}\right)^s \int_0^t \eta(\tau) a_{s-1}(\tau) e^{2ik(t-\tau)} d\tau \equiv -\sum_{s=2}^{\gamma-1} J_s(t,k), \tag{200}$$

$$I_4(t,k) = \frac{1}{2ik} \int_0^t \eta(\tau) a_\gamma(\tau)(1 - e^{2ik(t-\tau)}) d\tau. \tag{201}$$

Clearly, we only need to show that $I_j,\ 1 \leq j \leq 4$ can be expressed in the form

$$\sum_{j=1}^{\gamma-1} \left(\frac{1}{2ik}\right)^j \alpha_j(t) + \left(\frac{1}{2ik}\right)^\gamma \alpha_\gamma(t,k) \tag{202}$$

with $\alpha_j : R \to R,\ 1 \leq j \leq \gamma - 1$ satisfying condition (194) and $\alpha_\gamma : R \times C^+ \to C$ satisfying condition (195). Obviously, I_1 and I_4 are already in the form (202). We now use Lemma 3.3 to show that I_2, I_3 can also be expanded in the form (202). Observing that $\eta(t) = 0$ for all

$t \notin (0, T_1)$, $\eta^{(\gamma-2)}$ is absolutely continuous (see Lemma 3.7), and that $a_j^{(\gamma-j)}, 1 \le j \le \gamma - 1$ are absolutely continuous (due to the assumption of the induction), we can use formula (36) in Lemma 3.3 to expand I_2 and each term J_s $(s = 1, \ldots, \gamma - 1)$ of I_3 as

$$I_2(t,k) = \sum_{j=2}^{\gamma-1} \left(\frac{1}{2ik}\right)^j \eta^{(j-2)}(t) + \left(\frac{1}{2ik}\right)^\gamma b_1(t,k), \tag{203}$$

$$J_s(t,k) = \left(\frac{1}{2ik}\right)^s \int_0^t \eta(\tau) a_{s-1}(\tau) e^{2ik(t-\tau)} d\tau \tag{204}$$

$$= -\sum_{j=s+1}^{\gamma-1} \left(\frac{1}{2ik}\right)^j \frac{d^{(j-s-1)}}{dt^{(j-s-1)}} (\eta(\tau) a_{(s-1)} - \left(\frac{1}{2ik}\right)^\gamma b_s(t,k) \tag{205}$$

with $b_s : R \times C^+ \to C$ uniformly bounded on $R \times C^+$ (see Lemma 3.3). Therefore, I_2 is in the form (202) due to (203), and I_3 is of the form (202) due to (205), (200). Thus, $m_{l+1}(t,k)$ can indeed be written in the form (193) satisfying conditions (194), (195). □

Corollary 4.2 *Suppose that for any $a \ge 0$, the region $K(a) \subset C$ is defined by the formulae*

$$K(a) = \{k | k \in C, Im(k) \ge 0, |k| \ge a\}. \tag{206}$$

Suppose further that the functions $m, n, m_\gamma, n_\gamma : R \times C^+ \to C$ are defined by the formulae (173), (174), (192) and

$$n_\gamma(t,k) = m_\gamma(t,k) - \frac{1}{2ik} \int_0^t \eta(\tau) e^{2ik(t-\tau)} m_\gamma(\tau,k) d\tau \tag{207}$$

respectively. Then under the conditions of the preceding lemma, there exist positive numbers A, c_1, c_2, c_3 such that

$$|m(t,k) - m_\gamma(t,k)| \le \frac{c_1}{|k|^\gamma}, \tag{208}$$

$$|n(t,k) - n_\gamma(t,k)| \le \frac{c_2}{|k|^\gamma} \tag{209}$$

for all $(t,k) \in R \times K(A)$, and

$$\left| \frac{n(t,k)}{m(t,k)} - 1 \right| \le \frac{c_3}{|k|^\gamma} \tag{210}$$

for all $(t,k) \in [T_1, \infty) \times K(A)$.

Proof. Due to (182), the norm of the integral operator F_k in (192) is of the order $O(|k|^{-1})$ for any $k \in C^+$, from which we observe that there exists $A > 0$, such that (208) is true.

Subtracting (207) from (174), we obtain

$$n(t,k) - n_\gamma(t,k) = m(t,k) - m_\gamma(t,k) - \frac{1}{2ik} \int_0^t \eta(\tau) e^{2ik(t-\tau)} (m(\tau,k) - m_\gamma(\tau,k)) d\tau. \tag{211}$$

Now, the estimate (209) is a direct consequence of (211), (208) and the fact that in (211), the expression

$$\frac{1}{2ik}\eta(\tau)e^{2ik(t-\tau)} \tag{212}$$

is uniformly bounded for all $k \in K(A)$, $-\infty < \tau \le t < \infty$.

We now prove (210) by showing that there exists a positive number c_3 such that

$$\left|\frac{n_\gamma(t,k)}{m_\gamma(t,k)} - 1\right| \le \frac{c_3}{|k|^\gamma} \tag{213}$$

for all $(t,k) \in [T_1,\infty) \times K(A)$. According to Lemma 4.7, $m_\gamma(t,k)$ can be expressed in the form

$$m_\gamma(t,k) = 1 + \sum_{j=1}^{\gamma-1}\left(\frac{1}{2ik}\right)^j a_j(t) + \left(\frac{1}{2ik}\right)^\gamma a_\gamma(t,k), \tag{214}$$

with $a_j, j = 1,\ldots,\gamma$ satisfying conditions (194), (195). Therefore, we can assume that the constant A has been chosen such that for all $(t,k) \in R \times K(A)$,

$$|m_\gamma(t,k)| \ge \frac{1}{2}. \tag{215}$$

Combining (207) with (214), we obtain

$$n_\gamma(t,k) = m_\gamma(t,k) + I_2(t,k) + I_3(t,k) + I_5(t,k), \tag{216}$$

with $I_2, I_3(t,k)$ defined by (199), (200), and $I_5(t,k)$ defined by the formula

$$I_5(t,k) = \left(\frac{1}{2ik}\right)^{\gamma+1}\int_0^t \eta(\tau)a_\gamma(\tau,k)e^{2ik(t-\tau)})d\tau. \tag{217}$$

Noticing that $\eta(t) = 0$ for all $t \ge T_1$, we have

$$I_2(t,k) = \left(\frac{1}{2ik}\right)^\gamma b_1(t,k), \tag{218}$$

$$J_s(t,k) = \left(\frac{1}{2ik}\right)^\gamma b_s(t,k) \tag{219}$$

for all $(t,k) \in [T_1,\infty) \times K(A)$, due to (203), (205). Consequently, there exists $c > 0$ such that

$$|I_2(t,k) + I_3(t,k) + I_5(t,k)| \le \frac{c}{|k|^\gamma} \tag{220}$$

for all $(t,k) \in [T_1,\infty) \times K(A)$, since $a_\gamma(t,k)$, $b_s(t,k)$ are bounded for all $(t,k) \in [T_1,\infty) \times K(A)$, and $s = 1,\ldots,\gamma-1$.

Now, (213) follows immediately from (216), (220) and (215). The estimate (210) is a direct consequence of (213), (208) and (209). □

Lemma 4.8 *Suppose that $q \in c_0^\gamma([0,1])$, $\gamma \geq 2$, $q^{(\gamma)}$ is absolutely continuous and $q(x) > -1$ for all $x \in R$. Then there exists a positive number c such that*

$$|p_-(x,k) - 1| \leq \frac{c}{|k|^\gamma} \tag{221}$$

for all $x \geq 1$, $k \in C^+$.

Proof. According to Corollary 3.1 and formula (68),

$$p_-(x,k) = \frac{\psi_-'(t,k)}{-ik\psi_-(t,k)} \tag{222}$$

for all $t \geq T_1$ (i.e., for all $x \geq 1$), $k \in C^+$. According to (173), (174) and (222)

$$p_-(x,k) = \frac{m(t,k)}{n(t,k)} \tag{223}$$

for all $t \geq T_1$, $k \in C^+$. Now, the lemma follows immediately from (223) and (210). □.

Remark 4.2. By a similar calculation, one can show that under the conditions of the preceding lemma, there exist positive numbers $A > 0$, $c > 0$ such that

$$|p_+(x,k) - 1| \leq \frac{c}{|k|^\gamma} \tag{224}$$

for all $x \leq 0$, $k \in C^+$.

Theorem 4.1 *Suppose that $q \in c_0^2([0,1])$, $q(x) > -1$ for all $x \in R$ and q'' is absolutely continuous. Suppose further that*

$$D = \{(x,k) | x \in R, Im(k) \geq 0\}. \tag{225}$$

Then
(a) ϕ_+ and ϕ_- are continuous functions of (x,k) and analytic functions of k for all $x \in R$ and $k \in C$;
(b) p_+ and p_- are continuous functions of (x,k) and analytic functions of k in D;
(c) there exists a positive number c such that for all $(x,k) \in D$

$$p_+(x,k) = \sqrt{1+q(x)} - \frac{q'(x)}{4(1+q(x))} \cdot \frac{1}{ik} + \epsilon_+(x,k), \tag{226}$$

$$p_-(x,k) = \sqrt{1+q(x)} + \frac{q'(x)}{4(1+q(x))} \cdot \frac{1}{ik} + \epsilon_-(x,k), \tag{227}$$

with $\epsilon_+, \epsilon_- : D \to C$ continuous functions such that

$$|\epsilon_+(x,k)| \leq \frac{c}{|k|^2}, \tag{228}$$

$$|\epsilon_-(x,k)| \leq \frac{c}{|k|^2}. \tag{229}$$

Proof. We only give the proof for ϕ_-, p_- since the proof for ϕ_+, p_+ is identical. We introduce two auxiliary functions ϕ and ϕ_1 via the formulae

$$\phi(x,k) = \phi_-(x,k), \tag{230}$$
$$\phi_1(x,k) = \phi'_-(x,k), \tag{231}$$

so that the equation (1) and the initial condition (5) for ϕ_- can be rewritten as a system of linear ODEs

$$\phi'(x,k) = \phi_1(x,k), \tag{232}$$
$$\phi'_1(x,k) = -k^2 n^2(x)\phi(x,k), \tag{233}$$

subject to initial conditions

$$\phi(0,k) = 1, \tag{234}$$
$$\phi_1(0,k) = -ik. \tag{235}$$

According to Lemma 3.10, ϕ, ϕ_1 are continuous functions of (x,k) and entire functions of k for all $x \in R$ and $k \in C$, from which (a) follows immediately. Similarly, we obtain (b) by combining (a) with (16) and the fact that $\phi_-(x,k) \neq 0$ for all $(x,k) \in D$ (see Remark 4.1, Lemma 4.3 and Corollary 4.1).

The expansion (227) and the estimate (229) follow immediately from (68) (see Corollary 3.1 in Section 3), (167), (168), (171), and (172) (see Lemma 4.6). □

Corollary 4.3 *Denote by p p_+ or p_-. Then under the conditions of the preceding theorem, there exist positive number c_1, c_2 such that*

$$\left| e^{2ik\int_t^x p(\tau,k)d\tau} \right| \leq c_1, \tag{236}$$

for all $t, x \in [0,1]$, $k \in R$, or for all $0 \leq t \leq x \leq 1$, $k \in C^+$, and

$$|p'(x,k)| \leq c_2, \tag{237}$$

for all $x \in R$, $k \in C^+$.

Proof. Due to Statements (b), (c) of Theorem 4.1, the real part of the function

$$2ik\int_t^x p(\tau,k)d\tau \tag{238}$$

is uniformly bounded from above for $t, x \in [0,1]$, $k \in R$, or for all $0 \leq t \leq x \leq 1$, $k \in C^+$, from which (236) follows immediately. Estimate (237) is a direct consequence of Statement (c) of Theorem 4.1, and formulae (77), (78). □

Global upper and lower bounds for the impedance functions will be established in Theorem 4.2. We first obtain a partial result in the following lemma.

Lemma 4.9 *Suppose that for any positive numbers a, α, the domain $K(a,\alpha) \subset C$ is defined by the formula*

$$K(a,\alpha) = \{k | k \in C, Re(k) \in [-a,a], Im(k) \in [0,\alpha]\}. \tag{239}$$

Then under the conditions of the preceding theorem, for any $A > 0$, there exist positive numbers B, b, δ such that

$$|p_+(x,k)| \leq B, \tag{240}$$
$$|p_-(x,k)| \leq B, \tag{241}$$
$$Re(p_+(x,k)) \geq b, \tag{242}$$
$$Re(p_-(x,k)) \geq b, \tag{243}$$

in the domain $R \times K(A,\delta)$.

Proof. Since the proof of (240), (242) is identical to that of (241), (243), we only provide the latter. Denoting by u, v the real and imaginary parts of p_- so that

$$p_-(x,k) = u(x,k) + iv(x,k), \tag{244}$$

the Riccati equation (78) for p_- can be rewritten in the form

$$u' = -2kuv, \tag{245}$$
$$v' = -k(v^2 - u^2 + n^2), \tag{246}$$

for any $k \in R$. Integrating (245) on interval $[0,x]$ and observing that

$$u(x,k) = p_-(x,k) = 1 \tag{247}$$

for all $x \leq 0$, $k \in C$ (see (80)), we have

$$u(x,k) = e^{-2k\int_0^x v(t,k)dt} > 0 \tag{248}$$

for all $x, k \in R$. For any $A > 0$, $p_-, u = Re(p_-)$ are continuous functions of (x,k) in the compact domain $[0,1] \times K(A,\delta)$. Therefore, there exist positive numbers b_1, δ, B_1 such that

$$u(x,k) \geq b_1 > 0 \tag{249}$$
$$|p_-(x,k)| \leq B_1, \tag{250}$$

for all $(x,k) \in [0,1] \times K(A,\delta)$, which proves the estimates (241), (243).

We now prove the estimates (241), (243) for all $x \geq 1$ using the formula

$$p_-(x,k) = \frac{1 - b_-^2(k) + i2b_-(k)\sin(kx - \alpha_-(k))}{1 + b_-^2(k) + 2b_-(k)\cos(kx - \alpha_-(k))}, \tag{251}$$

(see Remark 2.2). According to Remark 2.2, $b(k) \geq 0$ is a real-valued continuous function of $k \in C$. We observe that

$$0 \le b(k) < 1 \tag{252}$$

for all k in the close domain $K(A,\delta)$ since otherwise if $b(k) \ge 1$, the real part of $p_-(1,k)$

$$u(1,k) = \frac{1 - b_-^2(k)}{1 + b_-^2(k) + 2b_-(k)\cos(kx - \alpha_-(k))} \tag{253}$$

will be non-positive, contradicting (249). Due to (252), (251), there exist positive numbers b_2, B_2 such that

$$u(x,k) \ \ge \ b_2, \tag{254}$$
$$|p_-(x,k)| \ \le \ B_2, \tag{255}$$

for all $x \ge 1$, $k \in K(A,\delta)$.

Now, (241), (243) follow immediately from (249), (250), (254), (255), and (247). □

Theorem 4.2 *Suppose that $q \in c_0^2([0,1])$, $q(x) > -1$ for all $x \in R$ and the second derivative of q is absolutely continuous. Then there exist real numbers $B > 0$, $b > 0$ such that*

$$|p_+(x,k)| \ \le \ B, \tag{256}$$
$$|p_-(x,k)| \ \le \ B, \tag{257}$$
$$Re(p_+(x,k)) \ \ge \ b, \tag{258}$$
$$Re(p_-(x,k)) \ \ge \ b, \tag{259}$$

in the domain

$$D = \{(x,k) | x \in R, Im(k) \ge 0\}. \tag{260}$$

Proof. Since the proof of (256), (258) is identical to that of (257), (259), we only provide the latter. According to the high-frequency asymptotics (227) in Theorem 4.1, there exist positive numbers A, b_1 such that

$$Re(p_-(x,k)) \ge b_1, \tag{261}$$

in the domain $D_1 \subset D$ defined by

$$D_1 = \{(x,k) | x \in R, |k| \ge A, Im(k) \ge 0\}. \tag{262}$$

Since $p_-(x,k)$ is a continuous function of $(x,k) \in D_1$, there exists a positive number B_1 such that

$$|p_-(x,k)| \le B_1, \tag{263}$$

for all $(x,k) \in D_1$. For such a number $A > 0$, according to Lemma 4.9, there exist positive numbers δ, B_2, b_2 such that

$$|p_-(x,k)| \ \le \ B_2, \tag{264}$$
$$Re(p_-(x,k)) \ \ge \ b_2, \tag{265}$$

in the domain $D_2 \subset D$ defined by the formula

$$D_2 = \{(x,k)|x \in R, Re(k) \in [-A,A], Im(k) \in [0,\delta]\}. \tag{266}$$

Now, according to Lemmas 4.4, 4.5, there exist positive numbers B_3, b_3 such that

$$|p_-(x,k)| \le B_3, \tag{267}$$
$$Re(p_-(x,k)) \ge b_3, \tag{268}$$

in the domain $D_3 \subset D$ defined by

$$D_3 = \{(x,k)|x \in R, Re(k) \in [-A,A], Im(k) \ge \delta\}. \tag{269}$$

The estimates (257), (259) for $(x,k) \in D$ follow immediately from the estimates for $(x,k) \in D_1$, D_2, D_3 since $D = D_1 \cup D_2 \cup D_3$. □

The following theorem furnishes the analytical apparatus for the error analysis of the truncated trace formula (see (282)).

Theorem 4.3 *Suppose that $q \in c_0^m([0,1])$, $m \ge 2$, $q^{(m)}$ is absolutely continuous and $q(x) > -1$ for all $x \in R$. Then there exists a positive number a such that*

$$\left|\overline{p_+(x,k)} - p_-(x,k)\right| \le \frac{a}{|k|^m} \tag{270}$$

for all $(x,k) \in R \times C^+$.

Proof. According to Lemma 4.8 and Remark 4.2, (270) is true for all $x \notin (0,1)$. In order to prove the theorem for $x \in (0,1)$, we observe that $\overline{p_+}$ and p_- obey the same Riccati equation (78) due to (77), (78). The difference, $s = \overline{p_+} - p_-$, satisfies the ODE

$$s'(x,k) = ik\,(\overline{p_+} + p_-)\,s \tag{271}$$

with the solution

$$s(x,k) = e^{-ik\int_0^x \left(\overline{p_+(t,k)}+p_-(t,k)\right)dt} s(0,k). \tag{272}$$

Corollary 4.3 indicates that there exists constant $b > 0$ such that

$$\left|e^{-ik\int_0^x \left(\overline{p_+(t,k)}+p_-(t,k)\right)dt}\right|, \tag{273}$$

for all $(x,k) \in [0,1] \times R$. Due to Remark 4.2, there exists a positive number c such that for all $k \in R$,

$$|s(0,k)| = |p_+(0,k) - p_-(0,k)| = |p_+(0,k) - 1| \le \frac{c}{|k|^m}. \tag{274}$$

Now, (270) for $x \in (0,1)$ follows immediately from (272), (273), (274). □

4.3 Trace Formulae

In this subsection, we prove Theorem 4.4, which is both the purpose of this section, and the principal analytical tool of this paper. Theorem 4.4 describes the so-called trace formulae for the impedance functions p_+, p_- (for a more detailed discussion of the term "trace formulae", see, for example, [7]). In fact, only the formula (278) is to be used by the reconstruction algorithm of the following section. We present the formulae (275), (276), (277) for completeness, since some of them appear to be well-known, and attempts have been made to use them in reconstruction algorithms (see, for example, [8]). See also Subsection 5.1 below for a more detailed discussion of the use of trace formulae in reconstruction schemes

Theorem 4.4 *(Trace formulae) Suppose that $q \in c_0^m([0,1])$, $m \geq 2$, $q^{(m)}$ is absolutely continuous and $q(x) > -1$ for all $x \in R$. Then*
(a)

$$\sqrt{1+q(x)} = \lim_{a \to +\infty} \frac{1}{2a} \int_{-a}^{a} p_+(x,k)dk. \tag{275}$$

(b)

$$q'(x) = \lim_{a \to +\infty} \frac{2}{ia}(1+q(x)) \int_{-a}^{a} k \cdot p_+(x,k)dk. \tag{276}$$

(c)

$$\sqrt{1+q(x)} = \lim_{a \to +\infty} \frac{1}{4a} \int_{-a}^{a} (p_+(x,k) + p_-(x,k))dk. \tag{277}$$

(d)

$$q'(x) = \frac{2}{\pi}(1+q(x)) \int_{-\infty}^{\infty} (p_+(x,k) - p_-(x,k))dk. \tag{278}$$

More precisely, there exist positive numbers c_1, c_2, c_3, c_4 such that

$$\left|\sqrt{1+q(x)} - \frac{1}{2a} \int_{-a}^{a} p_+(x,k)dk\right| \leq \frac{c_1}{a}, \tag{279}$$

$$\left|q'(x) - \frac{2}{ia}(1+q(x)) \int_{-a}^{a} k \cdot p_+(x,k)dk\right| \leq \frac{c_2}{a}, \tag{280}$$

$$\left|\sqrt{1+q(x)} - \frac{1}{4a} \int_{-a}^{a} (p_+(x,k) + p_-(x,k))dk\right| \leq \frac{c_3}{a^2}, \tag{281}$$

$$\left|q'(x) - \frac{2}{\pi}(1+q(x)) \int_{-a}^{a} (p_+(x,k) - p_-(x,k))dk\right| \leq \frac{c_4}{a^{(m-1)}}. \tag{282}$$

for all $x \in R$.

Proof. Since the proofs of trace formulae (a),(b),(c), and (d) are similar, we only present that of (d). According to statement (c) of Theorem 4.1, there exists $c > 0$ such that

$$\left|(p_+(x,k)-p_-(x,k))-\left(-\frac{q'(x)}{2(1+q(x))}\frac{1}{ik}\right)\right|\le\frac{c}{|k|^2}. \tag{283}$$

for all $(x,k)\in R\times C^+$. Denoting by Γ the upper half circle of radius A, with clockwise orientation, in the complex k-plane, i.e.,

$$\Gamma=\{k|k\in C^+,\ |k|=A\}, \tag{284}$$

and noting that p_+-p_- is an analytic function of $k\in C^+$, we obtain

$$\int_{-A}^{A}(p_+(x,k)-p_-(x,k))dk=\int_\Gamma(p_+(x,k)-p_-(x,k))dk. \tag{285}$$

Substituting (283) into (285), we have

$$\int_{-A}^{A}(p_+(x,k)-p_-(x,k))dk=\frac{\pi q'(x)}{2(1+q(x))}+O(k^{-1}) \tag{286}$$

from which (278) follows immediately.

In order to prove the estimate (282), we rewrite (278) as

$$q'(x)=\frac{2}{\pi}(1+q(x))\int_{-a}^{a}(p_+(x,k)-p_-(x,k))dk+I(a) \tag{287}$$

with $I(a)$ given by the formula

$$I(a)=\frac{2}{\pi}(1+q(x))\left(\int_{-\infty}^{-a}+\int_a^\infty\right)(p_+(x,k)-p_-(x,k))dk. \tag{288}$$

Now, formula (158) implies

$$I(a)=\frac{2}{\pi}(1+q(x))\left(\int_{-\infty}^{-a}+\int_a^\infty\right)\left(\overline{p_+(x,k)}-p_-(x,k)\right)dk, \tag{289}$$

and according to (270), there exists a constant c_4 such that

$$|I(a)|\le\frac{c_4}{|k|^{(m-1)}}, \tag{290}$$

from which (282) follows immediately. □

5 The Reconstruction Algorithm

5.1 Reconstruction via trace formulae–an informal description

An examination of the formulae (275)- (278) in combination with the Riccati equations (77), (78) immediately suggests an algorithm for the reconstruction of the parameter q given the impedance function $p_+(x_0,k)$ measured at some point $x_0\in R$ outside the scatterer. Namely, one is tempted to substitute one of the formulae (275)- (278) (for example, (275)) into (77), obtaining

$$p'_+(x,k) = -ik\left(p_+^2(x,k) - \lim_{a\to}\left(\frac{1}{2a}\int_{-a}^{a} p_+(x,k)dk\right)^2\right), \tag{291}$$

and attempt to view (291) as a differential equation for the function $p: R^1 \times R^1 \to C$.

Needless to say, standard existence and uniqueness theorems are not applicable to 'differential equations' of the form (291). Furthermore, in order to be numerically useful, the integral in (291) would have to be replaced with some finite quadrature formulae. The latter procedure is significantly complicated by the fact that the function p_+ is defined on the whole real line, and its domain of definition has to be truncated before discretization. It turns out that the solution of (291) is not unique, except in a very carefully chosen class of functions p. Such a class of functions has been successfully specified (see, for example, [8]). The resulting numerical scheme is, however, quite expensive, and the construction is not rigorous, though we believe that this could be made so. The same problem arises if one attempts to use the trace formulae (276), (277), and the conceptual reason for this situation is summarized in the following observation.

Observation 5.1. An immediate consequence of the formula (275) is

$$\sqrt{1+q(x)} = \lim_{a\to+\infty}\frac{1}{2a}\cdot\left(\int_{-a}^{-b} p_+(x,k)dk + \int_{b}^{a} p_+(x,k)dk\right), \tag{292}$$

for any positive real b. Thus, the 'differential equation' (291) can be replaced with

$$p'_+(x,k) = -ik\left(p_+^2(x,k) - \left(\lim_{a\to+\infty}\frac{1}{2a}\left(\int_{-a}^{-b} p_+(x,k)dk + \int_{b}^{a} p_+(x,k)dk\right)\right)^2\right) \tag{293}$$

and a convergence, uniqueness, etc. proof valid for (291) would also be valid for (293), unless some extremely subtle phenomenon interfered.

However, given a smooth scatterer q, for any $\varepsilon > 0$, one can choose a sufficiently large b that

$$\left|\sqrt{1+q(x)} - p_+(x,k)\right| < \varepsilon \tag{294}$$

for any $k \geq b$. If the scattered data $p_+(x_0,k)$ have been collected at some point x_0 outside a smooth scatterer, (294) assumes the form

$$|1 - p_+(x,k)| < \varepsilon. \tag{295}$$

In other words, a reconstruction algorithm using the 'differential equation' (293) with a sufficiently large b would effectively reconstruct the parameter $q(x)$ for all $x \in [0,1]$ from a single measurement, the latter being equal to 1 (!). Another way to make this observation is to notice that the formula (275) is simply the WKB approximation to the impedance function p_+, and that in the WKB regime, the back-scattered field is absent. A similar problem arises if one attempts to combine formulae (276), (277) with (77), and view the result as a 'system of ordinary differential equations'.

In the case of a discontinuous scatterer q, the WKB expansions (226), (227) are invalid. On the other hand, the trace formulae (275), (278) are valid (if the limits in these formulae are interpreted properly), and can be combined with the equations (77), (78) to obtain a numerical scheme for detecting discontinuities in the scatterer. If q is piece-wise constant, such a scheme will reconstruct it effectively, and time-domain versions of this procedure are known as layer-stripping algorithms (see, for example, [12], [13], [14]).

While the authors failed to find the trace formulae (276, (277) in the literature, they appear to be well-known among specialists, being an immediate consequence of the WKB analysis of the equation (77). On the other hand, the formula (278) does appear to be new, and its combination with the equation (77) immediately leads to a robust reconstruction algorithm. While we postpone a detailed construction and analysis of such a scheme till Subsections 5.2, 5.3, in the following observation we summarize the conceptual reasons for its analytical and numerical effectiveness.

Observation 5.2. (282) means that approximating the trace formula (278) with its 'truncated' version

$$q'(x) \sim \frac{2}{\pi}(1+q(x)) \int_{-a}^{a} (p_+(x,k) - p_-(x,k))dk, \tag{296}$$

we make an error of the order $a^{-(m-1)}$, where m is the smoothness of the scatterer. Thus, for a sufficiently smooth scatterer and a sufficiently large a, (296) is an extremely good approximation to the trace formula (278).

Now, for the system of equations (77), (78), (296), it is not hard to prove existence, uniqueness, etc. theorems of the type valid for systems of ODEs (since now for a fixed value of x, the functions $p_+(x,k), p_-(x,k) : [-a,a] \to C$ are defined on a compact interval, as opposed to the whole line). The remainder of the paper is devoted to proving such facts (see Theorem 5.1 below), and to a numerical implementation of the resulting procedure. The latter is also quite straightforward, since it only involves constructing a quadrature formula for the evaluation of the integral in (278), where it is taken over an interval of finite length. Furthermore, for all practical purposes, the integrand vanishes at the ends of the domain of integration together with all its derivatives, completely obviating the issue of the choice of the quadrature formula, and leading to extremely accurate numerical procedures (see Remark 6.1 below).

5.2 Reconstruction via trace formulae–a formal description

Now, we are prepared to construct a system of integro-differential equations whose initial conditions are the values of the impedance functions p_+, p_- measured outside the scatterer, and whose solution reconstructs the potential q for all $x \in [0,1]$. We will consider a system of integro-differential equations

$$p'_{a+}(x,k) = -ik(p^2_{a+}(x,k) - (1+q_a(x))), \tag{297}$$

$$p'_{a-}(x,k) = ik(p^2_{a-}(x,k) - (1+q_a(x))), \tag{298}$$

$$q'_a(x) = \frac{2}{\pi}(1+q_a(x)) \int_{-a}^{a} (p_{a+}(x,z) - p_{a-}(x,z))dz, \tag{299}$$

with respect to the functions $p_{a+}, p_{a-} : [0,1] \times [-a,a] \to C$, $q_a : [0,1] \to R$, subject to the initial conditions

$$
\begin{aligned}
p_{a+}(0,k) &= p_0(k), && (300)\\
p_{a-}(0,k) &= 1, && (301)\\
q(0) &= 0. && (302)
\end{aligned}
$$

It turns out that for sufficiently large a, the system (297)–(302)has a unique solution for all $x \in [0,1]$, that this solution is stable with respect to small perturbations of the initial data $p_0(k)$, and that q_a converges to q as $a \to \infty$. The following theorem formalizes these facts.

Theorem 5.1 *(Convergence of the inversion algorithm) Suppose that $q \in c_0^m([0,1])$, $m \geq 4$, $q^{(m)}$ is absolutely continuous and $q(x) > -1$ for all $x \in R$. Then there exist constants $A > 0, c > 0$ such that*

$$
|q(x) - q_a(x)| \leq \frac{c}{a^{(m-1)}} \tag{303}
$$

for all $x \in [0,1]$, $a \geq A$.

Since the proof of this theorem is quite involved, we break its technical part into three lemmas which are then directly used in the proof of Theorem 5.1.

Lemma 5.1 *Suppose that $q \in c_0^m([0,1])$, $m \geq 4$, $q^{(m)}$ is absolutely continuous and $q(x) > -1$ for all $x \in R$. Suppose further that the function space Σ is defined by the formula*

$$
\Sigma = \{[\alpha, \beta, \gamma] | \alpha, \beta \in c([0,1] \times [-a,a]), \gamma \in c([0,1])\}, \tag{304}
$$

equipped with the norm

$$
\|f\| = \max(\|\alpha\|, \|\beta\|, \|\gamma\|), \tag{305}
$$

with $f = [\alpha, \beta, \gamma] \in \Sigma$. Finally, suppose that for any $a > 0$, the functions $f_a, w, \epsilon_a : R \to R$ are defined by the formulae

$$
\begin{aligned}
f_a(x) &= \frac{2}{\pi} \int_{-a}^{a} (p_+(x,k) - p_-(x,k))dk, && (306)\\
w(x) &= \frac{2}{\pi}(1 + q(x)), && (307)\\
\epsilon_a(x) &= -w(x) \left(\int_{-\infty}^{-a} + \int_{a}^{\infty} \right) (p_+(x,k) - p_-(x,k))dk. && (308)
\end{aligned}
$$

Then the error function $u = [e_+, e_-, h] \in \Sigma$ defined by the formulae

$$
\begin{aligned}
e_+(x,k) &= p_{a+}(x,k) - p_+(x,k), && (309)\\
e_-(x,k) &= p_{a-}(x,k) - p_-(x,k), && (310)\\
h(x) &= q_a(x) - q(x) && (311)
\end{aligned}
$$

satisfies the equation

$$L(u)(x,k) = N(u)(t,k) + [0,0,\epsilon_a(t)], \tag{312}$$

where $L, N : \Sigma \to \Sigma$ *are defined by the formulae*

$$L(u) = \begin{bmatrix} e_+(x,k) - ik\int_0^x h(t)e^{-2ik\int_t^x p_+(\tau,k)d\tau}dt \\ e_-(x,k) + ik\int_0^x h(t)e^{2ik\int_t^x p_-(\tau,k)d\tau}dt \\ h(x) - \int_0^x h(t)f_a(t)dt - \int_0^x w(t)\int_{-a}^a (e_+(t,z) - e_-(t,z)dzdt \end{bmatrix}, \tag{313}$$

$$N(u) = \begin{bmatrix} -ik\int_0^x e_+^2(t,k)e^{-2ik\int_t^x p_+(\tau,k)d\tau}dt \\ ik\int_0^x e_-^2(t,k)e^{2ik\int_t^x p_-(\tau,k)d\tau}dt \\ \frac{2}{\pi}\int_0^x h(t)\int_{-a}^a (e_+(t,z) - e_-(t,z)dzdt \end{bmatrix}. \tag{314}$$

Proof. We know that the functions p_+, p_-, q satisfy the ODEs

$$p'_+(x,k) = -ik(p_+^2(x,k) - (1+q(x)), \tag{315}$$
$$p'_-(x,k) = ik(p_-^2(x,k) - (1+q(x)), \tag{316}$$
$$q'(x) = \frac{2}{\pi}(1+q(x))\int_{-\infty}^{\infty}(p_+(x,k) - p_-(x,k))dk, \tag{317}$$

for all $x \in R$, any $k \in C^+$, and that the functions p_{a+}, p_{a-}, q_a satisfy the ODEs

$$p'_{a+}(x,k) = -ik(p_{a+}^2(x,k) - (1+q_a(x))), \tag{318}$$
$$p'_{a-}(x,k) = ik(p_{a-}^2(x,k) - (1+q_a(x))), \tag{319}$$
$$q'_a(x) = \frac{2}{\pi}(1+q_a(x))\int_{-a}^{a}(p_{a+}(x,z) - p_{a-}(x,z))dz. \tag{320}$$

for all $(x,k) \in [0,1] \times [-a,a]$, subject to initial conditions

$$p_{a+}(0,k) = p_+(0,k), \tag{321}$$
$$p_{a-}(0,k) = p_-(0,k) = 1, \tag{322}$$
$$q_a(0) = q(0) = 0. \tag{323}$$

for all $k \in [-a,a]$. Subtracting equations (315), (316), (317) from equations (318), (319), (320) respectively, we observe that $[e_+, e_-, h]$ (see (309), (310), (311)) satisfies the ODEs

$$e'_+(x,k) = -ik\left(2p_+(x,k)e_+(x,k) + e_+^2(x,k) - h(x)\right), \tag{324}$$
$$e'_-(x,k) = ik\left(2p_-(x,k)e_-(x,k) + e_-^2(x,k) - h(x)\right), \tag{325}$$
$$h'(x) = h(x)f_a(x) + w(x)\int_{-a}^{a}(e_+(x,z) - e_-(x,z))dz + \frac{2}{\pi}h(x)\int_{-a}^{a}(e_+(x,z) - e_-(x,z))dz + \epsilon_a(x), \tag{326}$$

subject to the initial conditions

$$e_+(0,k) = e_-(0,k) = h(0) = 0. \tag{327}$$

We now convert the initial value problem (318)-(323) as a system of integral equations. Multiplying (324) by the function

$$e^{2ik\int_0^x p_+(t,k)dt}, \tag{328}$$

we have

$$\frac{d}{dx}\left(e^{2ik\int_0^x p_+(t,k)dt}e_+(x,k)\right) = -ik\cdot e^{2ik\int_0^x p_+(t,k)dt}\left(e_+^2(x,k)-h(x)\right). \tag{329}$$

Integrating the result over the interval $[0,x]$, we obtain

$$e_+(x,k) - ik\int_0^x h(t)e^{-2ik\int_t^x p_+(\tau,k)d\tau}dt = -ik\int_0^x e_+^2(t,k)e^{-2ik\int_t^x p_+(\tau,k)d\tau}dt. \tag{330}$$

A similar calculation reduces (325) to the equation

$$e_-(x,k) + ik\int_0^x h(t)e^{2ik\int_t^x p_-(\tau,k)d\tau}dt = ik\int_0^x e_-^2(t,k)e^{2ik\int_t^x p_-(\tau,k)d\tau}dt. \tag{331}$$

An integration of (326) over the interval $[0,x]$ converts (326) into the integral equation

$$\begin{aligned} & h(x) - \int_0^x h(t)f_a(t)dt - \int_0^x w(t)\int_{-a}^a (e_+(t,z)-e_-(t,z))dzdt \\ = & \frac{2}{\pi}\int_0^x h(t)\int_{-a}^a (e_+(t,z)-e_-(t,z)dzdt + \int_0^x \epsilon_a(t)dt. \end{aligned} \tag{332}$$

Clearly, equations (330), (331), (332) is equivalent to (312), which completes the proof. □

Lemma 5.2 *Under the conditions of Lemma 5.1, there exists a positive number c_1 such that for any $f,g\in\Sigma$, there exist continuous functions $\delta_1,\delta_2:[0,1]\times[-a,a]\to C$, $\delta_3:[0,1]\to C$ such that*

$$N(f)(x,k)-N(g)(x,k) = [\delta_1(x,k),\delta_2(x,k),\int_0^x \delta_3(t)dt], \tag{333}$$

and

$$\max(\|\delta_1\|,\|\delta_2\|,\|\delta_3\|) \le c_1\cdot a\cdot\max(\|f\|,\|g\|)\|f-g\|. \tag{334}$$

Proof. Formula (333) is a direct consequence of (314). In fact, we have

$$\delta_1(x,k) = -ik\int_0^x (f_1^2(t,k)-g_1^2(t,k))e^{-2ik\int_t^x p_+(\tau,k)d\tau}dt, \tag{335}$$

$$\delta_2(x,k) = ik\int_0^x (f_2^2(t,k)-g_2^2(t,k))e^{2ik\int_t^x p_-(\tau,k)d\tau}dt, \tag{336}$$

$$\delta_3(x) = \frac{2}{\pi}\int_{-a}^a (f_3(x)(f_1(x,z)-f_2(x,z))-g_3(x)(g_1(x,z)-g_2(x,z)))\,dz \tag{337}$$

for any $f=[f_1,f_2,f_3]\in\Sigma$, $g=[g_1,g_2,g_3]\in\Sigma$. In order to prove (334), we first observe that due to Corollary 4.3, there exists a positive number c_4 such that

$$\left|e^{-2ik\int_t^x p_+(\tau,k)d\tau}\right| \leq c_4, \tag{338}$$

$$\left|e^{2ik\int_t^x p_-(\tau,k)d\tau}\right| \leq c_4, \tag{339}$$

for all $t,x\in[0,1]$, $k\in R$. Observing that $|k|\leq a$, $0\leq x\leq 1$, and using the estimate (338), we obtain the estimate

$$\|\delta_1\|\leq a\cdot c_4\|f+g\|\cdot\|f-g\|. \tag{340}$$

A similar calculation shows that

$$\|\delta_2\|\leq a\cdot c_4\|f+g\|\cdot\|f-g\|, \tag{341}$$

and we obtain the estimate for δ_3 by first regrouping (337):

$$\begin{aligned}\|\delta_3\| &= \frac{2}{\pi}\sup_{x\in[0,1]}\Bigg|\Bigg((f_3(t)-g_3(t))\int_{-a}^{a}(f_1(t,z)-f_2(t,z))dz\\ &\quad +g_3(t)\int_{-a}^{a}((f_1(t,z)-g_1(t,z))-(f_2(t,z)-g_2(t,z)))dz\Bigg)\Bigg|\\ &\leq \frac{2}{\pi}\left(\|f-g\|\int_{-a}^{a}\|f+g\|dz+\|g\|\int_{-a}^{a}2\|f-g\|dz\right)\\ &\leq a\cdot\frac{4}{\pi}\|f-g\|\left(\|f+g\|+2\|g\|\right)\end{aligned} \tag{342}$$

Now, (334) follows immediately from (340), (341), (342). □

Lemma 5.3 *Under the conditions of Lemma 5.1, there exist positive numbers c_2,c_3 such that for any $\delta\in\Sigma$ of the form*

$$\delta(x,k)=[\delta_1(x,k),\delta_2(x,k),\int_0^x\delta_3(t)dt], \tag{343}$$

the linear equation

$$L(v)=\delta \tag{344}$$

has a unique solution $v=[v_1,v_2,v_3]\in\Sigma$. Furthermore,

$$\|v\|\leq c_2\cdot a\max(\|\delta_1\|,\|\delta_2\|)+c_3\|\delta_3\|. \tag{345}$$

Proof. We only need to prove (345), since the existence and uniqueness of the solution v of the linear equation (344) is a direct consequence of the estimate (345). Due to (313), (343), the equation (344) can be rewritten in the form

$$
\begin{aligned}
v_1(x,k) &= ik\int_0^x v_3(t)e^{-2ik\int_t^x p_+(\tau,k)d\tau}dt + \delta_1(x,k), && (346)\\
v_2(x,k) &= -ik\int_0^x v_3(t)e^{2ik\int_t^x p_-(\tau,k)d\tau}dt + \delta_2(x,k) && (347)\\
v_3(x) &= \int_0^x v_3(t)f_a(t)dt + \int_0^x w(t)\int_{-a}^a (v_1(t,z) - v_2(t,z)dzdt + \int_0^x \delta_3(t)dt. && (348)
\end{aligned}
$$

In order to prove (345), we first eliminate v_1, v_2 from (348) and obtain an estimate for v_3. Subtracting (347) from (346), and integrating the result over the interval $[-a,a]$, we obtain

$$
\begin{aligned}
&\int_{-a}^a (v_1(x,z) - v_2(x,z))\,dz\\
&= \int_0^x v_3(t)\int_{-a}^a iz\left(e^{-2iz\int_t^x p_+(\tau,z)d\tau} + e^{2iz\int_t^x p_-(\tau,z)d\tau}\right)dzdt\\
&\quad + \int_{-a}^a (\delta_1(x,z) - \delta_2(x,z))dz\\
&= \int_0^x g_a(x,t)v_3(t)dt + 4a\cdot s_a(x), \qquad (349)
\end{aligned}
$$

with $g_a : [0,1]\times[0,1] \to C$, $s_a : [0,1] \to C$ given by the formulae

$$
\begin{aligned}
g_a(x,t) &= \int_{-a}^a iz\left(e^{2iz\int_t^x p_+(\tau,z)d\tau} + e^{2iz\int_t^x p_-(\tau,z)d\tau}\right)dz, && (350)\\
s_a(x) &= \frac{1}{4a}\int_{-a}^a (\delta_1(x,z) - \delta_2(x,z))dz. && (351)
\end{aligned}
$$

Combining (348) with (349), we obtain

$$
\begin{aligned}
v_3(x) &= \int_0^x v_3(t)f_a(t)dt + \int_0^x w(t)\int_0^t g_a(t,\tau)v_3(\tau)d\tau dt\\
&\quad + 4a\int_0^x w(t)s_a(t)dt + \int_0^x \delta_3(t)dt. \qquad (352)
\end{aligned}
$$

We will obtain the estimate (362) for v_3 (see below) by first proving (353), (354), (355), and (359) for functions f_a, w, g_a, s_a. Obviously, there exist constants $c_5 > 0, c_6 > 0$ such that

$$|w(x)| \le c_5 \qquad (353)$$

for all $x \in R$ due to (307), and

$$|f_a(x)| \le c_6 \qquad (354)$$

for all $x \in [0,1]$, any $a > 0$, due to (306), (282), and

$$|s_a(x)| \le \max(\|\delta_1\|, \|\delta_2\|). \qquad (355)$$

due to (351). Observing that

$$\int_{-a}^{a} z \cdot e^{-2iz\int_t^x p_+(\tau,z)d\tau} dz = -\int_{-a}^{a} z \cdot e^{2iz\int_t^x \overline{p_+(\tau,z)}d\tau} dz \tag{356}$$

due to (156), and combining (350) with (356), we have

$$g_a(x,t) = \int_{-a}^{a} iz \left(e^{2iz\int_t^x p_-(\tau,z)d\tau} - e^{2iz\int_t^x \overline{p_+(\tau,z)}d\tau} \right) dz. \tag{357}$$

According to Theorem 4.3, for any $x \in R$, the function

$$p_-(x,k) - \overline{p_+(x,k)} \tag{358}$$

decays uniformly like k^{-m}, for $k \in R$, and consequently, the integrand in (357) decays like $k^{-(m-2)}$ uniformly with respect to $t, x \in [0,1]$. Since we have assumed that $m \geq 4$, there exists a constant $c_7 > 0$ such that

$$|g_a(x,t)| \leq c_7, \tag{359}$$

for all $t, x \in [0,1]$, $a > 0$. Now, combining the integral equation (352) with the estimates (353), (354), (355), (359), we have

$$\begin{aligned} |v_3(x)| &\leq c_6 \int_0^x |v_3(t)|dt + c_5 \cdot c_7 \int_0^x \int_0^t |v_3(\tau)|dtd\tau \\ &\quad +4a \cdot c_5 \cdot x \max(\|\delta_1\|, \|\delta_2\|) + \|\delta_3\| \\ &\leq \int_0^x (c_6 + c_5 \cdot c_7)|v_3(t)|dt + 4a \cdot c_5 \cdot x \max(\|\delta_1\|, \|\delta_2\|) + \|\delta_3\| \end{aligned} \tag{360}$$

Now, the estimate for v_3 follows from Gronwall's inequality (see Lemma 3.9),

$$\begin{aligned} |v_3(x)| &\leq 4a \cdot c_5 \cdot x \max(\|\delta_1\|, \|\delta_2\|) + \|\delta_3\| \\ &\quad +c_8 \int_0^x (4a \cdot c_5 \cdot t \max(\|\delta_1\|, \|\delta_2\|) + \|\delta_3\|) e^{(x-t)(c_6+c_5\cdot c_7)} dt, \end{aligned} \tag{361}$$

with $c_8 = c_6 + c_5 \cdot c_7$. Clearly, there exist positive numbers c_9, c_{10} such that

$$|v_3(x)| \leq c_9 \max(\|\delta_1\|, \|\delta_2\|) + c_{10}\|\delta_3\|, \tag{362}$$

for all $x \in [0,1]$, we thus have the estimate for v_3 (see (345)).

In order to obtain similar estimates for v_1, v_2, we first provide an estimate for the derivative of v_3. Differentiating (352), we have

$$v_3'(x) = v_3(x)f_a(x)dt + w(x)\int_0^x g_a(x,t)v_3(t)dt + 4a \cdot w(x)s_a(x) + \delta_3(x). \tag{363}$$

Combining (363) with (362), we observe that there exist positive numbers c_{11}, c_{12} such that

$$|v_3'(x)| \leq c_{11} \max(\|\delta_1\|, \|\delta_2\|) + c_{12}\|\delta_3\|. \tag{364}$$

Integrating by parts in (346) yields

$$
\begin{aligned}
v_1(x,k) &= \frac{1}{2}\int_0^x \frac{v_3(t)}{p_+(t,k)} d\left(e^{-2ik\int_t^x p_+(\tau,k)d\tau}\right) + \delta_1(x,k) \\
&= \delta_1(x,k) + \frac{1}{2}\left(\frac{v_3(x)}{p_+(x,k)}\right. \\
&\quad \left. - \int_0^x \frac{v_3'(t)p_+(t,k) - v_3(t)p_+'(t,k)}{p_+^2(t,k)} e^{-2ik\int_t^x p_+(\tau,k)d\tau} dt\right). \qquad (365)
\end{aligned}
$$

For all $(x,k) \in R \times C^+$, p_+ is uniformly bounded, $Re(p_+)$ is uniformly bounded from below by a positive number (see Theorem 4.2). Dut to Corollary 4.3, p_+' and

$$
e^{-2ik\int_t^x p_+(\tau,k)d\tau} \qquad (366)
$$

are uniformly bounded for all $x, t \in [0,1]$, $k \in R$. Therefore, combining (365) with (362), (364), (236), we observe that there exist positive numbers c_{13}, c_{14} such that

$$
|v_1(x,k)| \le c_{13} \max(||\delta_1||, ||\delta_2||) + c_{14}||\delta_3||, \qquad (367)
$$

and a similar calculation shows that

$$
|v_2(x,k)| \le c_{13} \max(||\delta_1||, ||\delta_2||) + c_{14}||\delta_3|| \qquad (368)
$$

for all $x \in [0,1]$, $k \in [-a,a]$. Now, the estimate (345) follows immediately from (362), (367), (368). □

Using Lemmas 5.1, 5.2, 5.3, we now proceed with the proof of Theorem 5.1.

Proof of Theorem 5.1. Theorem 4.4 implies that there exists positive numbers b_1, b_2 such that

$$
\begin{aligned}
|f_a(x)| &\le b_1, \qquad (369) \\
|\epsilon_a(x)| &\le \frac{b_2}{a^{(m-1)}}. \qquad (370)
\end{aligned}
$$

We prove the theorem by showing that there exist positive numbers A, c such that for all $a \ge A$, the solution $u = [e_+, e_-, h] \in \Sigma$ exists (see Lemma 5.1 for the definitions of u, Σ), and that

$$
||u|| \le \frac{c}{a^{(m-1)}}. \qquad (371)
$$

We will obtain the solution u of the equation (312) via the following iterative procedure:

$$
\begin{aligned}
u_0 &= 0, \qquad (372) \\
L(u_{n+1}) &= N(u_n) + [0, 0, \epsilon_a(t)], \qquad (373)
\end{aligned}
$$

with L, N defined by the formulae (313), (314), respectively. Clearly, we only need to show that there exist positive numbers A, c such that for all $a \ge A$

$$\|u_n\| \le \frac{c}{a^{(m-1)}}, \tag{374}$$

and the sequence u_n, $n = 0, 1, \ldots$ converges (to the solution u). The first iterate u_1 satisfies the equation

$$L(u_1) = [0, 0, \epsilon_a(t)], \tag{375}$$

and according to (308), (282), there exists a constant b such that

$$\|\epsilon_a\| \le \frac{b}{a^{(m-1)}}. \tag{376}$$

Combining (375) with (345) and (376), we observe that there exist constant c_4 such that

$$\|u_1\| \le \frac{c_4}{a^{(m-1)}}. \tag{377}$$

Now, we choose a constant $A > 0$ such that

$$a \cdot c_1(c_2 \cdot a + c_3)\|u_1\| \le \frac{1}{4} \tag{378}$$

for all $a \ge A$. Defining $u_{-1} = 2u_1$ for convenience, we prove by induction that

$$\|u_{n+1} - u_n\| \le \frac{1}{2}\|u_n - u_{n-1}\|, \tag{379}$$

$$\|u_{n+1}\| \le 2\|u_1\|, \tag{380}$$

for all $n \ge 0$, $a \ge A$.

The case $n = 0$ is a trivial one. For $n \ge 1$, (373) indicates that

$$L(u_{n+1} - u_n) = N(u_n) - N(u_{n-1}), \tag{381}$$

Due to Lemma 5.2, there exist continuous functions $\delta_1, \delta_2 : [0,1] \times [-a, a] \to C$, $\delta_3 : [0,1] \to C$ such that

$$N(u_n) - N(u_{n-1}) = [\delta_1(x,k), \delta_2(x,k), \int_0^x \delta_3(t)dt], \tag{382}$$

Now, combining (381), (382) with (345), (334), and the assumption of the induction, we obtain

$$\begin{aligned} \|u_{n+1} - u_n\| &\le c_2 \cdot a \max(\|\delta_1\|, \|\delta_2\|) + c_3\|\delta_3\| \\ &\le a \cdot c_1(c_2 \cdot a + c_3) \max(\|u_n\|, \|u_{n-1}\|)\, \|u_n - u_{n-1}\| \\ &\le \frac{1}{2}\|u_n - u_{n-1}\|, \end{aligned} \tag{383}$$

which proves (379). The estimate (380) is a direct consequence of (379).

Finally, the sequence u_n, $n = 0, 1, \ldots$ converges to the solution u due to (379), and therefore

$$\|u\| \le \frac{2c_4}{a^{(m-1)}}, \tag{384}$$

for all $a \geq A$ due to (380), (377), which was to be proved. □

Remark 5.2. The proof above requires that $q \in c^m(R)$, with $m \geq 4$. At the expense of a considerable increase in the complexity of the proof, it is not difficult to extend this result to $m \geq 2$. However, our numerical experiments (see the following section) indicate that the scheme works quite well for continuous, piecewise continuously differentiable q, and even for piecewise continuously differentiable q with finite number of jumps. In the latter two cases, the rates of convergence of the algorithm are $1/a$ and $1/\sqrt{a}$, respectively.

6 Implementation and Numerical Results

6.1 Implementation

In implementing the algorithm of this paper (see Section 5.2), the integral

$$\int_{-a}^{a} (p_+(x,k) - p_-(x,k))dk \tag{385}$$

in equation (299) is approximated by the trapezoidal sum

$$\begin{aligned} T(h) &= h \sum_{j=-M+1}^{M-1} (p_+(x,k_j) - p_-(x,k_j)) \\ &\quad + \frac{h}{2}((p_+(x,-a) - p_-(x,-a)) + (p_+(x,a) - p_-(x,a))), \end{aligned} \tag{386}$$

with $h = a/M$, $k_j = jh$, $j = -M, \ldots, M$. Since for real k, $p_+(x,-k) = \overline{p_+(x,k)}$, $p_-(x,-k) = \overline{p_-(x,k)}$ (see Observation 4.1), the ODEs (297), (298), (299) are discretized in the k-space using $M+1$ nodes $k_j = jh$, $j = 0, \ldots, M$, leading to a system of $2M+3$ ODEs

$$p'_{h+}(x,k_j) = -ik_j \left(p^2_{h+}(x,k_j) - (1 + q_h(x))\right), \tag{387}$$

$$p'_{h-}(x,k_j) = ik_j \left(p^2_{h-}(x,k_j) - (1 + q_h(x))\right), \tag{388}$$

$$\begin{aligned} q'_h(x) &= \frac{4h}{\pi}(1 + q_h(x)) \left(\sum_{j=1}^{M-1} Re(p_{h+}(x,k_j) - p_{h-}(x,k_j)) \right. \\ &\quad \left. + \frac{1}{2} \left(Re(p_{h+}(x,0) - p_{h-}(x,0)) + Re(p_{h+}(x,a) - p_{h-}(x,a)) \right) \right). \end{aligned} \tag{389}$$

subject to the initial conditions

$$p_{h+}(0,k_j) = p_0(k_j), \tag{390}$$

$$p_{h-}(0,k_j) = 1, \tag{391}$$

$$q_h(0) = 0 \tag{392}$$

(see (300)– (302)). These ODEs are then solved using a standard 4-th order Runge-Kutta scheme.

When an integral is discretized via a quadrature formula, the rate of convergence of the quadrature is critical to the numerical performance of the algorithm. It turns out that while the estimate

$$\overline{p_+(x,k)} - p_-(x,k) = O(a^{-m}) \tag{393}$$

(see Theorem 4.3) ensures a rapid convergence of q_a to q as a grows (see Theorem 5.1), it also guarantees a rapid convergence of the trapezoidal quadrature (386) to the integral (385). This fact is formalized in the following lemma. Its proof is based on the Euler-Maclaurin summation formula (see, for example, [11]), and is ommitted, since it is quite involved, and incidental to the purpose of this paper.

Lemma 6.1 *Suppose that $q \in c_0^m([0,1])$, $m \geq 2$, $q^{(m)}$ is absolutely continuous and $q(x) > -1$ for all $x \in R$. Then there exist positive numbers c_n, $n = 0, \ldots$ such that*

$$\left| \frac{d^n \left(\overline{p_+(x,a)} - p_-(x,a) \right)}{dk^n} \right| \leq \frac{c_n}{|k|^m}. \tag{394}$$

Furthermore, for any $\beta > 0, b > 0$, there exists a constant $c > 0$ such that

$$\left| \int_{-a}^{a} (p_+(x,k) - p_-(x,k)) dk - T\left(\frac{b}{a^\beta}\right) \right| \leq \frac{c}{a^m}. \tag{395}$$

Using the estimate (395), and reproducing the proof of Theorem 5.1 almost verbatim, one can prove the following theorem.

Theorem 6.1 *Suppose that $q \in c_0^m([0,1])$, $m \geq 4$, $q^{(m)}$ is absolutely continuous and $q(x) > -1$ for all $x \in R$. Suppose further that for given $r > 0, s > 0$, $q(r,s,x)$ denotes the solution q_h of the system (387)– (392) with $h = r/a^s$. Then for any $\alpha > 0, \beta > 0$, there exist constants $A > 0, c > 0$ such that*

$$|q(x) - q(\alpha,\beta,x)| \leq \frac{c}{a^{(m-1)}}. \tag{396}$$

for all $x \in [0,1]$, $a \geq A$

6.2 Numerical Results

We have applied the algorithm of the present paper to the reconstruction of several types of scatterers, from infinitely differentiable q to discontinuous q. In the four groups of examples presented below, unless specified otherwise, all scatterers have compact support $[0, 2\pi]$. The computations were performed in double precision on a SPARC computer.

The results of four classes of numerical experiments are presented in this section.

In the first class (Examples 1–2.2) are scatterers satisfying the smoothness conditions of Theorem 6.1. In the second class (Example 3) is a scatterer q violating the smoothness conditions only mildly (it is continuous, but its derivative is discontinuous at two points). In the third class (Examples 4.1, 4.2) are scatterers that strongly violate the smoothness conditions by being discontinuous. Finally, in Example 5 we perform a crude test of stability of the algorithm by truncating all measured data $p_0(k_j)$, $j = 1, \ldots, M$ after $1, 2, 3, 4$ or 5 digits.

In Tables 1–6, h_k denotes step size of the trapezoidal rule in the k-interval $[0, a]$, N_x denotes the number of points in the x-interval $[0, 2\pi]$, E^2, E^∞ represent the relative L^2 and maximum norm of error of the reconstructed scatterer, respectively. In Figures 1–6, dotted lines denote the exact solution, while solid lines denote the numerical reconstruction. In all examples, for a given a, h_k and N_x were chosen such that further decrease of h_k and increase of N_x brought no improvements on the accuracy of the reconstruction.

Remark 6.1. In order to obtain the scattering data $p_+(0, k)$ for the Examples 1–3, the scattered field ϕ_{scat+} was obtained as a solution of the boundary value problem (10), (8), (9) via a high order algorithm described in [15]. The parameters in the scheme were chosen in such a manner that at least 14-digit accuracy was always maintained. Formulae (4), (30) were then used to obtain $p_+(0, k)$ from ϕ_{scat+}.

In Examples 4.1 and 4.2, a standard procedure for the solution of the initial value problem (1), (26) (for ϕ_+) with piecewise constant q was used (see, for example, [16]). Here, the solutions were obtained with at least 15 correct digits. The scattering data $p_+(0, k)$ were obtained from ϕ_+ via formula (30).

Remark 6.2. In the examples below, no effort was made to optimize the code used, either from the algorithmic or from the programming point of view. For example, we used the Runge-Kutta scheme to solve ODEs (387), (388), (389). While it produced satisfactory results in our experiments, it is by no means the most efficient scheme for the solution of problems of this type.

Example 1. Reconstruction of a Gaussian distribution

$$q(x) = e^{-\left(\frac{x-\pi}{\sigma}\right)^2} \tag{397}$$

where the variant σ given by the formula

$$\sigma = \frac{\pi}{4}\sqrt{\log_{10}(e)} = 0.5175854235\ldots \tag{398}$$

was chosen such that the function is effectively zero to double precision outside the interval $[0, 2\pi]$. The results of this numerical experiment are depicted in Table 1 and Figure 1. For all practical purposes, the scatterer (397) is a c^∞-function in R with the support on the interval $[0, 2\pi]$, and therefore the algorithm should be expected to converge extremely rapidly. In fact, the graphs of the two reconstructions are indistinguishable from each other, and from the graph of the original q, and we only provide one of the reconstructions (Figure 1).

Table 1. CPU Times and Accuracies for Example 1

a	h_k	N_x	E^2	E^∞	t (*sec.*)
5	0.1	80	0.146×10^{-2}	0.153×10^{-2}	0.600
10	0.1	300	0.354×10^{-5}	0.415×10^{-5}	4.41
10	0.05	600	0.177×10^{-5}	0.183×10^{-5}	16.7
10	0.05	1200	0.175×10^{-5}	0.184×10^{-5}	34.2
20	0.05	2400	0.759×10^{-9}	0.108×10^{-8}	141
20	0.05	4000	0.988×10^{-10}	0.143×10^{-9}	235
20	0.025	4000	0.982×10^{-10}	0.142×10^{-9}	498

In the following two examples, we construct oscillatory scatterers of the form

$$q(x) = \sum_{j=1}^{3} c_j(1 - \cos(n_j x)), \tag{399}$$

with n_j, c_j, $j = 1, 2, 3$ given below. For given n_j, the coefficients c_j were chosen in such a manner that q is five times continuously differentiable for all $x \in R$, so that the rapid convergence of the reconstruction algorithm is guaranteed (see Theorems 5.1, 6.1)

Example 2.1. A less complicated scatterer is given by the formula

$$q(x) = 0.3\left((1 - \cos(2x)) - \frac{16}{21}(1 - \cos(3x)) + \frac{5}{28}(1 - \cos(4x)\right). \tag{400}$$

Reconstructions were performed with $a = 7,\ 14$. The results of this experiment are depicted in Table 2.1 and Figure 2.1(a). Since the scatterer is smooth, $\overline{p_+(x,k) - p_-(x,k)}$ decays rapidly as k grows. In particular, $Re(p_0(k))$ approaches 1 rapidly, as can be seen in Figure 2.1(b).

Table 2.1. CPU Times and Accuracies for Example 2.1

a	h_k	N_x	E^2	E^∞	t (*sec.*)
7	0.1	100	0.523×10^{-2}	0.983×10^{-2}	1.05
7	0.05	600	0.516×10^{-2}	0.833×10^{-2}	11.9
14	0.1	300	0.648×10^{-4}	0.172×10^{-3}	6.04
14	0.05	600	0.568×10^{-4}	0.948×10^{-4}	23.7
28	0.05	2000	0.231×10^{-7}	0.625×10^{-7}	170
28	0.025	4000	0.106×10^{-7}	0.155×10^{-7}	243

Example 2.2. A more complicated scatterer is given by the formula

$$q(x) = 0.4\left(1 - \cos(3x)) - \frac{1215}{2783}(1 - \cos(11x)) + \frac{7}{23}(1 - \cos(12x)\right). \tag{401}$$

Reconstructions were performed with $a = 10,\ 20$. The results of this experiment are depicted in Table 2.2 and Figure 2.2.

Table 2.2. CPU Times and Accuracies for Example 2.2

a	h_k	N_x	E^2	E^∞	t (*sec.*)
10	0.1	300	0.288×10^{-1}	0.376×10^{-1}	4.41
10	0.025	600	0.281×10^{-1}	0.367×10^{-1}	35.3
10	0.025	1200	0.281×10^{-1}	0.367×10^{-1}	70.4
20	0.1	400	0.395×10^{-2}	0.754×10^{-2}	11.4
20	0.025	800	0.127×10^{-2}	0.226×10^{-2}	98.7
20	0.025	1600	0.127×10^{-2}	0.220×10^{-2}	197
40	0.025	800	0.788×10^{-4}	0.300×10^{-3}	202
40	0.025	1600	0.878×10^{-5}	0.290×10^{-4}	404

Example 3. In this example, we reconstruct a scatterer defined by the formula

$$q(x) = 0.2 \cdot \sin(x). \tag{402}$$

Note that q' is discontinuous at the points $x = 0, 2\pi$, and as a result $\overline{p_+(x,k)} - p_-(x,k)$ decays like $1/k$, as can be seen in Figure 3(b). We have not proven a convergence theorem for such scatterers, but the algorithm seems to perform quite well in this case, and its rate of convergence oned.texto be linear (see Table 3 and Figure 3(a)).

Table 3. CPU Times and Accuracies for Example 3

a	h_k	N_x	E^2	E^∞	t (*sec.*)
5	0.1	75	0.482×10^{-1}	0.829×10^{-1}	0.590
10	0.1	150	0.239×10^{-1}	0.462×10^{-1}	2.19
20	0.1	300	0.119×10^{-1}	0.283×10^{-1}	8.47

Example 4.1. Here, we reconstruct a scatterer defined by the formula

$$q(x) = \begin{cases} 0.4 & \text{if } x \in [1,2], \\ 0 & \text{otherwise.} \end{cases} \tag{403}$$

In this example, the scatterer is discontinuous, and the conditions of Theorems 5.1, 6.1) are violated. In fact, in this case the integrand $p_+ - p_-$ does not even converge to zero as $k \to \infty$. The results of this experiment are depicted in Figures 4.1(a),(b) and Table 4.1.

Table 4.1. CPU Times and Accuracies for Example 4.1

a	h_k	N_x	E^2	t (*sec.*)
10	0.4	50	0.165	0.230
20	0.4	200	0.119	1.51
40	0.4	400	0.843×10^{-1}	6.03

Example 4.2. In this example, we reconstruct a staircase-shaped scatterer defined by the formula

$$q(x) = \begin{cases} 0 & x \in (-\infty, 0.5] \\ 0.1 & x \in (0.5, 1.0] \\ 0.2 & x \in (1.0, 1.5] \\ 0.4 & x \in (1.5, 2.0] \\ 0.6 & x \in (2.0, 2.5] \\ 0.5 & x \in (2.5, 3.0] \\ 0.3 & x \in (3.0, 3.5] \\ 0.1 & x \in (3.5, 4.0] \\ -0.1 & x \in (4.0, 4.5] \\ -0.3 & x \in (4.5, 5.0] \\ -0.2 & x \in (5.0, 5.5] \\ -0.1 & x \in (5.5, 6.0] \\ 0 & x \in (6.0, \infty) \end{cases} \tag{404}$$

This example is similar to the preceding one, but the shape of the scatterer is more complicated. The results of this experiment are shown in Table 4.2 and Figure 4.2.

Table 4.2. CPU Times and Accuracies for Example 4.2

a	h_k	N_x	E^2	t (*sec.*)
5	0.2	100	0.149	0.430
10	0.2	150	0.936×10^{-1}	1.18
20	0.2	300	0.682×10^{-1}	4.40

In the following example, we investigate the sensitivity of the reconstruction to perturbations of the initial data. In a somewhat crude test, we perturb the initial data for the algorithm by truncating it after a specified number of decimal digits (both the real and the imaginary parts). Clearly, after such a truncation, the maximum relative error is of the order 10^{D-1} (for example, when the number 1.999 is truncated after $D = 1$ digits, the result is 1).

Example 5. Tables 5 and 6 demonstrate the numerical results of the reconstruction of Examples 2.1 and 3, respectively, with various truncations of the input data. In each case, a was chosen sufficiently large that the error from the truncation of the trace formula due to finite a (see (278), (282)) is negligible compared to the error due to the finite number D of digits retained. For a given a, the parameters h_k, N_x were chosen such that accuracy of the reconstruction was not improved by a further decrease of h_k and/or increase of N_x. Also see Figures 5.1–6.2 comparing the scatterers reconstructed using the perturbed data with the prescribed ones.

Table 5. Accuracies for Example 2.1 with Truncated Data

D	a	h_k	N_x	E^2	E^∞
1	7	0.1	100	0.410	0.474
1	14	0.1	300	0.412	0.473
2	7	0.1	100	0.126	0.156
2	14	0.1	300	0.128	0.157
3	7	0.1	100	0.174×10^{-1}	0.265×10^{-1}
3	14	0.1	300	0.187×10^{-1}	0.256×10^{-1}
4	14	0.1	300	0.126×10^{-2}	0.151×10^{-2}
4	28	0.05	600	0.118×10^{-2}	0.132×10^{-2}
5	14	0.1	300	0.297×10^{-3}	0.426×10^{-3}
5	28	0.05	600	0.250×10^{-3}	0.324×10^{-3}

Table 6. Accuracies for Example 3 with Truncated Data

D	a	h_k	N_x	E^2	E^∞
1	10	0.1	150	0.647	0.863
1	20	0.1	300	0.640	0.852
2	10	0.1	150	0.121	0.173
2	20	0.1	300	0.113	0.164
3	10	0.1	150	0.314×10^{-1}	0.602×10^{-1}
3	20	0.1	300	0.206×10^{-1}	0.439×10^{-1}

The following observations can be made from Tables 1–6 and Figures 1–6.

1. When the scatterer satisfies the conditions of Theorems 5.1, 6.1, the accuracy of the reconstruction is somewhat better than that predicated by these theorems (see Example 2.1). This indicates that (as expected) the estimates (303), (396) are somewhat pessimistic.

2. When the scatterer violates the conditions of Theorems 5.1, 6.1 mildly (by having discontinuous derivative at the points $0, 2\pi$), the reconstruction algorithm still converges. Qualitatively, the reconstructions in Figure 3(a) should be described as good. A careful examination of Table 3 (and other data not presented in this paper) shows that the error of the reconstruction for such scatterers is proportional to $1/a$.

3. When the scatterer is discontinuous (Examples 4.1, 4.2), the algorithm produces results depicted in Figures 4.1, 4.2. The oscillatory behavior near the discontinuities resembles the well known Gibbs phenomenon. A careful examination of Tables 4.1, 4.2 (and other data not presented in the paper) shows that in this case, the point-wise convergence is absent. In the L^2-norm, the error of the reconstruction behaves as $1/\sqrt{a}$.

4. When the initial data are perturbed, the resulting error of the reconstruction appears to be proportional to the magnitude of the perturbation, and the proportionality coefficient is close to 1. This is a much better estimate than the one of Lemma 5.3 which bounds the condition

number of the algorithm by a. Qualitatively, it can be said that the algorithm is not sensitive to errors in the initial data.

6.3 Generalizations and Conclusions

An algorithm has been presented for the solution of the inverse scattering problem for the Helmholtz equation in one dimension. The algorithm is based on a combination of the standard Riccati equation for the impedance function with a newly constructed trace formula for the derivative of the potential, and leads to extremely accurate and efficient numerical schemes for smooth scatterers. The principal differences between this scheme and various layer-stripping techniques (see [12], [13], [14]) are:

1. Our algorithm operates in the frequency domain, while other efficient schemes are time-domain ones.

2. While the layer-stripping algorithms assume (at least conceptually) that the scatterer is piece-wise constant, and are best in this regime, our algorithm assumes that the scatterer is continuously differentiable. When the scatterer has a sufficient number of derivatives, our algorithm converges almost instantaneously (see Theorems 5.1, 6.1).

3. The principal drawback of the layer-stripping algorithms is the fact that they are an essentially one-dimensional techniques, and the authors are not aware of any successful attempts to generalize them to higher dimensions. We believe that our techniques do generalize to two and three dimensions, and in fact an implementation of a two-dimensional version of the procedure is in progress.

Following is a short discussion of other possible generalizations of the techniques of this paper.

1. In their present form, Theorems 5.1, 6.1 require that the scatterer have at least four continuous derivatives. Numerical examples 3–4 of the preceding subsection make it abundantly clear that this is a superflous requirement. Obviously, Theorems 5.1, 6.1 can be generalized to at least include the scatterers of the type reconstructed in examples 3–4. Including the scatterers of examples 3–4 will be somewhat more involved, and will require a significant reformulation of Theorems 5.1, 6.1.

2. The algorithm of this paper can be generalized for the Schrödinger equation. The generalization is faily straightforward and will be reported at a later date.

3. In the present paper, we are reconstructing a scalar potential q given the scattered data for a single Helmholtz equation. In many problems of physical interest, the potential has several components (such as the compressional and shear speeds of sound in a medium), and the scattered data correspond to a system of Helmholtz equations (such as equations of elastic scattering, or Maxwell's equations in the frequency domain). An extension of our techniques to these cases appears to be relatively straightforward, and will be reported at a later date.

6.4 Acknowledgments

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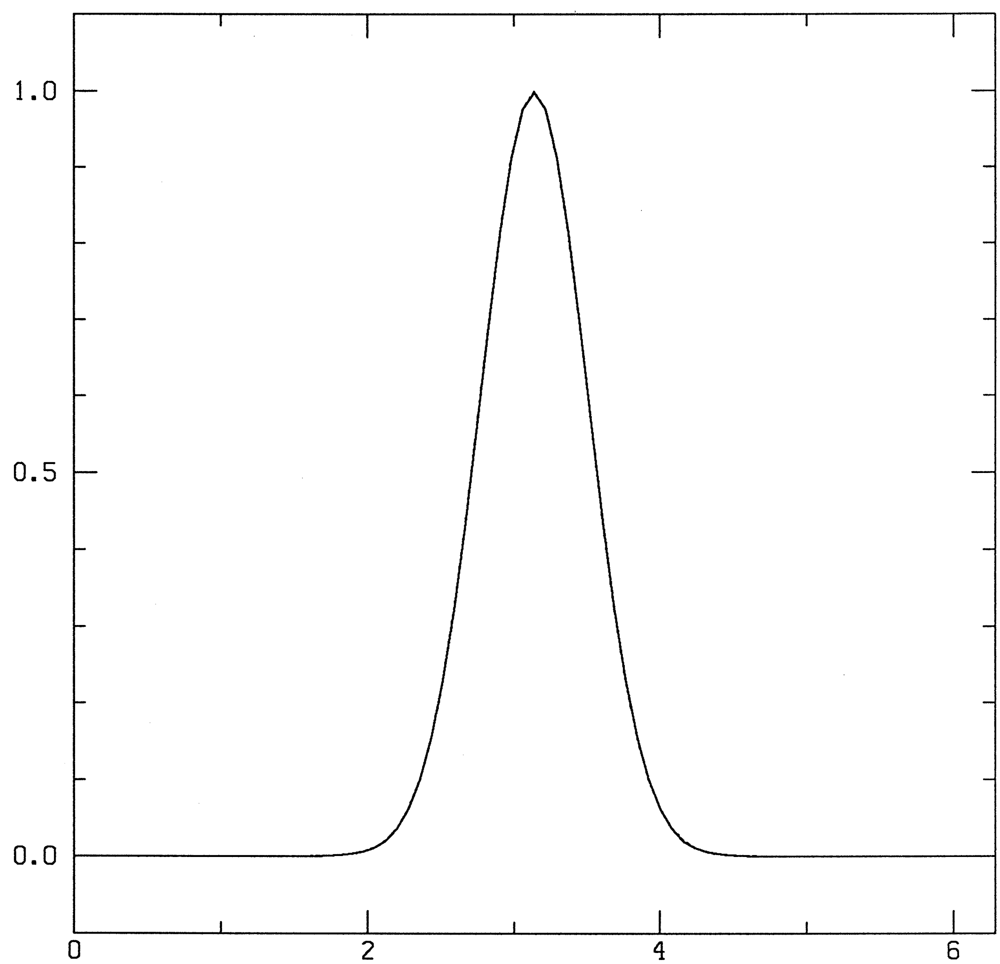

Figure 1. Reconstruction of Example 1 with a=5

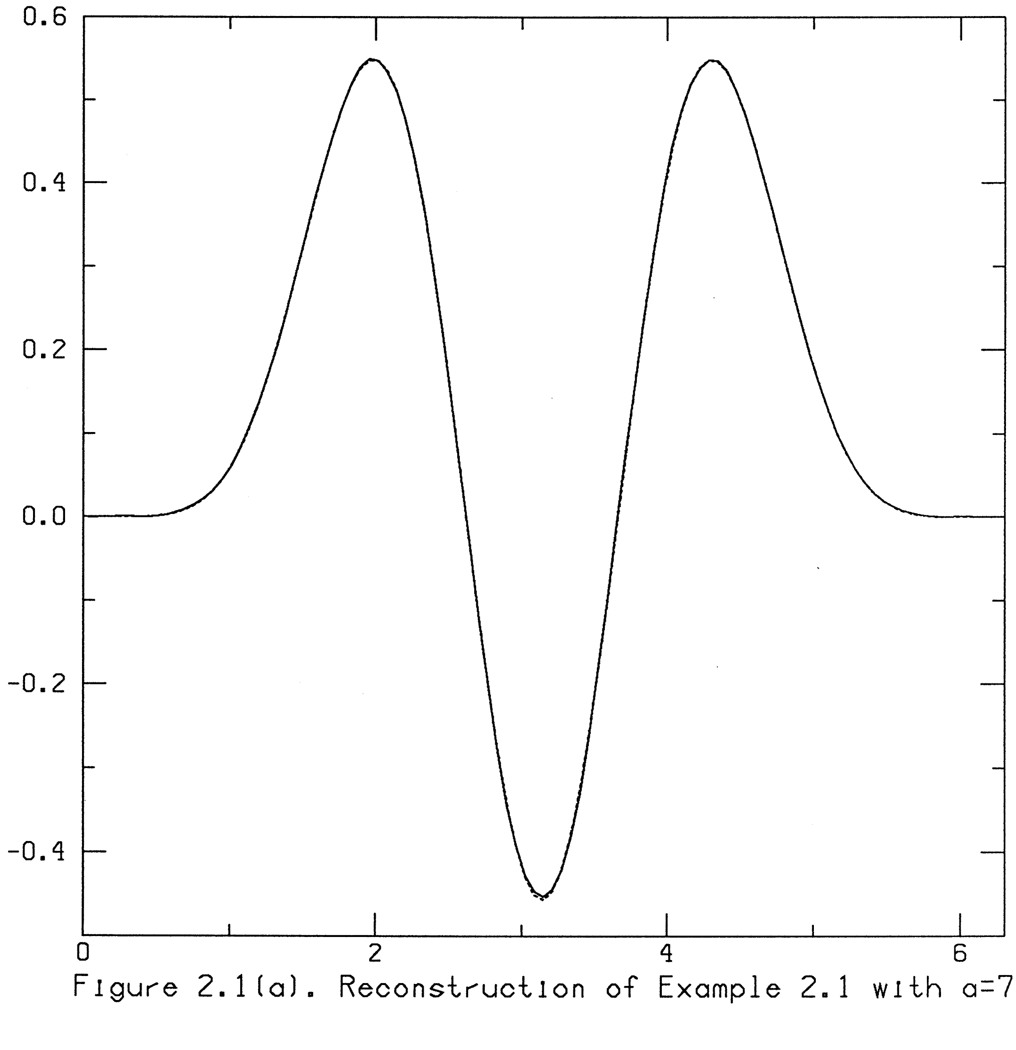

Figure 2.1(a). Reconstruction of Example 2.1 with a=7

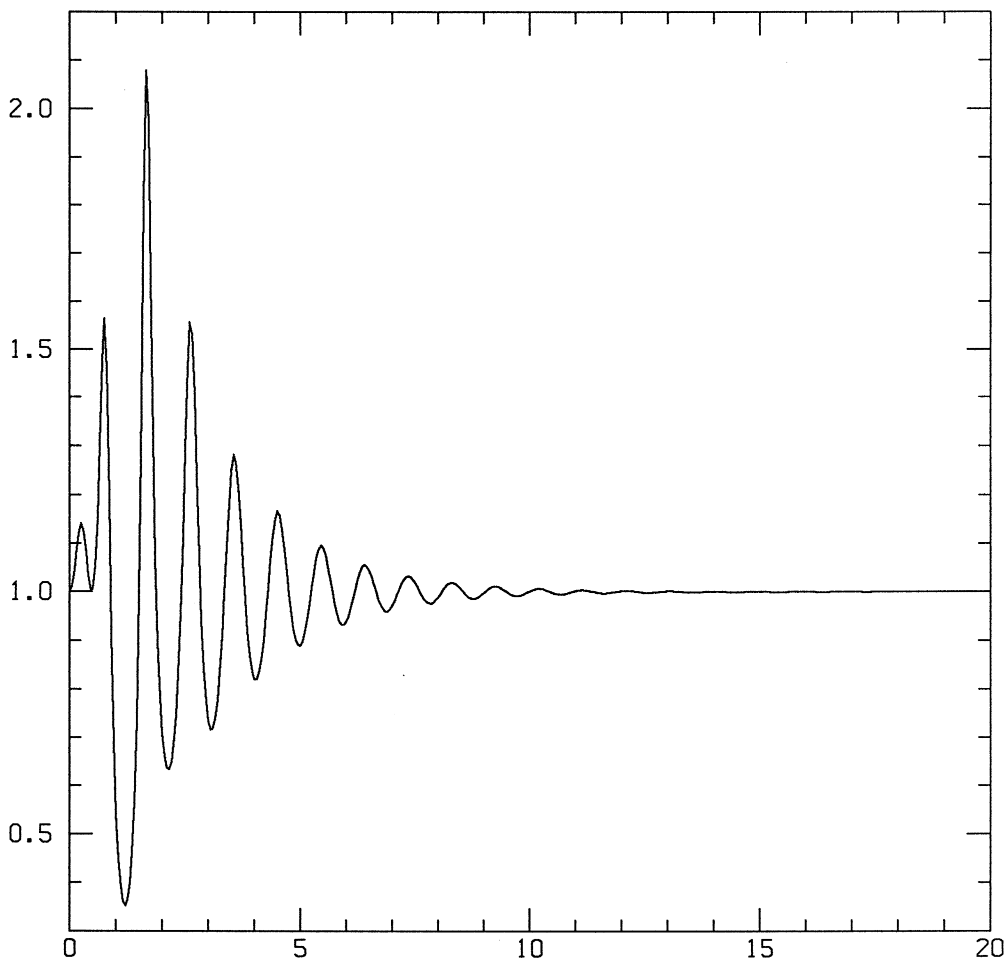

Figure 2.1(b). Real Part of p0(K) for Example 2.1

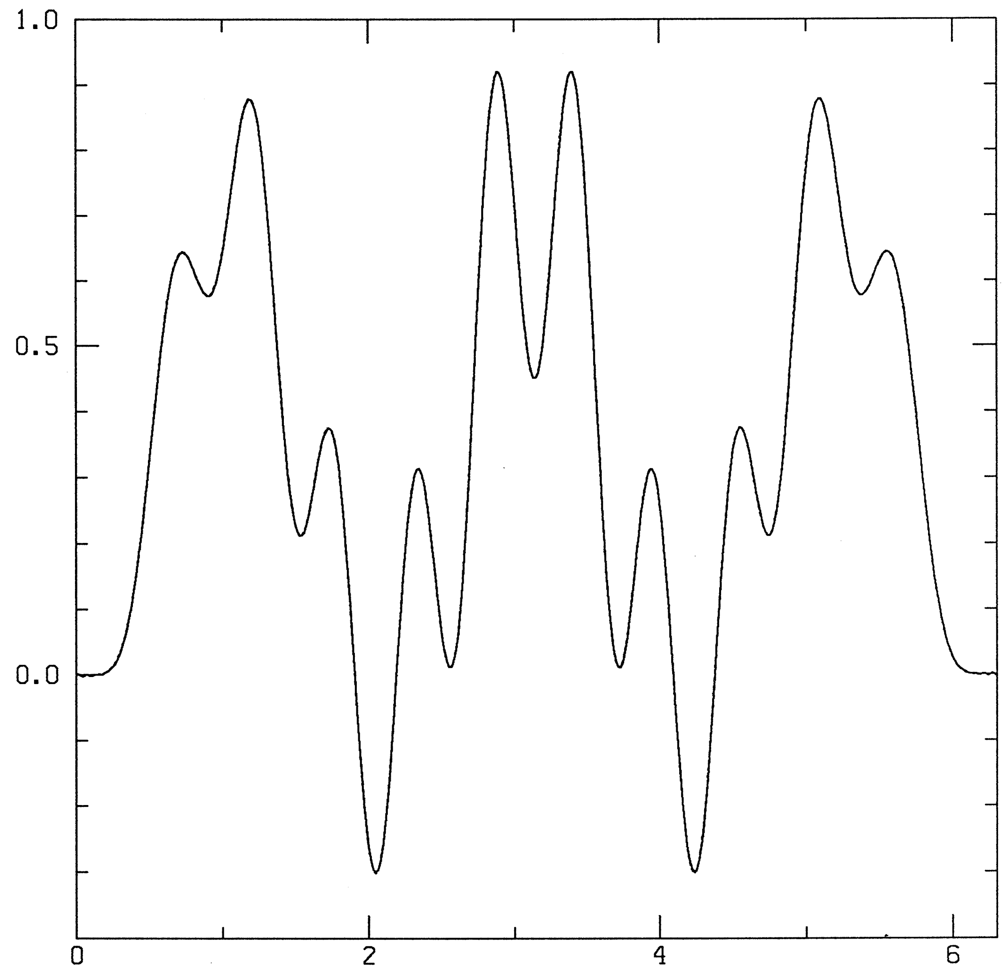

Figure 2.2. Reconstruction of Example 2.2 with a=20

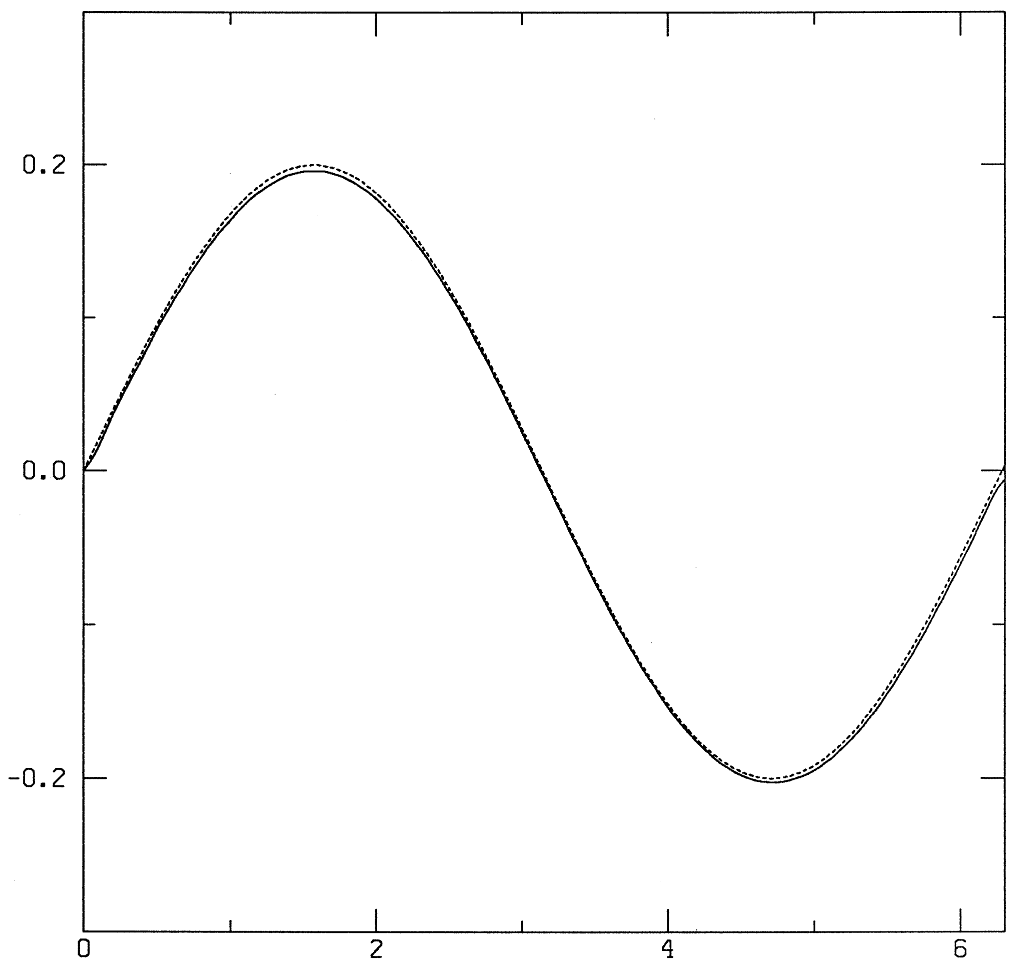

Figure 3(a). Reconstruction of Example 3 with a=10

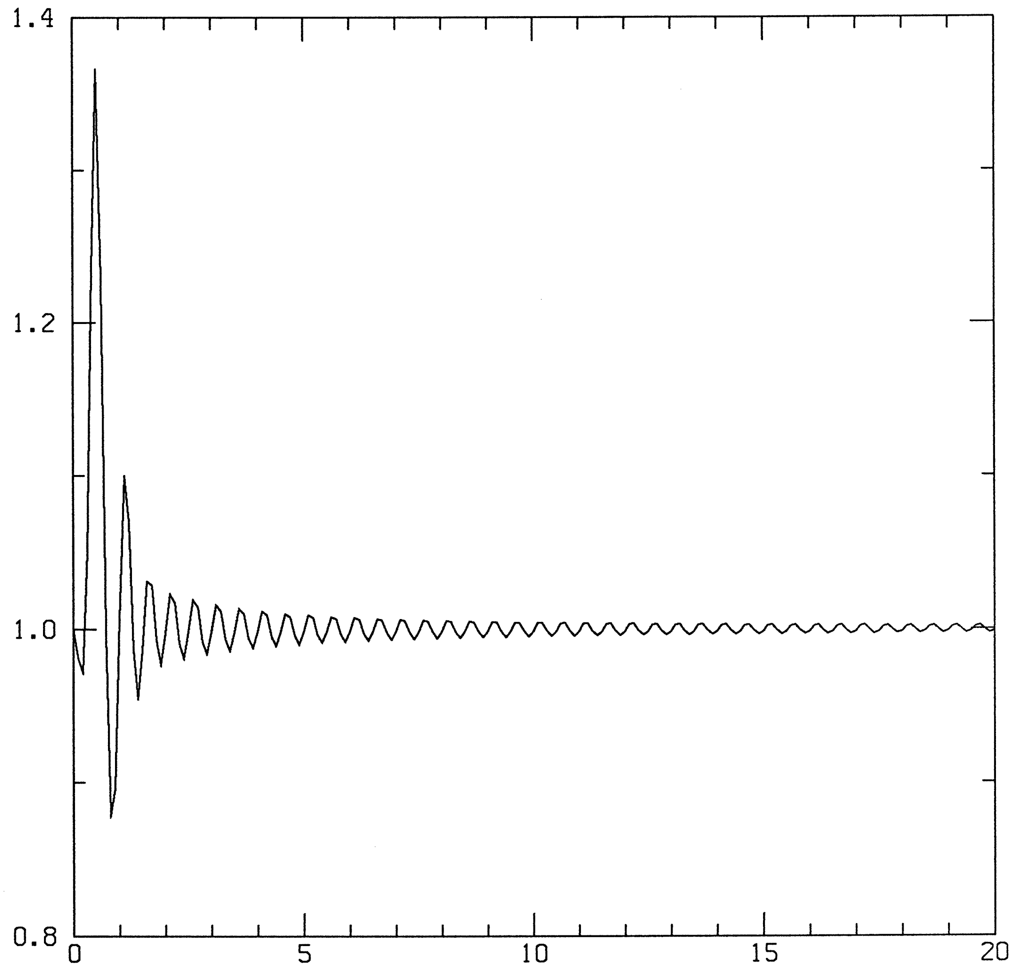

Figure 3(b). Real Part of p0(K) for Example 3

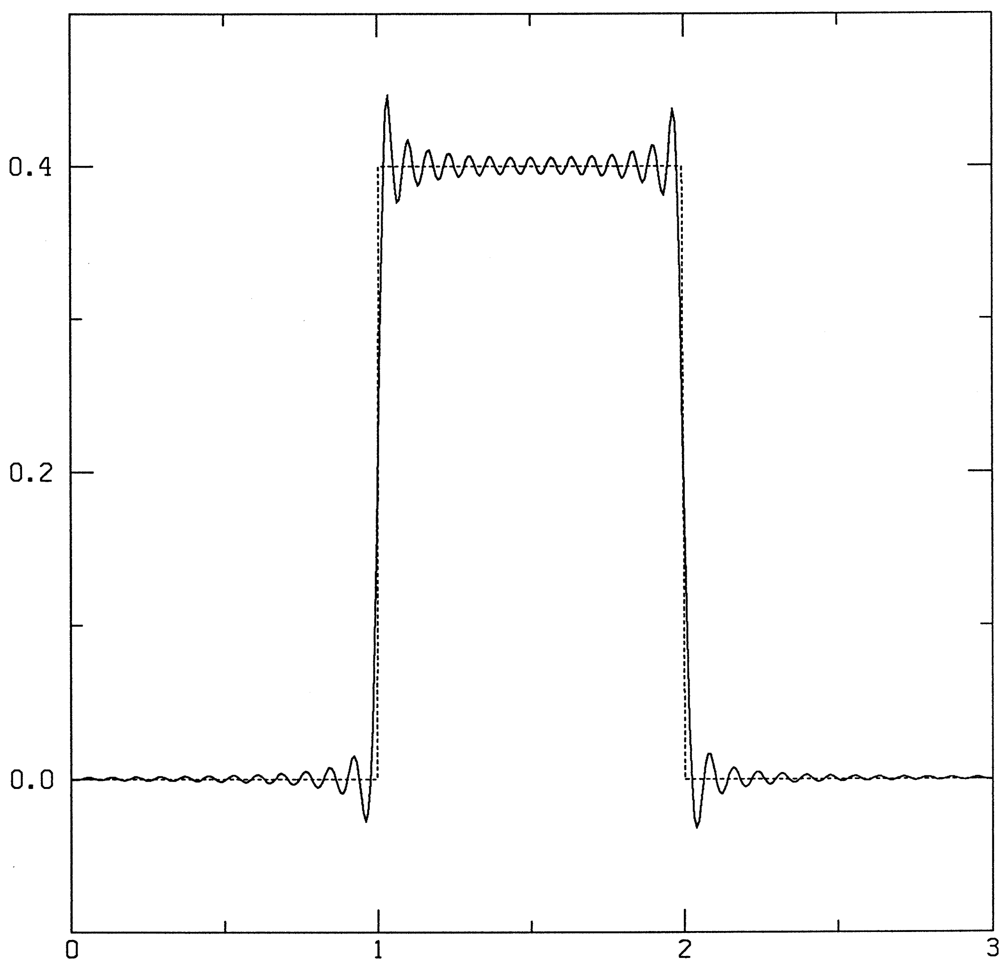

Figure 4.1(a). Reconstruction of Example 4.1 with a=40

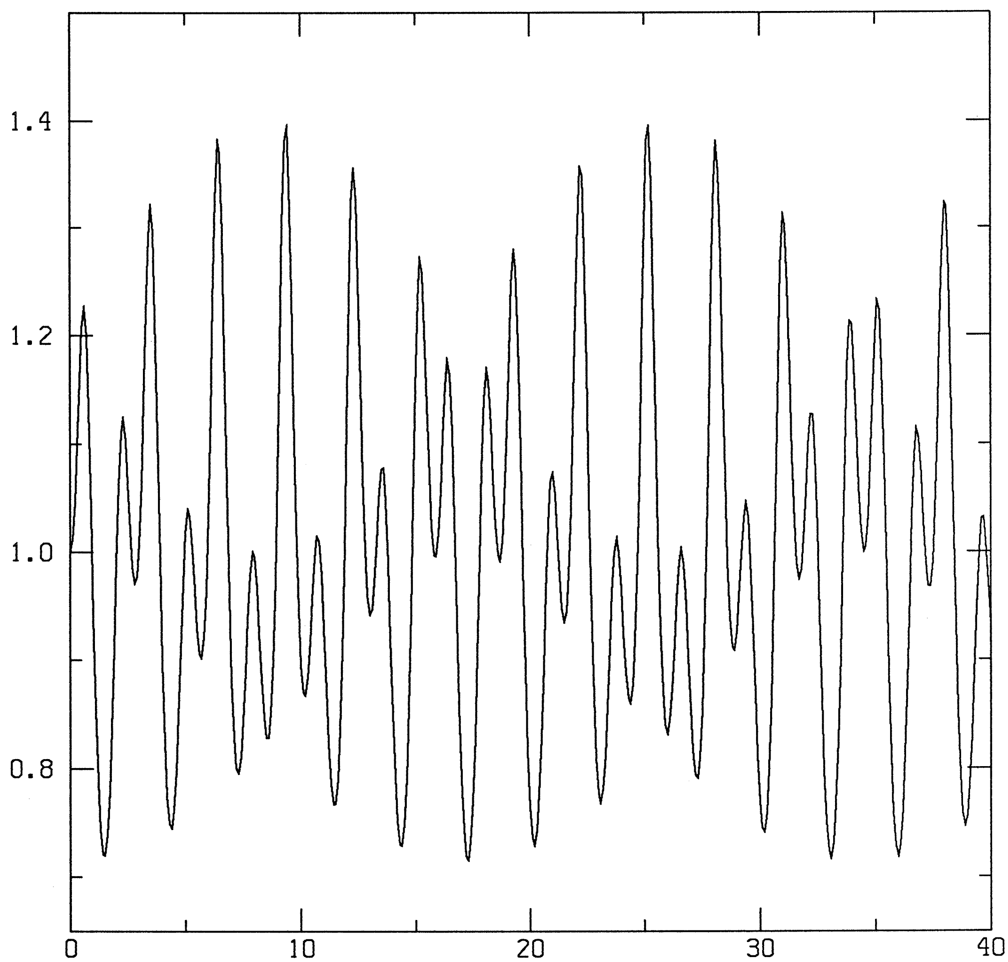

Figure 4.1(b). Real Part of p0(K) for Example 4.1

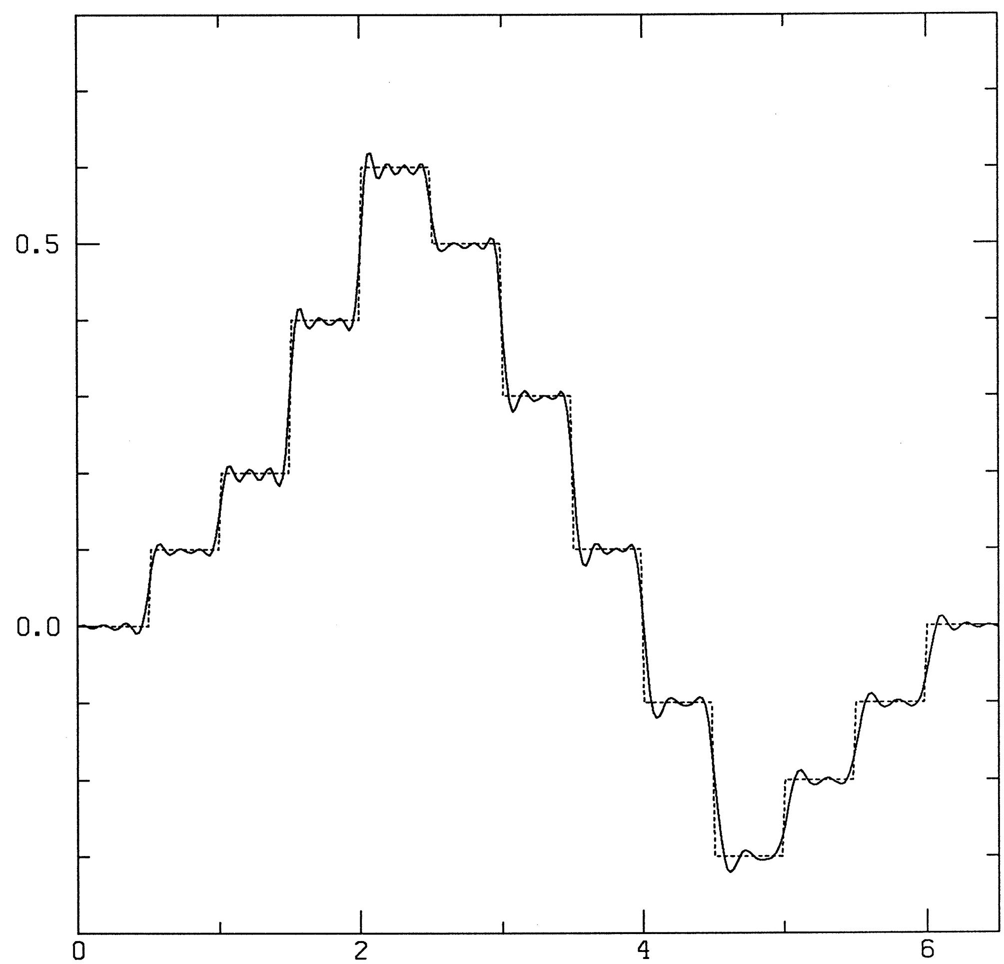

Figure 4.2. Reconstruction of Example 4.2 with a=20

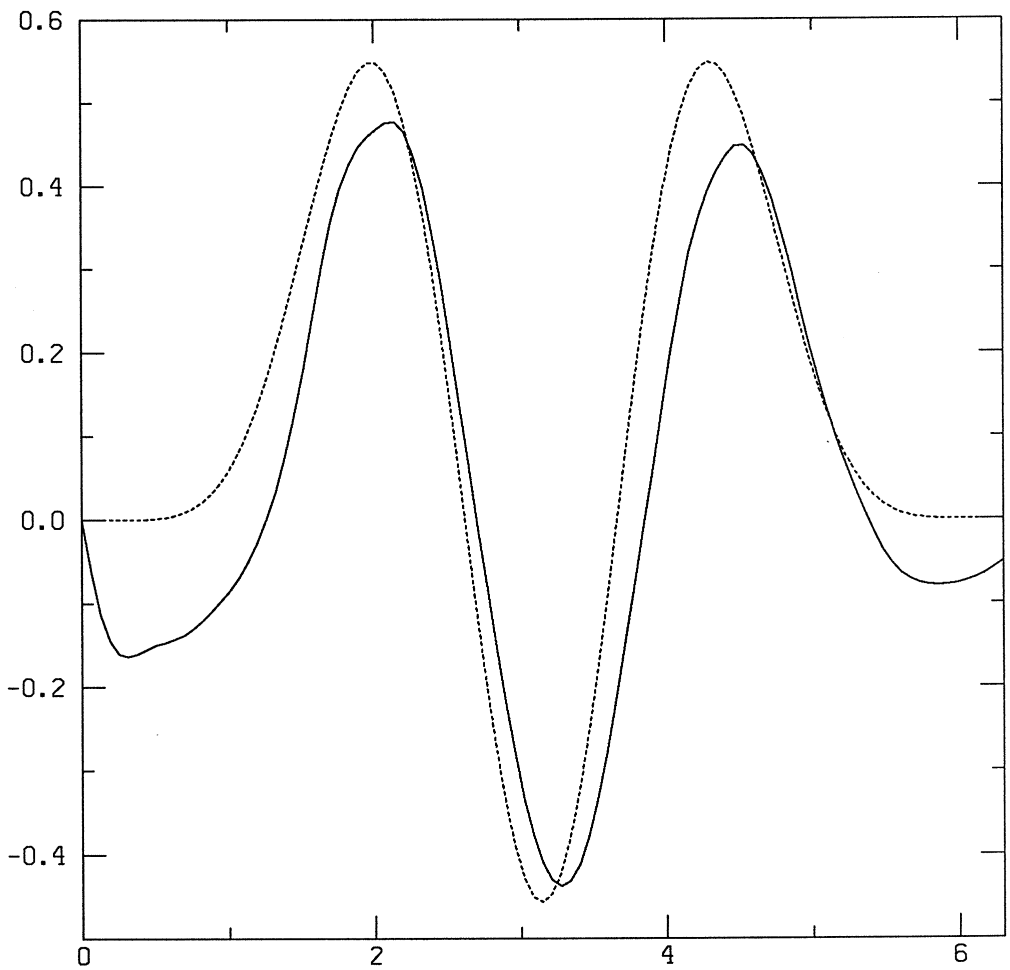

Figure 5.1. Reconstruction of Example 2.1 with D=1

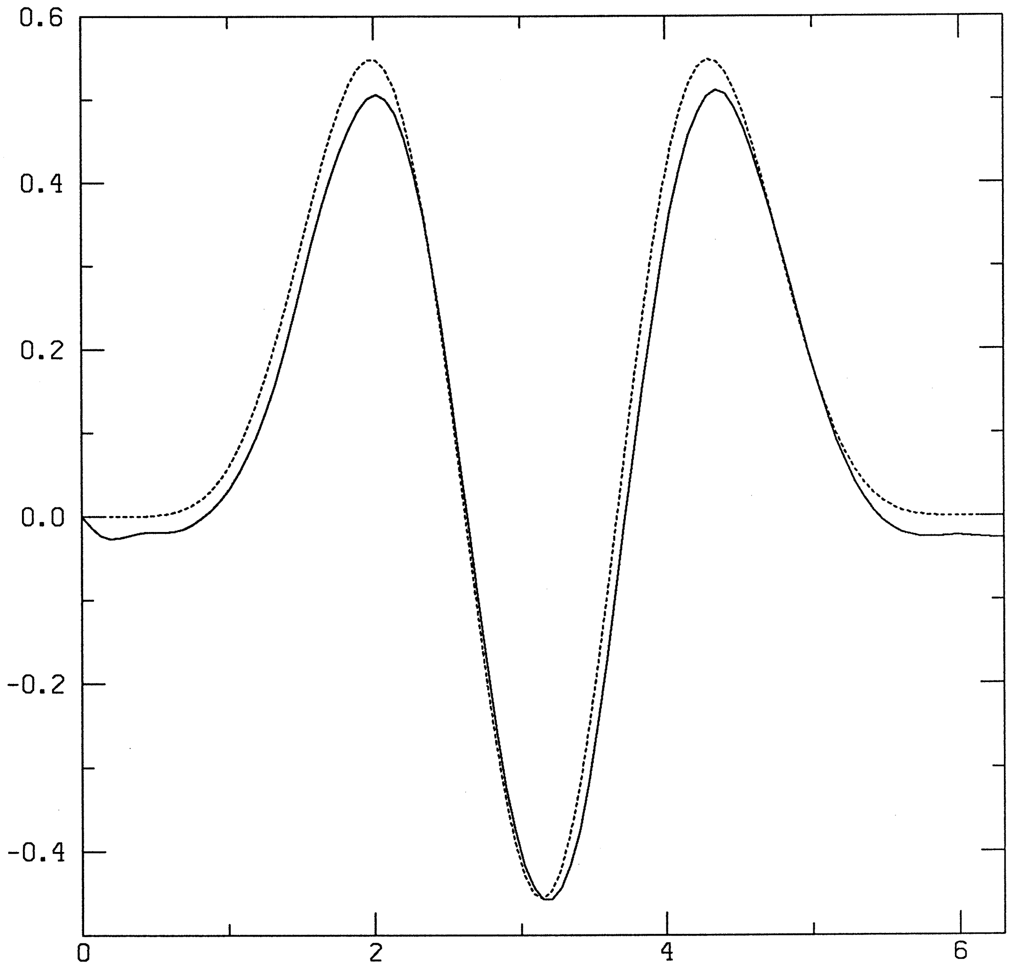

Figure 5.2. Reconstruction of Example 2.1 with D=2

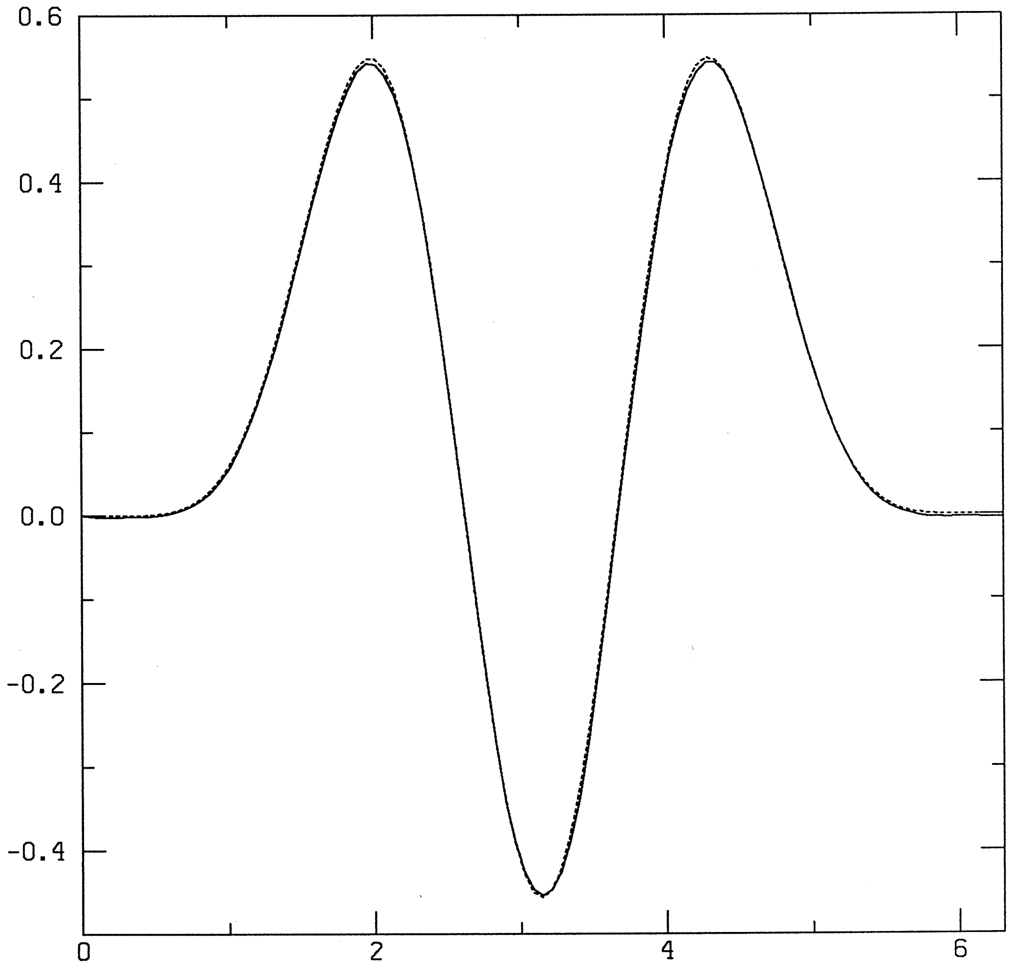

Figure 5.3. Reconstruction of Example 2.1 with D=3

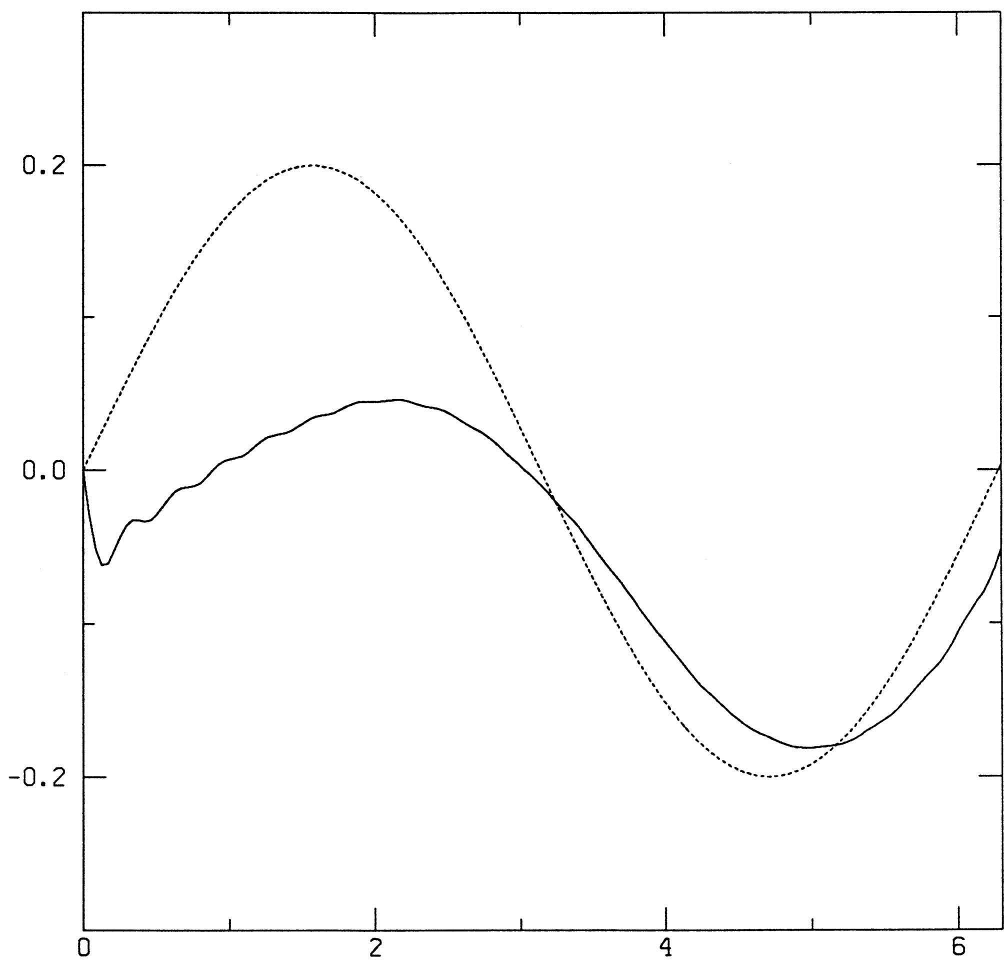

Figure 6.1. Reconstruction of Example 3 with D=1

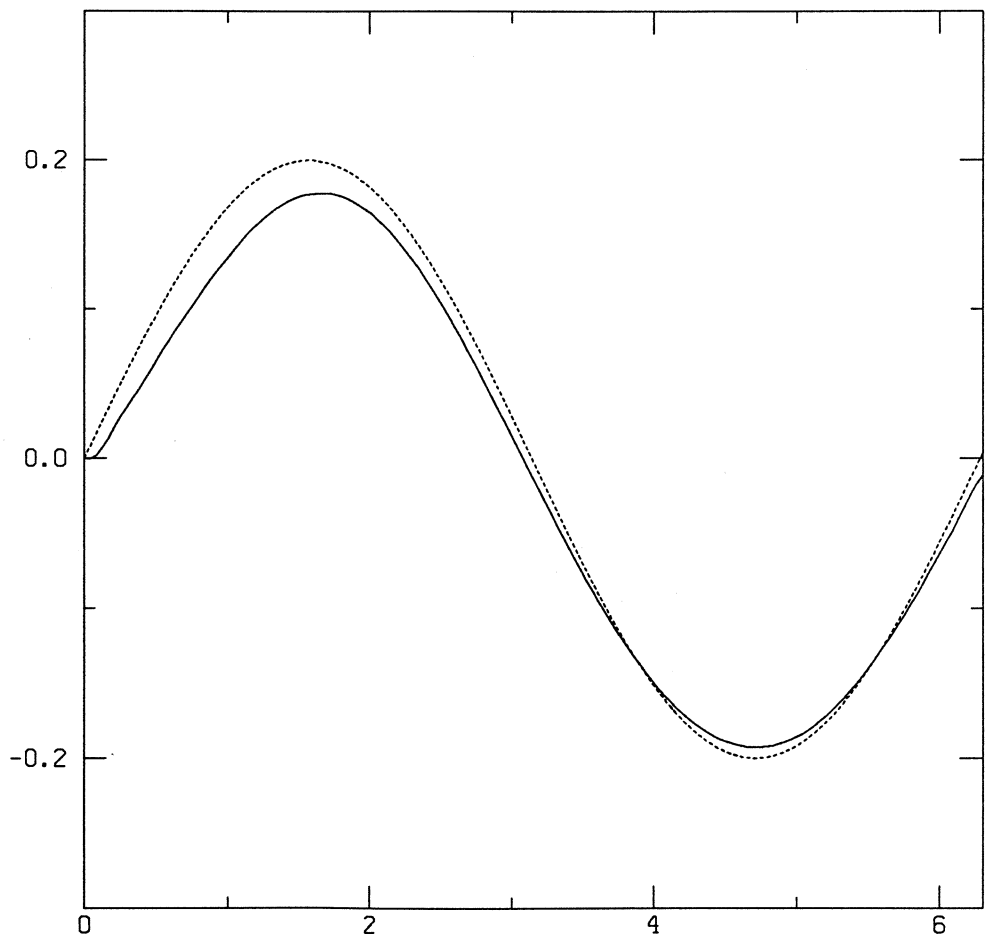

Figure 6.2. Reconstruction of Example 3 with D=2