Abstract

We study the effects of component-wise relative perturbations of a symmetric matrix on its eigenvalues and of a general matrix on its singular values. We characterize a class of matrices whose eigenvalues or singular values incur small relative changes under such perturbations. Up to a small constant factor, our results are optimal and agree with those of Barlow and Demmel for the eigenvalues of a scaled diagonally dominant matrix, and those of Demmel and Veselič for the eigenvalues of a symmetric positive definite matrix. Our results are asymptotically optimal and are more general than those of Demmel and Veselič for the singular values.

Relative Perturbation Theory for Eigenproblems †

Ming Gu and Stanley C. Eisenstat

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1. Introduction

Let H and $H + \delta H$ be real symmetric matrices with λ_i and λ'_i being the i-th eigenvalue of H and $H + \delta H$, respectively. The standard perturbation result [3] is

$$|\lambda'_i - \lambda_i| \leq \|\delta H\|_2 \quad . \tag{1.1}$$

This bound is attainable in that for any given H, for any of its eigenvalues, and for any $\eta > 0$, there exists a perturbation δH with $\|\delta H\|_2 = \eta$ such that equality holds.

Inequality (1.1) gives an upper bound on the *absolute* perturbation of each eigenvalue of H under the perturbation δH. To measure the *relative* perturbation of each eigenvalue, (1.1) can be rewritten as

$$\frac{|\lambda'_i - \lambda_i|}{|\lambda_i|} \leq \frac{\|\delta H\|_2}{|\lambda_i|} \quad . \tag{1.2}$$

This inequality is attainable in the same sense as (1.1). It predicts that the relative perturbation is small for large eigenvalues but can be large for small eigenvalues.

However, inequality (1.2) can be quite conservative since it relates the *relative* perturbation in λ_i to the *absolute* perturbation in H. If we restrict δH to be a component-wise relative perturbation of H, then (1.2) is not necessarily attainable. Indeed, when elements of H have very different magnitudes, (1.2) can be a severe over-estimate of the relative perturbation for small eigenvalues.

For a general symmetric matrix H, if we assume that $H = DAD$ and $\delta H = D(\delta A)D$ for some diagonal scaling matrix D and symmetric matrices A and δA, then a component-wise relative perturbation of H is a component-wise relative perturbation of A. Thus we are interested in an error bound of the form

$$\frac{|\lambda'_i - \lambda_i|}{|\lambda_i|} \leq \kappa \|\delta A\|_2 \quad , \tag{1.3}$$

where κ is independent of δA. κ can be chosen to be roughly $\|A^{-1}\|_2$ when A is diagonally dominant or positive definite [1, 2]. In these two cases (1.3) can be significantly better than (1.2) when D is badly scaled.

In this paper we establish error bounds similar to (1.3). In a well-defined sense, our results are optimal up to a small constant factor. In the case where A is either diagonally dominant or positive definite, our bounds agree with those in [1, 2] within a factor close to 2. We also generalize the relative perturbation theory for eigenvectors in [1, 2] to a general symmetric matrix[1].

We also consider the relative perturbation theory for the singular values of a general real matrix. Given $G = DBF$, where D and F are diagonal scaling matrices, we consider the relative perturbation of the singular values of G under the perturbation $\delta G = D(\delta B)F$. In

[1] Veselić and Slapničar [5] have established error bounds similar to (1.3) using a very different approach. In the case where A is either diagonally dominant or positive definite, these bounds also agree with those in [1, 2] within a factor related to the matrix size.

a well-defined sense, our results are again optimal up to a small constant factor. In the case where F is the identity matrix and G is fat, our results agree with those in [2] asymptotically.

Section 2 introduces some notation and preliminaries; Section 3 presents our relative perturbation results for eigenvalues and eigenvectors; Section 4 presents our relative perturbation results for singular values.

2. Notation and Preliminaries

A, H and W are symmetric matrices. B, G and K are general matrices. D, F, J and S are diagonal matrices. I is the identity matrix. u, v, x, y and z are vectors.

$\aleph$ and $\Re$ are index sets, i.e., subsets of the set of positive integers. $\emptyset$ is the empty index set. $|\aleph|$ denotes the size of $\aleph$. $A_\aleph$ denotes the principle submatrix of A such that the (i,j) element of A is an element of $A_\aleph$ if and only if $i,j \in \aleph$.

$\lambda_{\min}(A)$ denotes the smallest eigenvalue of A in magnitude. Define

$$\kappa_\aleph(A) = \min\{|\lambda_{\min}(A_\Re)| : \Re \supseteq \aleph \text{ and } \Re \neq \emptyset.\}$$

In particular, $\kappa_\emptyset(A)$ is the absolute value of the smallest eigenvalue in magnitude among all principle submatrices of A; and $\kappa_\emptyset(A) \leq \kappa_\aleph(A)$ for any $\aleph$.

LEMMA 1. *Let δA be a symmetric perturbation of A, and let $\aleph$ be an index set. Then*

$$\kappa_\aleph(A+\delta A) \geq \kappa_\aleph(A) - \|\delta A\|_2 \quad .$$

Proof: By definition, there exists a non-empty index set $\Re \supseteq \aleph$ such that $\kappa_\aleph(A+\delta A) = |\lambda_{\min}(A_\Re + \delta A_\Re)|$. Using (1.1), we have

$$\kappa_\aleph(A+\delta A) = |\lambda_{\min}(A_\Re + \delta A_\Re)| \geq |\lambda_{\min}(A_\Re)| - \|\delta A_\Re\|_2 \geq \kappa_\aleph(A) - \|\delta A\|_2 \quad .$$

∎

We say that an $m \times n$ matrix B is *fat* if $m \leq n$ and *skinny* if $m \geq n$. $\sigma_{\min}(B)$ denotes the smallest singular value of B. Define

$$\chi_1(B) = \min\{\sigma_{\min}(B_1) : B_1 \text{ is a square submatrix of } B\}$$

and

$$\chi_2(B) = \begin{cases} \sigma_{\min}(B) & \text{if } m \leq n \ , \\ \min\{\sigma_{\min}(B_2) : B_2 \text{ is an } n \times n \text{ submatrix of } B\} & \text{if } m > n \ . \end{cases}$$

In particular, $\chi_1(B) \leq \sigma_{\min}(B_1)$ for any submatrix B_1 of B; and $\chi_2(B) \leq \sigma_{\min}(B_2)$ for any submatrix B_2 of B with column dimension n; and $\chi_1(B) \leq \chi_2(B)$. When B is a square matrix, $\chi_2(B) = \sigma_{\min}(B)$. The following lemma is similar to Lemma 1.

LEMMA 2. *Let δB be a perturbation of B. Then*

$$\chi_1(B+\delta B) \geq \chi_1(B) - \|\delta B\|_2 \quad \textit{and} \quad \chi_2(B+\delta B) \geq \chi_2(B) - \|\delta B\|_2 \quad .$$

3. Relative Perturbation Theory for Symmetric Eigenproblems

3.1. Relative Perturbation Theory for Eigenvalues

The following theorem is the basis for the results that follow. The case where A is diagonally dominant is proved in [1].

THEOREM 3. *Let $H = DAD$, and let $\delta H = \eta DED$ be a symmetric perturbation of H with $\eta > 0$ and $\|E\|_2 = 1$. Assume that $\lambda(\xi) \neq 0$ is the i-th eigenvalue of $H(\xi) = H + \xi DED$ with unit eigenvector $x(\xi)$ for $0 \leq \xi \leq \eta$. If there exists a continuous function $f(\xi)$ such that*

$$\|Dx(\xi)\|_2^2 \leq f(\xi)\ |\lambda(\xi)|$$

for $0 \leq \xi \leq \eta$, then

$$\exp\left(-\int_0^\eta f(\xi)\,d\xi\right) \leq \frac{\lambda(\eta)}{\lambda(0)} \leq \exp\left(\int_0^\eta f(\xi)\,d\xi\right) \quad . \tag{3.1}$$

Proof: Suppose first that $\lambda(\xi)$ is simple for $0 \leq \xi \leq \eta$. Then $\lambda(\xi)$ is analytic, and it follows from standard perturbation theory [3] that

$$\frac{d}{d\xi}\lambda(\xi) = x(\xi)^T DEDx(\xi) \quad .$$

Thus

$$\left|\frac{1}{\lambda(\xi)}\frac{d}{d\xi}\lambda(\xi)\right| = \left|\frac{x(\xi)^T DEDx(\xi)}{\lambda(\xi)}\right| \leq f(\xi)$$

for $0 < \xi < \eta$, which implies that

$$\left|\frac{d}{d\xi}\log|\lambda(\xi)|\right| \leq f(\xi) \quad .$$

Integrating ξ over the interval $[0, \eta]$ yields (3.1). A continuation argument similar to that in [1] shows that (3.1) is still valid even when $\lambda(\xi)$ is a multiple eigenvalue. ∎

To estimate the function $f(\xi)$ in Theorem 3, we use the following lemma.

LEMMA 4. *Let W be a symmetric matrix and let $S = \mathrm{diag}(s_1, \ldots, s_n)$ be a positive diagonal matrix. Assume that a unit vector z and a positive scalar λ satisfy the relation $Wz = \lambda Sz$. Also assume that there exists an index set $\aleph$ such that $\|S_\aleph\|_2 \leq 1$. Then we have*

$$z^T W z \geq \frac{1}{2}(\kappa_\aleph(W) - (\sqrt{2}+1)\lambda) \quad ; \tag{3.2}$$

and in the special case where $\aleph = \emptyset$*, we have*[2]

$$z^T W z \geq \frac{1}{2}\kappa_\emptyset(W) \quad . \tag{3.3}$$

Proof: We assume without loss of generality that $W_\aleph$ is the $|\aleph| \times |\aleph|$ upper left block of W. We consider the problem of finding the infimum ϕ of $z^T W z$ over z and $\{s_j\}_{j>|\aleph|}$, under the constraints

$$z^T z = 1 \ , \quad Wz = \lambda S z \quad \text{and} \quad s_j \geq 0 \quad \text{for} \quad j > |\aleph| \quad , \tag{3.4}$$

where λ and $\{s_j\}_{j\leq|\aleph|}$ are fixed parameters with $0 < s_j \leq 1$. The infimum ϕ exists because $z^T W z = \lambda z^T S z \geq 0$.

Let $\{(z^{(t)}, S^{(t)})\}$ be a sequence satisfying (3.4) such that

$$\lim_{t\to\infty} z^{(t)^T} W z^{(t)} = \phi \quad .$$

Let $\bar{z} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_n)^T$ be a limit point of $\{z^{(t)}\}$. Then $\bar{z}^T W \bar{z} = \phi$. For any $j > |\aleph|$, if $\bar{\zeta}_j$ is non-zero, then the j-th diagonal element of $S^{(t)}$ converges to a finite number $(W\bar{z})_j/(\lambda\bar{\zeta}_j)$; and if $\bar{\zeta}_j$ is zero, then we can reduce the size of the problem by dropping the j-th column and row of W and the j-th equation in the constraint $W\bar{z} = \lambda S\bar{z}$. When the problem size is reduced, the value of $\bar{z}^T W \bar{z}$ does not change; neither do the remaining constraints. The infimum of the reduced problem is no larger than ϕ.

By repeating this reduction as needed, we can bound ϕ below by the solution of the following minimization problem over $\tilde{z}$ and $\widetilde{S}_2$, which must achieve its infimum at a finite point:

$$\text{minimize} \qquad \tilde{z}^T \widetilde{W} \tilde{z} \quad \text{subject to} \quad \widetilde{W}\tilde{z} = \lambda \widetilde{S}\tilde{z}, \ \tilde{z}^T\tilde{z} = 1, \ \widetilde{S}_2 > 0 \quad ,$$

where

$$\widetilde{W} = \begin{pmatrix} \widetilde{W}_{00} & \widetilde{W}_{01} & \widetilde{W}_{02} \\ \widetilde{W}_{10} & \widetilde{W}_{11} & \widetilde{W}_{12} \\ \widetilde{W}_{20} & \widetilde{W}_{21} & \widetilde{W}_{22} \end{pmatrix} \quad , \quad \widetilde{S} = \begin{pmatrix} \widetilde{S}_0 & & \\ & 0 & \\ & & \widetilde{S}_2 \end{pmatrix} \quad , \quad \text{and} \quad \tilde{z} = \begin{pmatrix} \tilde{z}_0 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \quad ;$$

$\widetilde{W}_{00}$ and $\widetilde{S}_0$ are the upper-left $|\aleph| \times |\aleph|$ blocks of W and S, respectively; $\widetilde{W}$ and $\widetilde{S}$ are principle submatrices of W and S, respectively, up to a symmetric permutation (this permutation is chosen such that no component of $\widetilde{S}_2$ is zero); and $\tilde{z}$ is a subvector of z up to the same permutation (no component of $\tilde{z}_1$ or $\tilde{z}_2$ is zero).

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[2] When W is positive definite, $\kappa_\emptyset(W) = \lambda_{\min}(W)$ and inequality (3.3) can be improved to

$$z^T W z \geq \kappa_\emptyset(W) \quad .$$

The minimum is a stationary point of the Lagrangian

$$\mathcal{L}(\tilde{z},\widetilde{S}) = \frac{1}{2}\tilde{z}^T\widetilde{W}\tilde{z} + v^T(\widetilde{W}\tilde{z} - \lambda\widetilde{S}\tilde{z}) - \frac{1}{2}\mu(\tilde{z}^T\tilde{z} - 1)$$

with $v = (v_0^T, v_1^T, v_2^T)^T$ and μ being Lagrange multipliers. The only term in $\mathcal{L}(\tilde{z},\widetilde{S})$ involving $\widetilde{S}_2$ is $v_2^T\widetilde{S}_2\tilde{z}_2$. Since $\partial\mathcal{L}/\partial\widetilde{S}_2 = 0$ and each component of $\tilde{z}_2$ is non-zero at the minimum, we must have $v_2 = 0$. On the other hand, since $\partial\mathcal{L}/\partial\tilde{z} = 0$, we must have

$$\widetilde{W}(\tilde{z} + v) - \lambda\widetilde{S}v = \mu\tilde{z} \quad .$$

Putting together these equations and those in the constraints, and disregarding equations that involve $\widetilde{S}_2$, we have

$$\begin{pmatrix} \widetilde{W}_{00} & \widetilde{W}_{01} & \widetilde{W}_{02} \\ \widetilde{W}_{10} & \widetilde{W}_{11} & \widetilde{W}_{12} \\ \widetilde{W}_{20} & \widetilde{W}_{21} & \widetilde{W}_{22} \end{pmatrix} \begin{pmatrix} \tilde{z}_0 + v_0 \\ \tilde{z}_1 + v_1 \\ \tilde{z}_2 \end{pmatrix} - \lambda \begin{pmatrix} \widetilde{S}_0 v_0 \\ 0 \\ 0 \end{pmatrix} = \mu \begin{pmatrix} \tilde{z}_0 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \quad ,$$

and

$$\begin{pmatrix} \widetilde{W}_{00} & \widetilde{W}_{01} & \widetilde{W}_{02} \\ \widetilde{W}_{10} & \widetilde{W}_{11} & \widetilde{W}_{12} \end{pmatrix} \begin{pmatrix} \tilde{z}_0 \\ \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = \lambda \begin{pmatrix} \widetilde{S}_0\tilde{z}_0 \\ 0 \end{pmatrix} \quad .$$

These equations can be rewritten as

$$C(\aleph) \begin{pmatrix} \tilde{z}_0 + v_0 \\ \tilde{z}_1 + v_1 \\ \tilde{z}_2 \\ v_0 \\ v_1 \end{pmatrix} = \mu \begin{pmatrix} \tilde{z}_0 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_0 \\ \tilde{z}_1 \end{pmatrix} \quad , \tag{3.5}$$

where

$$C(\aleph) = \begin{pmatrix} \widetilde{W}_{00} & \widetilde{W}_{01} & \widetilde{W}_{02} & -\lambda\widetilde{S}_0 & \\ \widetilde{W}_{10} & \widetilde{W}_{11} & \widetilde{W}_{12} & & \\ \widetilde{W}_{20} & \widetilde{W}_{21} & \widetilde{W}_{22} & & \\ \lambda\widetilde{S}_0 & & & \widetilde{W}_{00} - 2\lambda\widetilde{S}_0 & \widetilde{W}_{01} \\ & & & \widetilde{W}_{10} & \widetilde{W}_{11} \end{pmatrix} \quad .$$

We also have

$$\mu = \mu\tilde{z}^T\tilde{z} = \tilde{z}^T(\widetilde{W}(\tilde{z} + v) - \lambda\widetilde{S}v) = \tilde{z}^T\widetilde{W}\tilde{z} + v^T(\widetilde{W}\tilde{z} - \lambda\widetilde{S}\tilde{z}) = \tilde{z}^T\widetilde{W}\tilde{z} \geq 0 \quad ,$$

and so μ is a lower bound on ϕ.

Taking norms on both sides of (3.5), we have

$$\sigma_{\min}(C(\aleph)) \left\| \begin{pmatrix} \tilde{z}_0 + v_0 \\ \tilde{z}_1 + v_1 \\ \tilde{z}_2 \\ v_0 \\ v_1 \end{pmatrix} \right\|_2 \leq \mu \left\| \begin{pmatrix} \tilde{z}_0 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_0 \\ \tilde{z}_1 \end{pmatrix} \right\|_2 . \tag{3.6}$$

On the other hand, since $\tilde{z}$ is a unit vector,

$$\left\| \begin{pmatrix} \tilde{z}_0 + v_0 \\ \tilde{z}_1 + v_1 \\ \tilde{z}_2 \\ v_0 \\ v_1 \end{pmatrix} \right\|_2 \geq 1/\sqrt{2} \quad \text{and} \quad \left\| \begin{pmatrix} \tilde{z}_0 \\ \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_0 \\ \tilde{z}_1 \end{pmatrix} \right\|_2 \leq \sqrt{2} .$$

Combining these relations,

$$\frac{1}{2}\sigma_{\min}(C(\aleph)) \leq \mu . \tag{3.7}$$

To get (3.2), we note that

$$C(\aleph) = \begin{pmatrix} \widetilde{W}_{00} & \widetilde{W}_{01} & \widetilde{W}_{02} & & \\ \widetilde{W}_{10} & \widetilde{W}_{11} & \widetilde{W}_{12} & & \\ \widetilde{W}_{20} & \widetilde{W}_{21} & \widetilde{W}_{22} & & \\ & & & \widetilde{W}_{00} & \widetilde{W}_{01} \\ & & & \widetilde{W}_{10} & \widetilde{W}_{11} \end{pmatrix} + \begin{pmatrix} 0 & & & -\lambda\widetilde{S}_0 & \\ & 0 & & & \\ & & 0 & & \\ \lambda\widetilde{S}_0 & & & -2\lambda\widetilde{S}_0 & \\ & & & & 0 \end{pmatrix} .$$

On the right-hand side, the smallest singular value of the first matrix is no smaller than $\kappa_\aleph(W)$, and the norm of the second matrix is no larger than $(\sqrt{2}+1)\lambda$. Thus

$$\sigma_{\min}(C(\aleph)) \geq \kappa_\aleph(W) - (\sqrt{2}+1)\lambda .$$

Plugging this relation into (3.7) we have

$$\frac{1}{2}(\kappa_\aleph(W) - (\sqrt{2}+1)\lambda) \leq \mu .$$

Since μ is a lower bound on ϕ, this relation implies (3.2).

To prove (3.3), we set $\aleph = \emptyset$ in (3.5). Then all the entries in $C(\aleph)$ with subscript 0 disappear, and

$$C(\emptyset) = \begin{pmatrix} \widetilde{W}_{11} & \widetilde{W}_{12} & \\ \widetilde{W}_{21} & \widetilde{W}_{22} & \\ & & \widetilde{W}_{11} \end{pmatrix} ,$$

whence $\sigma_{\min}(C(\emptyset)) \geq \kappa_\emptyset(W)$. Thus (3.7) simplifies to

$$\frac{1}{2}\kappa_\emptyset(W) \leq \mu ,$$

which implies (3.3). ∎

Example 1 illustrates the tightness of (3.3).

EXAMPLE 1. Let W be any symmetric matrix with $\kappa_\emptyset(W) > 0$. Then, up to a symmetric permutation,

$$W = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \quad ,$$

where $|\lambda_{\min}(W_{22})| = \kappa_\emptyset(W)$. Let $H = DWD$, where $D = diag(dI_1, I_2)$ is blocked conformally with W.

Assume that $\lambda_{\min}(W_{22}) > 0$; if $\lambda_{\min}(W_{22}) < 0$ then we consider the matrix $-W$. Then there exists a unit vector x_2 such that $W_{22}x_2 = \kappa_\emptyset(W)x_2$. According to [4], H has an eigenvalue ω satisfying

$$|\omega - \kappa_\emptyset(W)| \le \left\| H \begin{pmatrix} 0 \\ x_2 \end{pmatrix} - \kappa_\emptyset(W) \begin{pmatrix} 0 \\ x_2 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} dW_{12}x_2 \\ 0 \end{pmatrix} \right\|_2 \le d\|W\|_2 \quad .$$

Thus for $d > 0$ sufficiently small, we have

$$\omega \ge \kappa_\emptyset(W) - d\|W\|_2 > 0 \quad .$$

For any unit vector $y = (y_1^T, y_2^T)^T$ satisfying $Wy = \omega D^{-2}y$, we have

$$W_{11}y_1 + W_{12}y_2 = \frac{\omega}{d^2}y_1 \quad ,$$

whence

$$\|y_1\|_2 = \frac{d^2}{\omega} \left\| (W_{11} \quad W_{12}) \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \right\|_2 \le \frac{\|W\|_2 d^2}{\omega} \quad .$$

Thus

$$y^T W y = \omega y^T D^{-2} y \le \omega \left(1 + \left(\frac{\|W\|_2 d}{\omega} \right)^2 \right) \quad .$$

As d goes to zero, ω goes to $\kappa_\emptyset(W)$ as does the upper bound for y^TWy. Thus equality in (3.3) can be achieved for some y, λ and $K = D^{-2}$ up to a factor close to 2.

COROLLARY 5. *Let $H = DAD$ with $\kappa_\emptyset(A) > 0$. Let $\delta H = D(\delta A)D$ be a symmetric perturbation of H with $\|\delta A\|_2 \equiv \eta < \kappa_\emptyset(A)$. Let λ_i and λ_i' be the i-th eigenvalue of H and $H + \delta H$, respectively. Then*[3]

$$-\frac{\eta(2\kappa_\emptyset(A) - \eta)}{\kappa_\emptyset^2(A)} \le \frac{\lambda_i' - \lambda_i}{\lambda_i} \le \frac{\eta(2\kappa_\emptyset(A) - \eta)}{(\kappa_\emptyset(A) - \eta)^2} \quad . \tag{3.9}$$

[3] When A is positive definite (3.9) can be improved to

$$-\frac{\eta}{\kappa_\emptyset(A)} \le \frac{\lambda_i' - \lambda_i}{\lambda_i} \le \frac{\eta}{\kappa_\emptyset(A) - \eta} \quad . \tag{3.8}$$

Proof: Assume that $\lambda_i \geq 0$; if $\lambda_i < 0$, then we consider the matrix $-H$. Let $E = \delta A/\|\delta A\|_2$, let $H(\xi) = D(A+\xi E)D$, and let $\lambda_i(\xi)$ be the i-th eigenvalue of $H(\xi)$ with corresponding unit eigenvector $x_i(\xi)$ for $0 \leq \xi \leq \eta$. Then $\lambda_i(0) = \lambda_i$ and $\lambda_i(\eta) = \lambda_i'$. According to Lemma 1, we have $\kappa_\emptyset(A+\xi E) \geq \kappa_\emptyset(A) - \xi > 0$. It follows that $\lambda_i(\xi)$ is positive for $0 \leq \xi \leq \eta$. Applying (3.3) in Lemma 4 with $W = A + \xi E$, $S = D^{-2}$, $z = Dx_i(\xi)/\|Dx_i(\xi)\|_2$ and $\lambda_i(\xi)$, we have

$$\frac{1}{2}\kappa_\emptyset(A+\xi E) \leq \frac{\lambda_i(\xi)}{\|Dx_i(\xi)\|_2^2} \quad ,$$

or

$$\|Dx_i(\xi)\|_2^2 \leq \frac{2}{\kappa_\emptyset(A+\xi E)}\,|\lambda_i(\xi)| \leq \frac{2}{\kappa_\emptyset(A)-\xi}\,|\lambda_i(\xi)| \quad ,$$

Theorem 3 then gives

$$\left(\frac{\kappa_\emptyset(A)-\eta}{\kappa_\emptyset(A)}\right)^2 \leq \frac{\lambda_i'}{\lambda_i} \leq \left(\frac{\kappa_\emptyset(A)}{\kappa_\emptyset(A)-\eta}\right)^2 \quad ,$$

which is equivalent to (3.9). ∎

Inequality (3.9) gives lower and upper bounds on the ratio $(\lambda_i' - \lambda_i)/\lambda_i$ that are independent of D. Because of the tightness of (3.3), the bounds in (3.9) are optimal up to a factor close to 2 for η sufficiently small. Moreover, inequality (3.9) can be rewritten as

$$\left|\frac{\lambda_i'-\lambda_i}{\lambda_i}\right| \leq \frac{2\eta}{\kappa_\emptyset(A)} + O\left(\left(\frac{\eta}{\kappa_\emptyset(A)}\right)^2\right) \quad .$$

EXAMPLE 2. Let A be positive definite. Demmel and Veselič [2] show that

$$\left|\frac{\lambda_i'-\lambda_i}{\lambda_i}\right| \leq \frac{\eta}{\lambda_{\min}(A)} \quad .$$

Since $\kappa_\emptyset(A) = \lambda_{\min}(A)$, inequality (3.8) agrees with this result asymptotically.

EXAMPLE 3. Let $A = \Delta + N$, where Δ is diagonal with elements ± 1 and N has a zero diagonal. Assume that A is diagonally dominant with respect to the 1-norm, 2-norm or ∞-norm; that is

$$\|N\|_p \leq \gamma < 1 \quad ,$$

where p is one of $1, 2$ and ∞. Barlow and Demmel [1] show that

$$\left|\frac{\lambda_i'-\lambda_i}{\lambda_i}\right| \leq \frac{\eta}{1-\gamma} \quad .$$

Since $\kappa_\emptyset(A) \geq 1-\gamma$, inequality (3.9) agrees with this result up to a factor close to 2 for η sufficiently small.

We also have the following result.

COROLLARY 6. *Let $H = DAD$ with $\|D_{\aleph}^{-1}\|_2 \leq 1$ and $\kappa_{\aleph}(A) > 0$ for some $\aleph$. Let $\delta H = D(\delta A)D$ be a symmetric perturbation of H with $0 < \|\delta A\|_2 \equiv \eta < \kappa_{\aleph}(A)/10$. Let λ_i and λ_i' be the i-th eigenvalue of H and $H + \delta H$, respectively. If $|\lambda_i| \leq \kappa_{\aleph}(A)/10$, then*

$$\left|\frac{\lambda_i' - \lambda_i}{\lambda_i}\right| \leq \frac{10\eta}{\kappa_{\aleph}(A)} \quad . \tag{3.10}$$

Proof: Assume that $\lambda_i \geq 0$; if $\lambda_i < 0$, then we consider the matrix $-H$. Let $E = \delta A/\|\delta A\|_2$, let $H(\xi) = D(A+\xi E)D$, and let $\lambda_i(\xi)$ be the i-th eigenvalue of $H(\xi)$ with corresponding unit eigenvector $x_i(\xi)$ for $0 \leq \xi \leq \eta$. Then $\lambda_i(0) = \lambda_i$ and $\lambda_i(\eta) = \lambda_i'$. According to Lemma 1, we have $\kappa_{\aleph}(A + \xi E) \geq \kappa_{\aleph}(A) - \xi > 0$. It follows that $\lambda_i(\xi)$ is positive for $0 \leq \xi \leq \eta$. As in the proof of Theorem 3, we only consider the case where $\lambda_i(\xi)$ is simple for $0 \leq \xi \leq \eta$. Applying (3.2) in Lemma 4 with $W = A + \xi E$, $S = D^{-2}$, $z = Dx_i(\xi)/\|Dx_i(\xi)\|_2$, $\lambda_i(\xi)$ and index set $\aleph$, we have

$$\frac{1}{2}(\kappa_{\aleph}(A + \xi E) - (\sqrt{2}+1)\lambda_i(\xi)) \leq \frac{\lambda_i(\xi)}{\|Dx_i(\xi)\|_2^2} \quad ,$$

or

$$\|Dx_i(\xi)\|_2^2 \leq \frac{2}{\kappa_{\aleph}(A + \xi E) - (\sqrt{2}+1)\lambda_i(\xi)} \; |\lambda_i(\xi)| \quad .$$

Thus as in the proof of Theorem 3, we have

$$\left|\frac{d}{d\xi}\,\lambda_i(\xi)\right| \leq \frac{2\lambda_i(\xi)}{\kappa_{\aleph}(A + \xi E) - (\sqrt{2}+1)\lambda_i(\xi)} \quad . \tag{3.11}$$

In the following we prove (3.10) by contradiction. Assume that there exists $0 < \xi_0 < \eta$ such that

$$\lambda_i(\xi) < \left(1 + \frac{10\eta}{\kappa_{\aleph}(A)}\right)\lambda_i(0) \quad , \quad 0 \leq \xi < \xi_0 \tag{3.12}$$

and

$$\lambda_i(\xi_0) = \left(1 + \frac{10\eta}{\kappa_{\aleph}(A)}\right)\lambda_i(0) \quad . \tag{3.13}$$

Recall that $\lambda_i(0) < \kappa_{\aleph}(A)/10$ and $\eta < \kappa_{\aleph}(A)/10$. Then for $0 \leq \xi < \xi_0$,

$$\begin{aligned}
&\kappa_{\aleph}(A + \xi E) - (\sqrt{2}+1)\lambda_i(\xi) \\
&\qquad \geq \kappa_{\aleph}(A) - \xi - (\sqrt{2}+1)\left(1 + \frac{10\eta}{\kappa_{\aleph}(A)}\right)\lambda_i(0) \\
&\qquad \geq \kappa_{\aleph}(A) - \kappa_{\aleph}(A)/10 - (\sqrt{2}+1)\cdot 2 \cdot \kappa_{\aleph}(A)/10 \\
&\qquad \geq \frac{2}{5}\kappa_{\aleph}(A) \quad .
\end{aligned}$$

Thus (3.11) and (3.12) give

$$\begin{aligned}\left|\frac{d}{d\xi}\,\lambda_i(\xi)\right| &\le \frac{2\lambda_i(\xi)}{\kappa_{\aleph}(A+\xi E)-(\sqrt{2}+1)\lambda_i(\xi)}\\ &< \frac{5}{\kappa_{\aleph}(A)}\cdot\left(1+\frac{10\eta}{\kappa_{\aleph}(A)}\right)\lambda_i(0)\\ &\le \frac{5}{\kappa_{\aleph}(A)}\cdot 2\cdot\lambda_i(0)\\ &= \frac{10}{\kappa_{\aleph}(A)}\,\lambda_i(0)\end{aligned}$$

for $0\le\xi<\xi_0$, which contradicts (3.13). Thus we must have

$$\lambda_i(\xi)\le\left(1+\frac{10\eta}{\kappa_{\aleph}(A)}\right)\lambda_i(0)\quad,\qquad 0\le\xi<\eta\quad,$$

which is equivalent to the upper bound in (3.10). A similar proof gives the lower bound. ■

EXAMPLE 4. Consider the matrix $H = DAD$ with

$$A=\begin{pmatrix}0&1&0\\1&1&1\\0&1&1\end{pmatrix}\quad,$$

and $D = diag(1,1,10^{-5})$. Let $\aleph = \{1,2\}$. Then $\kappa_{\aleph}(A) = |\lambda_{\min}(A)| > 0.55$. H has a tiny eigenvalue of the order $O(10^{-10})$. A small component-wise relative perturbation of H only causes a small relative perturbation of this eigenvalue.

3.2. Perturbation Theory for Eigenvectors

Following [1, 2], we define the *relative gap* of the i-th eigenvalue of H as

$$relgap_H(\lambda_i)=\min_{j\ne i}\frac{|\lambda_j-\lambda_i|}{|\lambda_j\lambda_i|^{\frac{1}{2}}}\quad.$$

The following is a generalization of the results in [1, 2] which deal with the special case where A is diagonally dominant or positive definite.

THEOREM 7. *Let $H = DAD$ with $\kappa_{\emptyset}(A) > 0$. Let $\delta H = D(\delta A)D$ be a symmetric perturbation of H with $\|\delta A\|_2 \equiv \eta > 0$. Let λ_i and λ_i' be the i-th eigenvalue of H and $H+\delta H$, with corresponding unit eigenvector x_i and x_i', respectively. Assume that $relgap_H(\lambda_i) > 0$ and that*

$$\eta\le\frac{\kappa_{\emptyset}(A)\ relgap_H(\lambda_i)}{8(1+relgap_H(\lambda_i))}\quad.$$

Then

$$\|x_i'-x_i\|_2\le\frac{5\eta\sqrt{n-1}}{\kappa_{\emptyset}(A)\ relgap_H(\lambda_i)}\quad. \tag{3.14}$$

Proof: Since $relgap_H(\lambda_i) > 0$, λ_i is a simple eigenvalue of H. Let $E = \delta A/\|\delta A\|_2$, let $H(\xi) = D(A+\xi E)D$, and let $\lambda_j(\xi)$ denote the j-th eigenvalue of $H(\xi)$ with corresponding unit eigenvector $x_j(\xi)$ for $0 \le \xi \le \eta$. Then $\lambda_i(0) = \lambda_i$, $x_i(0) = x_i$, $\lambda_i(\eta) = \lambda_i'$ and $x_i(\eta) = x_i'$.

To estimate $relgap_H(\lambda_i(\xi))$, we note that Corollary 5 implies that

$$\left|\frac{\lambda_j(\xi)-\lambda_j(0)}{\lambda_j(0)}\right| \le \frac{\xi(2\kappa_\emptyset(A)-\xi)}{(\kappa_\emptyset(A)-\xi)^2} \quad \text{and} \quad \left|\frac{\lambda_j(0)}{\lambda_j(\xi)}\right| \ge \left(\frac{\kappa_\emptyset(A)-\xi}{\kappa_\emptyset(A)}\right)^2$$

for each j. We also have

$$\frac{|\lambda_i(0)|+|\lambda_j(0)|}{|\lambda_i(0)\lambda_j(0)|^{\frac{1}{2}}} \le \frac{|\lambda_i(0)-\lambda_j(0)|}{|\lambda_i(0)\lambda_j(0)|^{\frac{1}{2}}} + 2 \quad .$$

Thus

$$\begin{aligned}
\frac{|\lambda_j(\xi)-\lambda_i(\xi)|}{|\lambda_j(\xi)\lambda_i(\xi)|^{\frac{1}{2}}} &= \frac{|(\lambda_j(0)-\lambda_i(0))+(\lambda_j(\xi)-\lambda_j(0))-(\lambda_i(\xi)-\lambda_i(0))|}{|\lambda_j(0)\lambda_i(0)|^{\frac{1}{2}}}\left|\frac{\lambda_j(0)\lambda_i(0)}{\lambda_j(\xi)\lambda_i(\xi)}\right|^{\frac{1}{2}} \\
&\ge \left(\frac{|\lambda_j(0)-\lambda_i(0)|}{|\lambda_j(0)\lambda_i(0)|^{\frac{1}{2}}} - \frac{\xi(2\kappa_\emptyset(A)-\xi)}{(\kappa_\emptyset(A)-\xi)^2}\frac{|\lambda_j(0)+|\lambda_i(0)|}{|\lambda_j(0)\lambda_i(0)|^{\frac{1}{2}}}\right)\left|\frac{\lambda_j(0)\lambda_i(0)}{\lambda_j(\xi)\lambda_i(\xi)}\right|^{\frac{1}{2}} \\
&\ge \left(\left(1-\frac{\xi(2\kappa_\emptyset(A)-\xi)}{(\kappa_\emptyset(A)-\xi)^2}\right)\frac{|\lambda_j(0)-\lambda_i(0)|}{|\lambda_j(0)\lambda_i(0)|^{\frac{1}{2}}} - \frac{2\xi(2\kappa_\emptyset(A)-\xi)}{(\kappa_\emptyset(A)-\xi)^2}\right)\left(\frac{\kappa_\emptyset(A)-\xi}{\kappa_\emptyset(A)}\right)^2 \\
&= \left(1-\frac{4\xi}{\kappa_\emptyset(A)}+2\left(\frac{\xi}{\kappa_\emptyset(A)}\right)^2\right)\frac{|\lambda_j(0)-\lambda_i(0)|}{|\lambda_j(0)\lambda_i(0)|^{\frac{1}{2}}} - \frac{2\xi}{\kappa_\emptyset(A)}\left(2-\frac{\xi}{\kappa_\emptyset(A)}\right) \\
&\ge \left(1-\frac{4\xi}{\kappa_\emptyset(A)}\right)\frac{|\lambda_j(0)-\lambda_i(0)|}{|\lambda_j(0)\lambda_i(0)|^{\frac{1}{2}}} - \frac{4\xi}{\kappa_\emptyset(A)} \quad ,
\end{aligned}$$

which implies that

$$relgap_H(\lambda_i(\xi)) \ge \left(1-\frac{4\xi}{\kappa_\emptyset(A)}\right) relgap_H(\lambda_i(0)) - \frac{4\xi}{\kappa_\emptyset(A)} \ge relgap_H(\lambda_i(0))/2 \tag{3.15}$$

for $0 \le \xi \le \eta$.

Equation (3.15) implies that $\lambda_i(\xi)$ is a simple eigenvalue of $H(\xi)$ for $0 \le \xi \le \eta$. From [3] we have

$$\frac{d}{d\xi}\, x_i(\xi) = \sum_{j\ne i}\frac{x_j^T(\xi)DEDx_i(\xi)}{\lambda_i(\xi)-\lambda_j(\xi)}\, x_j(\xi) \quad .$$

Using the Cauchy-Schwartz inequality, we have

$$\left\|\frac{d}{d\xi}\, x_i(\xi)\right\|_2 \le \sqrt{n-1}\,\max_{j\ne i}\frac{\|Dx_i(\xi)\|_2\|Dx_j(\xi)\|_2}{|\lambda_i(\xi)-\lambda_j(\xi)|} \quad . \tag{3.16}$$

Applying (3.3) of Lemma 4 with $W = A+\xi E$, $S = D^{-2}$, $z = Dx_k(\xi)/\|Dx_k(\xi)\|_2$ and $\lambda_k(\xi)$, we have

$$\|Dx_k(\xi)\|_2 \le \sqrt{2|\lambda_k(\xi)|/\kappa_\emptyset(A+\xi E)}$$

for each k. Plugging this relation into (3.16), we have

$$\begin{aligned}\left\|\frac{d}{d\xi}\, x_i(\xi)\right\|_2 &\le \sqrt{n-1}\,\max_{j\neq i}\frac{\sqrt{2|\lambda_i(\xi)|/\kappa_\emptyset(A+\xi E)}\sqrt{2|\lambda_j(\xi)|/\kappa_\emptyset(A+\xi E)}}{|\lambda_i(\xi)-\lambda_j(\xi)|}\\ &= \frac{2\sqrt{n-1}}{\kappa_\emptyset(A+\xi E)\ relgap_H(\lambda_i(\xi))}\\ &\le \frac{2\sqrt{n-1}}{(\kappa_\emptyset(A)-\xi)\ relgap_H(\lambda_i(\xi))}\\ &\le \frac{5\sqrt{n-1}}{2\kappa_\emptyset(A)\ relgap_H(\lambda_i(\xi))}\quad, \end{aligned} \tag{3.17}$$

where we have used the fact that $\xi \le \eta \le \kappa_\emptyset(A)/5$. Plugging (3.15) into (3.17) and integrating over the interval $[0,\eta]$, we arrive at the conclusion. ∎

4. Relative Perturbation Theory for Singular Values

The following theorem parallels Theorem 3.

THEOREM 8. *Let $G = DBF$, and let $\delta G = \eta DEF$ be a perturbation of G with $\eta > 0$ and $\|E\|_2 = 1$. Assume that $\sigma(\xi)$ is a non-zero singular value of $G(\xi) = G + \xi DEF$ with unit left and right singular vectors $x(\xi)$ and $y(\xi)$ for $0 \le \xi \le \eta$. If there exists a continuous function $f(\xi)$ such that*

$$\|Dx(\xi)\|_2\|Fy(\xi)\|_2 \le f(\xi)\ \sigma(\xi)$$

for $0 \le \xi \le \eta$, then

$$\exp\left(-\int_0^\eta f(\xi)\,d\xi\right) \le \frac{\sigma(\eta)}{\sigma(0)} \le \exp\left(\int_0^\eta f(\xi)\,d\xi\right)\quad. \tag{4.1}$$

Proof: As in the proof of Theorem 3, we only consider the case where $\sigma(\xi)$ is simple for $0 \le \xi \le \eta$. Then $\sigma(\xi)$ is analytic, and it follows from standard perturbation theory [3] that

$$\frac{d}{d\xi}\,\sigma(\xi) = x(\xi)^T DEFy(\xi)\quad.$$

Thus

$$\left|\frac{1}{\sigma(\xi)}\frac{d}{d\xi}\,\sigma(\xi)\right| = \left|\frac{x(\xi)^T DEFy(\xi)}{\sigma(\xi)}\right| \le f(\xi)$$

for $0 < \xi < \eta$, which implies that

$$\left|\frac{d}{d\xi}\,\log\sigma(\xi)\right| \le f(\xi)\quad.$$

Integrating ξ over the interval $[0,\eta]$ yields (4.1). ∎

The following lemma estimates $f(\xi)$ for a matrix K when there are scalings on both sides.

LEMMA 9. *Let K be a general matrix. For any positive diagonal matrices J and S, and unit vectors w and z satisfying $Kz = Jw$ and $K^T w = Sz$, we have*

$$w^T K z \geq \frac{1}{2}\chi_1(K) \quad . \tag{4.2}$$

Proof: We assume that $\chi_1(K) > 0$; otherwise the result is trivial. Thus every submatrix of K has full rank. Parallel to the proof of Lemma 4, we consider the problem of finding the infimum ψ of $w^T K z$ over w, z, J and S, under the constraints

$$w^T w = 1 \ , \quad z^T z = 1 \ , \quad Kz = Jw \ , \quad K^T w = Sz \ , \quad J \geq 0 \quad \text{and} \quad S \geq 0 \quad . \tag{4.3}$$

The infimum ψ exists because $w^T S z = w^T J w \geq 0$.

Let $\{(w^{(t)}, z^{(t)}, J^{(t)}, S^{(t)})\}$ be a sequence satisfying the constraints (4.3) such that

$$\lim_{t\to\infty} w^{(t)^T} K z^{(t)} = \psi \quad .$$

Let $(\bar{w}, \bar{z})$ be a limit point of $\{(w^{(t)}, z^{(t)})\}$ with $\bar{w} = (\bar{\omega}_1, \ldots, \bar{\omega}_m)^T$ and $\bar{z} = (\bar{\zeta}_1, \ldots, \bar{\zeta}_n)^T$. Then $\bar{w}^T K \bar{z} = \psi$. For any j, if $\bar{\omega}_j \neq 0$, then the j-th diagonal element of $J^{(t)}$ converges to a finite number $(K\bar{z})_j/\bar{\omega}_j$; if $\bar{\omega}_j = 0$, then we can reduce the size of the problem by dropping the j-th row of K and the j-th equation in the constraint $K\bar{z} = J\bar{w}$. Similarly, for any k, if $\bar{\zeta}_k \neq 0$, then the k-th diagonal element of $S^{(t)}$ converges to a finite number $(K^T\bar{w})_k/\bar{\zeta}_k$; if $\bar{\zeta}_k = 0$, then we can reduce the size of the problem by dropping the k-th column of K and the k-th equation in the constraint $K^T\bar{w} = S\bar{z}$. When the problem size is reduced, the value of $\bar{w}^T K \bar{z}$ does not change; neither do the remaining constraints. Thus the infimum of the reduced problem is no larger than ψ.

As in the proof of Lemma 4, we can bound ψ below by the solution to the following minimization problem over $\tilde{w}$, $\tilde{z}$, $\widetilde{J}_2$ and $\widetilde{S}_2$, which achieves its infimum at a finite point:

$$\text{minimize} \qquad \tilde{w}^T \widetilde{K} \tilde{z}$$

$$\text{subject to} \quad \tilde{w}^T\tilde{w} = 1,\ \tilde{z}^T\tilde{z} = 1,\ \widetilde{J}_2 > 0,\ \widetilde{S}_2 > 0,\ \widetilde{K}\tilde{z} = \widetilde{J}\tilde{w},\ \widetilde{K}^T\tilde{w} = \widetilde{S}\tilde{z} \quad ,$$

where

$$\widetilde{K} = \begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} \\ \widetilde{K}_{21} & \widetilde{K}_{22} \end{pmatrix}, \quad \widetilde{J} = \begin{pmatrix} 0 & \\ & \widetilde{J}_2 \end{pmatrix}, \quad \widetilde{S} = \begin{pmatrix} 0 & \\ & \widetilde{S}_2 \end{pmatrix}, \quad \tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix}, \quad \tilde{z} = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} \ ;$$

$\widetilde{K}$ is a submatrix of K up to a row permutation and a column permutation (the row permutation is chosen such that no component of $\widetilde{J}_2$ is zero; and the column permutation is chosen such that no component of $\widetilde{S}_2$ is zero); and $\tilde{w}$ and $\tilde{z}$ are subvectors of w and z up to the same row and column permutations (and no component of $\tilde{w}$ and $\tilde{z}$ is zero).

The minimum is a stationary point of the Lagrangian

$$\mathcal{L}(\tilde{w},\tilde{z},\widetilde{J},\widetilde{S}) = \tilde{w}^T\widetilde{K}\tilde{z} + u^T(\widetilde{K}\tilde{z} - \widetilde{J}\tilde{w}) + v^T(\widetilde{K}^T\tilde{w} - \widetilde{S}\tilde{z}) - \frac{1}{2}\mu(\tilde{w}^T\tilde{w} - 1) - \frac{1}{2}\nu(\tilde{z}^T\tilde{z} - 1) \quad ,$$

where $u = (u_1^T, u_2^T)^T$, $v = (v_1^T, v_2^T)^T$, μ and ν are Lagrange multipliers. The only terms in $\mathcal{L}(\tilde{w},\tilde{z},\widetilde{J},\widetilde{S})$ involving $\widetilde{J}_2$ and $\widetilde{S}_2$ are $u_2^T\widetilde{J}_2\tilde{w}_2$ and $v_2^T\widetilde{S}_2\tilde{z}_2$, respectively. Since $\partial\mathcal{L}/\partial\widetilde{J}_2 = 0$ and $\partial\mathcal{L}/\partial\widetilde{S}_2 = 0$, and since each component of $\tilde{w}_2$ and $\tilde{z}_2$ is non-zero at the minimum, we must have $u_2 = 0$ and $v_2 = 0$. On the other hand, since $\partial\mathcal{L}/\partial\tilde{w} = 0$ and $\partial\mathcal{L}/\partial\tilde{z} = 0$, we must have

$$\widetilde{K}(\tilde{z} + v) - \widetilde{J}u = \mu\tilde{w} \quad \text{and} \quad \widetilde{K}^T(\tilde{w} + u) - \widetilde{S}v = \nu\tilde{z} \quad .$$

But

$$\widetilde{J}u = \begin{pmatrix} 0 & \\ & \widetilde{J}_2 \end{pmatrix}\begin{pmatrix} u_1 \\ 0 \end{pmatrix} = 0 \quad \text{and} \quad \widetilde{S}v = \begin{pmatrix} 0 & \\ & \widetilde{S}_2 \end{pmatrix}\begin{pmatrix} v_1 \\ 0 \end{pmatrix} = 0 \quad ,$$

so that

$$\widetilde{K}(\tilde{z} + v) = \mu\tilde{w} \quad \text{and} \quad \widetilde{K}^T(\tilde{w} + u) = \nu\tilde{z} \quad . \tag{4.4}$$

Putting together these equations and those in the constraints, and disregarding equations that involve $\widetilde{J}_2$ or $\widetilde{S}_2$, we arrive at the following equations:

$$\begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} \\ \widetilde{K}_{21} & \widetilde{K}_{22} \end{pmatrix}\begin{pmatrix} \tilde{z}_1 + v_1 \\ \tilde{z}_2 \end{pmatrix} = \mu\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} \quad , \quad \begin{pmatrix} \widetilde{K}_{11}^T & \widetilde{K}_{21}^T \\ \widetilde{K}_{12}^T & \widetilde{K}_{22}^T \end{pmatrix}\begin{pmatrix} \tilde{w}_1 + u_1 \\ \tilde{w}_2 \end{pmatrix} = \nu\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix}$$

$$\begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} \end{pmatrix}\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} = 0 \quad , \quad \begin{pmatrix} \widetilde{K}_{11}^T & \widetilde{K}_{21}^T \end{pmatrix}\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} = 0 \quad .$$

These can be rewritten as

$$\begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} & \\ \widetilde{K}_{21} & \widetilde{K}_{22} & \\ & & \widetilde{K}_{11} \end{pmatrix}\begin{pmatrix} \tilde{z}_1 + v_1 \\ \tilde{z}_2 \\ v_1 \end{pmatrix} = \mu\begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_1 \end{pmatrix} \tag{4.5}$$

$$\begin{pmatrix} \widetilde{K}_{11}^T & \widetilde{K}_{21}^T & \\ \widetilde{K}_{12}^T & \widetilde{K}_{22}^T & \\ & & \widetilde{K}_{11}^T \end{pmatrix}\begin{pmatrix} \tilde{w}_1 + u_1 \\ \tilde{w}_2 \\ u_1 \end{pmatrix} = \nu\begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_1 \end{pmatrix} \quad . \tag{4.6}$$

From (4.4) we also have

$$\begin{aligned} \mu &= \tilde{w}^T\widetilde{K}(\tilde{z} + v) = \tilde{w}^T\widetilde{K}\tilde{z} + \tilde{z}^T\widetilde{S}v = \tilde{w}^T\widetilde{K}\tilde{z} \quad , \\ \nu &= \tilde{z}^T\widetilde{K}^T(\tilde{w} + u) = \tilde{z}^T\widetilde{K}^T\tilde{w} + \tilde{w}^T\widetilde{J}u = \tilde{z}^T\widetilde{K}^T\tilde{w} = \mu \quad , \end{aligned}$$

where we have used the fact that the constraints are satisfied and that $\widetilde{J}u = 0$ and $\widetilde{S}v = 0$. Thus $\mu = \nu$ is a lower bound on ψ.

Similar to the proof of Lemma 4, we estimate a lower bound on μ by taking norms in equations (4.5) and (4.6). Note that $\widetilde{K}_{11}$ and $\widetilde{K} = \begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} \\ \widetilde{K}_{21} & \widetilde{K}_{22} \end{pmatrix}$ are not necessarily square matrices, and so the coefficient matrices

$$\begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} & \\ \widetilde{K}_{21} & \widetilde{K}_{22} & \\ & & \widetilde{K}_{11} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \widetilde{K}_{11}^T & \widetilde{K}_{21}^T & \\ \widetilde{K}_{12}^T & \widetilde{K}_{22}^T & \\ & & \widetilde{K}_{11}^T \end{pmatrix}$$

in equations (4.5) and (4.6) can be structurally singular. Without loss of generality, we assume that $\widetilde{K}_{11}$ is a fat matrix; otherwise we consider the matrix $\widetilde{K}^T$.

We first assume that $\widetilde{K}$ is fat. The coefficient matrix in (4.6) is skinny and has full rank. Its smallest singular value is no less than $\chi_1(K)$. Taking norms in (4.6), we have

$$\chi_1(K) \left\| \begin{pmatrix} \tilde{w}_1 + u_1 \\ \tilde{w}_2 \\ u_1 \end{pmatrix} \right\|_2 \le \mu \left\| \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \\ \tilde{z}_1 \end{pmatrix} \right\|_2 \le \sqrt{2}\mu \quad .$$

But

$$\left\| \begin{pmatrix} \tilde{w}_1 + u_1 \\ \tilde{w}_2 \\ u_1 \end{pmatrix} \right\|_2 \ge 1/\sqrt{2} \quad .$$

Thus we have $\chi_1(K)/2 \le \mu$, which implies (4.2).

Now we consider the case where $\widetilde{K}$ is skinny. In this case the coefficient matrices in both equations (4.5) and (4.6) are singular. There exists an orthogonal matrix $Q = \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$ such that $\widetilde{K}_{11} = (\widehat{K}_{11}, 0) \begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix}$, where $\widehat{K}_{11}$ is a square matrix. Let

$$\begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} = \begin{pmatrix} Q_1 \tilde{z}_1 \\ Q_2 \tilde{z}_1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \hat{v}_1 \\ \hat{v}_2 \end{pmatrix} = \begin{pmatrix} Q_1 v_1 \\ Q_2 v_1 \end{pmatrix} \quad .$$

Substituting $\widetilde{K}_{11} v_1 = \widehat{K}_{11} \hat{v}_1$ into the left-hand side of (4.5) we have

$$\begin{pmatrix} \widetilde{K}_{11} & \widetilde{K}_{12} & \\ \widetilde{K}_{21} & \widetilde{K}_{22} & \\ & & \widehat{K}_{11} \end{pmatrix} \begin{pmatrix} \tilde{z}_1 + v_1 \\ \tilde{z}_2 \\ \hat{v}_1 \end{pmatrix} = \mu \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \\ \tilde{w}_1 \end{pmatrix} \quad .$$

The coefficient matrix of this equation is skinny, and its smallest singular value is no less than $\chi_1(K)$. Taking norms on both sides, we have

$$\chi_1(K) \left\| \begin{pmatrix} \tilde{z}_1 + v_1 \\ \tilde{z}_2 \\ \hat{v}_1 \end{pmatrix} \right\|_2 \le \sqrt{2}\mu \quad . \tag{4.7}$$

On the other hand, from (4.6) we have

$$\widetilde{K}_{11}^T u_1 = \mu \tilde{z}_1 \quad . \tag{4.8}$$

Since

$$Q\widetilde{K}_{11}^T = \begin{pmatrix} \widehat{K}_{11}^T \\ 0 \end{pmatrix} \quad \text{and} \quad Q\tilde{z}_1 = \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} \quad ,$$

applying Q to both sides of (4.8) we have

$$\begin{pmatrix} \widehat{K}_{11}^T \\ 0 \end{pmatrix} u_1 = \mu \begin{pmatrix} \hat{z}_1 \\ \hat{z}_2 \end{pmatrix} \quad ,$$

whence $0 = \mu \hat{z}_2$.

Note that

$$\mu = \tilde{w}^T \widetilde{K} \tilde{z} = \tilde{w}^T J \tilde{w} = \tilde{w}_2^T \widetilde{J}_2 \tilde{w}_2 = \tilde{z}_2^T \widetilde{S}_2 \tilde{z}_2 \quad .$$

Since the components of $\tilde{w}_2$ and $\tilde{z}_2$ are non-zero, and since $\widetilde{J}_2$ and $\widetilde{S}_2$ are positive and diagonal, μ can be 0 only when $\tilde{w}_2$, $\tilde{z}_2$, $\widetilde{J}_2$ and $\widetilde{S}_2$ are all of dimension 0. But in this case the constraints become $\widetilde{K}\tilde{z} = 0$ and $\widetilde{K}^T \tilde{w} = 0$, which contradict the fact that $\widetilde{K}$ has full rank. Thus $\mu > 0$ and hence $\hat{z}_2 = 0$.

Since

$$\left\| \begin{pmatrix} \tilde{z}_1 + v_1 \\ \tilde{z}_2 \\ \hat{v}_1 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} \hat{z}_1 + \hat{v}_1 \\ \hat{z}_2 + \hat{v}_2 \\ \tilde{z}_2 \\ \hat{v}_1 \end{pmatrix} \right\|_2 = \left\| \begin{pmatrix} \hat{z}_1 + \hat{v}_1 \\ \hat{v}_2 \\ \tilde{z}_2 \\ \hat{v}_1 \end{pmatrix} \right\|_2 \geq \left\| \begin{pmatrix} \hat{z}_1 + \hat{v}_1 \\ \tilde{z}_2 \\ \hat{v}_1 \end{pmatrix} \right\|_2 \geq 1/\sqrt{2} \quad ,$$

substituting this into (4.7) we have $\chi_1(K)/\sqrt{2} \leq \sqrt{2}\mu$, which implies (4.2). ■

Example 5 illustrates the tightness of (4.2).

EXAMPLE 5. Let K be any matrix with $\chi_1(K) > 0$. Then, up to a permutation,

$$K = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}$$

with $\sigma_{\min}(K_{22}) = \chi_1(K) > 0$. Let $G = DKF$, where $D = diag(dI_1, I_2)$ and $F = diag(fI_1', I_2')$ are blocked conformally with K.

There exist unit vectors x_2 and y_2 such that $K_{22}y_2 = \chi_1(K)x_2$ and $K_{22}^T x_2 = \chi_1(K)y_2$. According to [4], G has a singular value ω satisfying

$$|\omega - \chi_1(K)| \leq 1/\sqrt{2} \left\| \begin{pmatrix} & & dfK_{11} & dK_{12} \\ & & fK_{21} & K_{22} \\ dfK_{11}^T & fK_{21}^T & & \\ dK_{12}^T & K_{22}^T & & \end{pmatrix} \begin{pmatrix} 0 \\ x_2 \\ 0 \\ y_2 \end{pmatrix} - \chi_1(K) \begin{pmatrix} 0 \\ x_2 \\ 0 \\ y_2 \end{pmatrix} \right\|_2$$

$$= 1/\sqrt{2}\left\|\begin{pmatrix} dK_{12}y_2 \\ 0 \\ fK_{21}^T x_2 \\ 0 \end{pmatrix}\right\|_2$$

$$\leq \sqrt{\frac{d^2+f^2}{2}}\,\|K\|_2 \quad .$$

Thus for $d > 0$ and $f > 0$ sufficiently small, we have

$$\omega \geq \chi_1(K) - \sqrt{\frac{d^2+f^2}{2}}\,\|K\|_2 > 0 \quad .$$

For any unit vectors $x = (x_1^T, x_2^T)^T$ and $y = (y_1^T, y_2^T)^T$ satisfying $Gy = \omega x$ and $G^T x = \omega y$, we have

$$dfK_{11}y_1 + dK_{12}y_2 = \omega x_1 \quad \text{and} \quad fdK_{11}^T x_1 + fK_{21}^T x_2 = \omega y_1 \quad ,$$

whence

$$\|x_1\|_2 \leq \frac{\|K\|_2 d}{\omega} \quad \text{and} \quad \|y_1\|_2 \leq \frac{\|K\|_2 f}{\omega}$$

for $0 < d < 1$ and $0 < f < 1$. These relations imply that

$$\|Dx\|_2 \geq \|x_2\|_2 \geq \sqrt{1 - \left(\frac{\|K\|_2 d}{\omega}\right)^2} \quad \text{and} \quad \|Fy\|_2 \geq \|y_2\|_2 \geq \sqrt{1 - \left(\frac{\|K\|_2 f}{\omega}\right)^2} \quad .$$

Let

$$w = \frac{Dx}{\|Dx\|_2}, \quad z = \frac{Fy}{\|Fy\|_2}, \quad J = \frac{\omega\|Dx\|_2}{\|Fy\|_2} D^{-2} \quad \text{and} \quad S = \frac{\omega\|Fy\|_2}{\|Dx\|_2} F^{-2} \quad .$$

Then w and z are unit vectors satisfying

$$Kz = Jw \quad \text{and} \quad K^T w = Sz \quad ,$$

and

$$w^T K z = \frac{\omega}{\|Dx\|_2\,\|Fy\|_2} \leq \frac{\omega}{\sqrt{\left(1 - \left(\frac{\|K\|_2 d}{\omega}\right)^2\right)\left(1 - \left(\frac{\|K\|_2 f}{\omega}\right)^2\right)}} \quad .$$

As d and f go to zero, ω goes to $\chi_1(K)$ as does the last upper bound. Thus equality in (4.2) can be achieved up to a factor close to 2.

COROLLARY 10. *Let $G = DBF$ with $\chi_1(B) > 0$. Let $\delta G = D(\delta B)F$ be a perturbation of G with $\|\delta B\|_2 \equiv \eta < \chi_1(B)$. Let σ_i and σ_i' be the i-th singular value of G and $G + \delta G$, respectively. Then*

$$-\frac{\eta(2\chi_1(B) - \eta)}{\chi_1^2(B)} \leq \frac{\sigma_i' - \sigma_i}{\sigma_i} \leq \frac{\eta(2\chi_1(B) - \eta)}{(\chi_1(B) - \eta)^2} \quad . \tag{4.9}$$

Proof: Let $E = \delta B/\|\delta B\|_2$, let $G(\xi) = D(B+\xi E)F$, and let $\sigma_i(\xi)$ be the i-th singular value of $G(\xi)$ with corresponding unit left and right singular vectors $x_i(\xi)$ and $y_i(\xi)$ for $0 \le \xi \le \eta$. Then $\sigma_i(0) = \sigma_i$ and $\sigma_i(\eta) = \sigma_i'$. According to Lemma 2, we have $\chi_1(B+\xi E) \ge \chi_1(B) - \xi > 0$. It follows that $\sigma_i(\xi)$ is positive for $0 \le \xi \le \eta$. Applying (4.2) in Lemma 9 with

$$K = B + \xi E \ , \quad w = \frac{Dx_i(\xi)}{\|Dx_i(\xi)\|_2} \ , \quad z = \frac{Fy_i(\xi)}{\|Fy_i(\xi)\|_2} \ ,$$

and

$$J = \frac{\sigma_i(\xi)\|Dx_i(\xi)\|_2}{\|Fy_i(\xi)\|_2} D^{-2} \quad \text{and} \quad S = \frac{\sigma_i(\xi)\|Fy_i(\xi)\|_2}{\|Dx_i(\xi)\|_2} F^{-2} \ ,$$

we have

$$\frac{1}{2}\chi_1(B+\xi E) \le \frac{\sigma_i(\xi)}{\|Dx_i(\xi)\|_2\|Fy_i(\xi)\|_2} \ .$$

Applying Lemma 2,

$$\|Dx_i(\xi)\|_2\|Fy_i(\xi)\|_2 \le \frac{2\sigma_i(\xi)}{\chi_1(B+\xi E)} \le \frac{2\sigma_i(\xi)}{\chi_1(B) - \xi} \ .$$

Theorem 8 then gives

$$\left(\frac{\chi_1(B) - \eta}{\chi_1(B)}\right)^2 \le \frac{\sigma_i'}{\sigma_i} \le \left(\frac{\chi_1(B)}{\chi_1(B) - \eta}\right)^2 \ ,$$

which is equivalent to (4.9). ∎

Inequality (4.9) gives lower and upper bounds on the ratio $(\sigma_i' - \sigma_i)/\sigma_i$ that are independent of D and F. Because of the tightness of (4.2), the bounds in (4.9) are optimal, up to a factor close to 2 for η sufficiently small.

Now we estimate the function $f(\xi)$ in Theorem 8 when there are scalings on only one side of K.

LEMMA 11. *Let K be a general matrix. For any positive diagonal matrix J, positive scalar ζ, and unit vectors w and z satisfying $Kz = Jw$ and $K^T w = \zeta z$, we have*

$$w^T K z \ge \chi_2(K) \ .$$

Proof: We assume that $\chi_2(K) > 0$; otherwise the result is trivial. Parallel to the proof of Lemma 9, we consider the problem of finding the infimum ψ of $w^T K z$, under the constraints

$$w^T w = 1 \ , \quad z^T z = 1 \ , \quad Kz = Jw \ , \quad K^T w = \zeta z \ , \quad J \ge 0 \quad \text{and} \quad \zeta \ge 0 \ .$$

The infimum ψ exists because $x^T K z = w^T J w \ge 0$.

As in the proof of Lemma 9, we can bound ψ below by the solution to the following problem, which achieves its infimum at a finite point:

$$\begin{array}{rl} \text{minimize} & \tilde{w}^T \widetilde{K} \tilde{z} \\ \text{subject to} & \tilde{w}^T\tilde{w} = 1 \;,\; \tilde{z}^T\tilde{z} = 1 \;,\; \widetilde{J}_2 > 0 \;,\; \zeta \geq 0 \;,\; \widetilde{K}\tilde{z} = \widetilde{J}\tilde{w} \;,\; \widetilde{K}^T\tilde{w} = \zeta\tilde{z} \;, \end{array}$$

where

$$\widetilde{J} = \begin{pmatrix} 0 & \\ & \widetilde{J}_2 \end{pmatrix} \quad , \quad \tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \tilde{w}_2 \end{pmatrix} \quad \text{and} \quad \widetilde{K} = \begin{pmatrix} \widetilde{K}_1 \\ \widetilde{K}_2 \end{pmatrix} \quad ;$$

$\widetilde{K}$ is a submatrix of K up to a row permutation (this permutation is chosen such that no component of $\widetilde{J}_2$ is zero); and $\tilde{w}$ is a subvector of w up to the same permutation (and no component of $\tilde{w}$ is zero).

The minimum is a stationary point of the Lagrangian

$$\mathcal{L}(\tilde{w}, \tilde{z}, \widetilde{J}, \zeta) = \tilde{w}^T\widetilde{K}\tilde{z} + u^T(\widetilde{K}\tilde{z} - \widetilde{J}\tilde{w}) + v^T(\widetilde{K}^T\tilde{w} - \zeta\tilde{z}) - \frac{1}{2}\mu(\tilde{w}^T\tilde{w} - 1) - \frac{1}{2}\nu(\tilde{z}^T\tilde{z} - 1) \quad ,$$

where $u = (u_1, u_2)^T$, μ and ν are Lagrange multipliers.

We first show that ζ must be non-zero. Indeed, if ζ were zero, then $\widetilde{K}^T\tilde{w} = 0$ and thus

$$\tilde{w}_2^T\widetilde{J}_2\tilde{w}_2 = \tilde{w}^T\widetilde{J}\tilde{w} = \tilde{w}^T\widetilde{K}\tilde{z} = 0 \quad .$$

Since the components of $\tilde{w}_2$ are non-zero and since $\widetilde{J}_2$ is positive diagonal, this equation holds only when $\tilde{w}_2$ and $\widetilde{J}_2$ are of dimension 0. But in this case the constraints would become $\widetilde{K}\tilde{z} = 0$ and $\widetilde{K}^T\tilde{w} = 0$, which would imply that $\widetilde{K}$ does not have full rank, contradicting the fact that $\chi_2(K) > 0$. Thus ζ must be non-zero.

The only terms in $\mathcal{L}(\tilde{w}, \tilde{z}, \widetilde{J}, \zeta)$ involving ζ and $\widetilde{J}_2$ are $\zeta v^T\tilde{z}$ and $u_2^T\widetilde{J}_2\tilde{w}_2$, respectively. Since $\zeta > 0$, we must have $\partial\mathcal{L}/\partial\zeta = 0$, which implies that $v^T\tilde{z} = 0$; and since $\partial\mathcal{L}/\partial\widetilde{J}_2 = 0$ and each component of $\tilde{w}_2$ is non-zero, we must have $u_2 = 0$. On the other hand, since $\partial\mathcal{L}/\partial\tilde{w} = 0$ and $\partial\mathcal{L}/\partial\tilde{z} = 0$, we must have

$$\widetilde{K}(\tilde{z} + v) - \widetilde{J}u = \mu\tilde{w} \quad \text{and} \quad \widetilde{K}^T(\tilde{w} + u) - \zeta v = \nu\tilde{z} \quad . \tag{4.10}$$

We also have

$$\widetilde{J}u = \begin{pmatrix} 0 & \\ & \widetilde{J}_2 \end{pmatrix}\begin{pmatrix} u_1 \\ 0 \end{pmatrix} = 0 \quad .$$

Using (4.10) with $\widetilde{J}u = 0$, $v^T\tilde{z} = 0$ and the equations in the constraints, we get the following relations:

$$\begin{aligned} \mu &= \mu\tilde{w}^T\tilde{w} = \tilde{w}^T\widetilde{K}(\tilde{z} + v) = \zeta\tilde{z}^T(\tilde{z} + v) = \zeta \quad , \\ \nu &= \nu\tilde{z}^T\tilde{z} = \tilde{z}^T(\widetilde{K}^T(w + u) - \zeta v) = \tilde{z}^T\widetilde{K}^T(w + u) = \zeta\tilde{z}^T\tilde{z} + \tilde{w}^T\widetilde{J}u = \zeta \quad . \end{aligned}$$

Thus $\zeta = \mu = \nu$ is a lower bound on ψ. Further, the constraints and (4.10) imply that

$$\widetilde{K}^T\tilde{w} = \zeta\tilde{z} \quad \text{and} \quad \widetilde{K}(\tilde{z} + v) = \zeta\tilde{w} \quad . \tag{4.11}$$

Since $v^T \tilde{z} = 0$, we have

$$\|\tilde{z} + v\|_2 = \sqrt{\|\tilde{z}\|_2^2 + \|v\|_2^2} \geq 1 \quad .$$

One of $\widetilde{K}^T$ and $\widetilde{K}$ is skinny. Taking norms from the corresponding equation in (4.11), we arrive at our conclusion. ■

With the techniques used in proving Corollary 10, we can easily prove the following result by using Lemmas 2 and 11.

COROLLARY 12. *Let $G = DB$ with $\chi_2(B) > 0$. Let $\delta G = D(\delta)B$ be a perturbation of G with $\|\delta B\|_2 \equiv \eta < \chi_2(B)$. Let σ_i and σ_i' be the i-th singular value of G and $G + \delta G$, respectively. Then*

$$-\frac{\eta}{\chi_2(B)} \leq \frac{\sigma_i' - \sigma_i}{\sigma_i} \leq \frac{\eta}{\chi_2(B) - \eta} \quad . \tag{4.12}$$

When G is scaled from the right, we can apply Corollary 12 to G^T. Similar to (4.9), the bounds in inequality (4.12) are asymptotically optimal.

EXAMPLE 6. Let $G = DB$ with D diagonal and B fat. Demmel and Veseliĉ [2] show that

$$\left| \frac{\sigma_i' - \sigma_i}{\sigma_i} \right| \leq \frac{\eta}{\chi_2(B)} \quad .$$

Thus (4.12) agrees with this result asymptotically.

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