Some Results on Maximum a Posteriori Probability Parsing Algorithms[†]

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Research Report #98

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[†] This work was supported by The Office of Naval Research under Grant N00014-75-C-0752.

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In this paper we discuss the theory, performance and implications of a class of Syntactic Pattern Recognition algorithms which are optimal in a well defined sense.

Assume that it is desired to transmit a message consisting of a sequence of symbols chosen from a finite alphabet. Suppose further that any such message will be a well formed sentence in a language generated by a known formal grammar. The message is to be encoded and transmitted by sending a sequence of complex signals, one signal for each symbol in the message, through a noisy channel.

The corrupted message is decoded in two stages. First, the individual symbols are identified by a maximum a posteriori probability decision rule. The resulting string of symbols will not, in general, be a grammatically correct sentence. Thus, in stage two, a parser which finds that sentence in the language which maximizes the product of the individual symbol probabilities conditioned on the signals received at the output of the channel is used. The decoding thus obtained is the maximum likelihood estimator of the transmitted message.

Viterbi [1] treats the above described problem as one of estimating the state sequence of a finite Markov process. The desired estimator is obtained by a dynamic programming technique. Recently, Fung and Fu [2,3] have described an algorithm for the case of messages which are sentences in a context free language. Their procedure is based on an algorithm given by Younger [4]. We have derived efficient recursive procedures which solve the problem for regular, one-counter and context free languages. The space and time complexity of these algorithms in terms of n, the length of the input, is summarized in the table below.

$G(V_N,V_T,S,P)$	Space	Time
Regular	$\|V_N\|n$	$\|P\|n$
One-counter	$\|V_N\|n^2$	$\|P\|n^2$
Context free	$0\ (n^2)$	$\|P\|n^3$

In addition to the analysis, we have tested these algorithms for several formal grammars using both real and simulated channels. Because of the effieciency of these algorithms the tests were conducted on several thousands of sentences. Some of the test grammars had over 300 production rules. The tests were accomplished without special programming considerations.

In the course of our experiments with the algorithms, we discovered an empirical measure of the information content of formal languages. By making the signal to noise ratio of the channel very low, we can reduce the performance of the single symbol decoder to the extent that it makes a random choice for each symbol independent of the input to the channel.

Although the symbol accuracy of the MAP parser also degrades with decreasing SNR, its limiting value is greater than that of the single symbol decoder alone. The difference between the two limiting values is a measure of the information encoded in the grammar.

For a given grammar G (V_N, V_T, S, P) and the language, $L(G)$, it generates we define

$$L_n = \{w \in L(G) \mid |w| = n\}$$
$$V_n = \{w \in V_T^* \mid |w| = n\}$$

Then we define the entropy $H(L(G))$ of the language $L(G)$ by:

$$H(L(G)) = - \sum_n \frac{|Ln|}{|Vn|} \log_2 \left(\frac{|Ln|}{|Vn|}\right)$$

We have observed that for two grammars G_1 and G_2, $H(L(G_1))$ and $H(L(G_2))$ are in the same order as the differences between the limiting values for their single symbol decoding and MAP parsing accuracies.

Forney [5] has listed several important problems of the type described here and has suggested that they be solved by the Viterbi Algorithm. Because this algorithm has an exponential running time it is intractible for long inputs. Forney further suggests that secondary information be used to guide a heuristic search. Such backtracking procedures have been used by Neely and White [6], Walker [7] and Levinson [8] in speech recognition algorithms.

We have observed that by applying the appropriate one of our algorithms to a variety of pattern recognition problems both the high cost of the Viterbi algorithm and the obvious disadvantages of sub-optimal bactracking procedures can be avoided and the optimal solution still obtained.

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On the Halting of Tree Replacement Systems

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Research Report #99

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This work was supported by the Office of Naval Research under grant N00014-75-C-0752.

In algebraic simplification of expressions, code generation and other areas one is often given a finite set of rewriting rules which are to be applied to an expression until no further rules apply. For example,

> Given well-formed arithmetic expressions composed of parentheses and variables with operators of addition (+) and multiplication (x), apply
>
> $$(\alpha + \beta)\times\gamma \rightarrow (\alpha \times \gamma) + (\beta \times \gamma) \qquad (1)$$
>
> until no further applications are possible.

We assume that the Greek letters can match any well formed subexpression or variable, and that "apply" has its usual meaning in symbolic manipulation: when ever the left side of the rule matches a subexpression, the subexpression is to be replaced by the right side of the rule with the Greek letters consistently instantiated.

The key phrase in the forgoing problem statement is "until no further applications are possible." This raises a difficult question:

> How do we prove that a particular set of rules halts (i.e. no further applications are possible) for all expressions and all possible sequences of rule applications?

For rule (1) the problem was first answered by Iturriaga [1] and later a simpler proof was presented by Manna and Ness [2]. In this latter paper the standard proof method was used which we call the <u>well ordering</u> method. In particular, given a set of rules, one seeks a measure M on the expressions such that

$$M(\ldots\ (\alpha + \beta) \times \gamma\ \ldots) > M(\ldots\ (\alpha \times \gamma) + (\beta \times \gamma)\ldots).$$

The task of finding a measure M is not always as simple as it might appear -- especially when there are many rules that interact in non-trivial ways. In the present case, if the "local measure" of γ has a large value, then the measure of the entire expression after the rule has been applied may increase rather than decrease, since there are now two copies of γ.

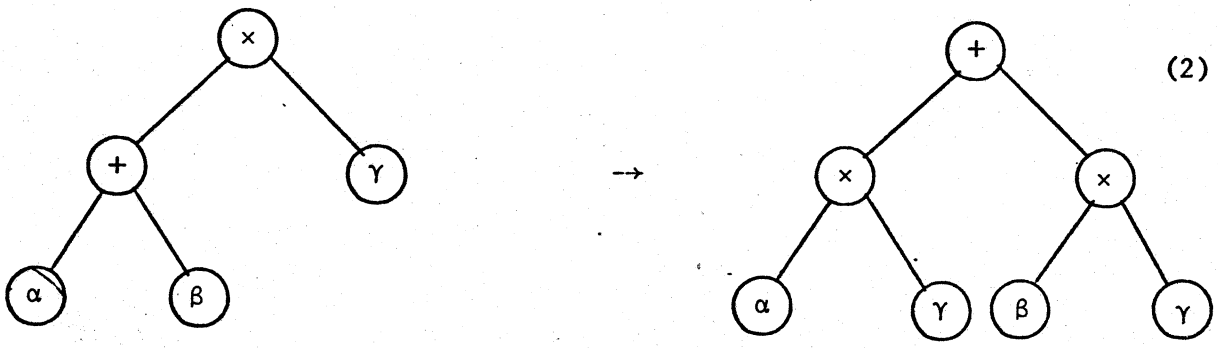
The main result of this paper is to present a new method, the value preserving method, for proving that a set of rules halts. The method is broadly applicable and as the name implies it takes advantage of the fact that rules for symbolic expression manipulation are often intended to be value preserving. Specifically, we prove a theorem stating that two properties of rule sets, value preserving and monotonicity are sufficient to imply that the rule set halts for all expressions and all sequences of rule applications. The exact statement of the general form of the theorem, as well as the proof itself require considerable technical development. For the purposes of this abstract we by-pass the tedious details and concentrate on examples, the generalization of which, we believe, will be clear.

Before proceeding, we note that we are presenting general sufficient conditions for a rule set to halt. The impossibility of finding necessary and sufficient conditions is implied by

Theorem: The problem of determining if a finite set of rules halts is undecidable, even if the set contains as few as 3 rules.

Thus, sufficient conditions are all that can be hoped for.

Turning now to a proof using the value preserving method for the distributive law (1), we assume that the expressions are given as trees (variables at the leaves, operators at non-leaf vertices). The distributive law is then written



(2)

where α, β and γ match any subtree.

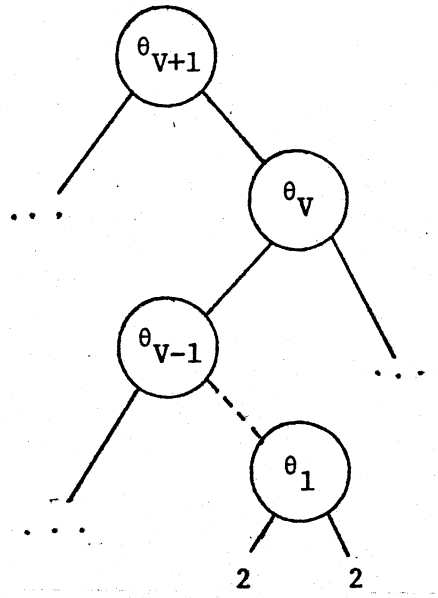
Let E be any expression in this tree form. Replace each leaf in E by the integer constant 2 and let the resulting tree be E'. Evaluate E' in the obvious way and let the resulting number be V. We note that the distributive law preserves this value, i.e. if E'_{i+1} is the result of applying (2) to E'_i then the values of each expression are equal to V.

Assume, for the purpose of contradiction, that (2) does not halt for E'. That is, there exists a sequence

$$E' = E'_0, E'_1, E'_2, \ldots$$

such that E'_{i+1} follows from E'_i by application of (2). If we denote by $\| E'_i \|$ the number of vertices of E'_i, then as $i \to \infty$, $\| E_i \| \to \infty$, i.e. the size of each expression increases without bound. This implies that for some E'_k there is a root-to-leaf path of length V, i.e.

E'_k =



where θ_j is either + or ×. But by the monotonic properties of + and ×, the subtree rooted at θ_{i+1} has value greater than the subtree rooted at θ_i ($1 \le i < V$) and thus the value of E'_k must be greater than V. Contradiction! The rules must halt. □

The rules rely on the facts

I. Both + and × are monotone in the sense that

$$a + b > \max(a,b)$$

$$a \times b > \max(a,b)$$

for all integers $a,b \ge 2$,

II. If E'_{i+1} follows from E'_i by application of the distributive law, then

$$\| E'_{i+1} \| > \| E'_i \|$$

and their <u>values</u> are equal.

Our theorem proves the substance of the forgoing argument for generalized statements of I and II. All that would be required to prove halting for the distributive law given the theorem would be to find an interpretation in which I and II were satisfied. Informally, the general monotone property can be stated:

> An operator is monotone if its value when applied to operends whose values are members of a subset (not necessarily proper) of its legal arguments yields a result which is (1) in the subset and (2) greater than the value of any of the operands.

Thus, for the present case the subset is the subset of integers {2,3,4,...} Notice also that the ordering $\|E'_i\| < \|E'_{i+1}\|$ is opposite of the that required by the well ordering method of proof. Thus, the value preserving method in one sense complements the well ordering method in that if the size increases as rules are applied, the value preserving method can be used, but if the size decreases then a well ordering proof is immediate.

As a second example of the value preserving method, consider the problem of showing that for

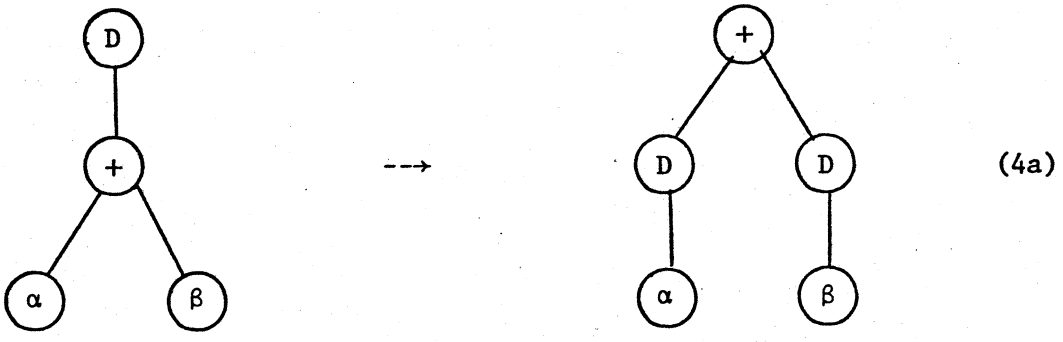
> arithmetic expressions formed from variables, the binary operators + and ×, and the monodic differential operator D, the rules

$$D(\alpha + \beta) \rightarrow D\alpha + D\beta \qquad (3a)$$

$$D(\alpha \times \beta) \rightarrow (\beta \times D\alpha) + (\alpha \times D\beta) \qquad (3b)$$

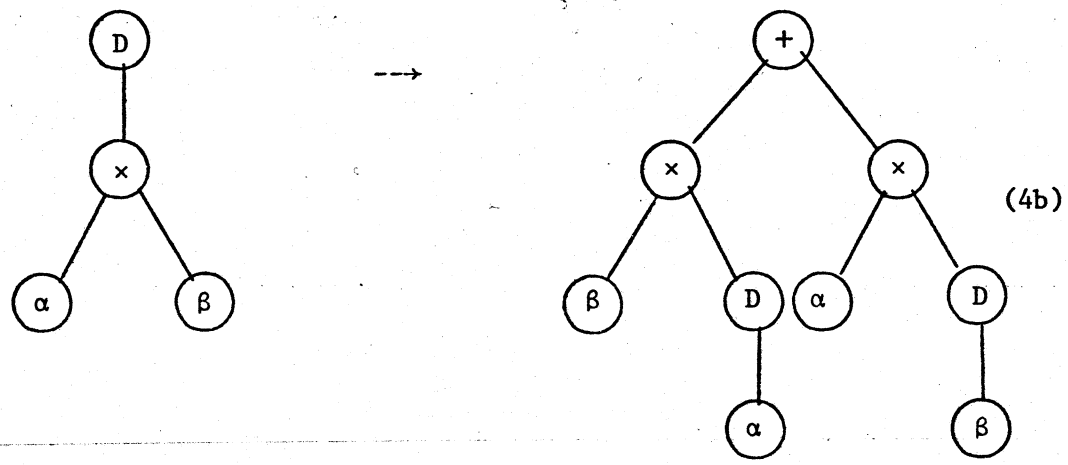
halt.

Following our earlier strategy we express these rules in tree form as



(4a)

and



(4b)

For the interpretation of expression E we select the usual interpretation for +, × and D and we replace the leaves by the functional quantity $2e^{2x}$. The value of the expression E' so interpreted is its value when $x=0$.

For monotonicity it is easy to see that

I. (a) $Df > f$

(b) $f + g > f$ and g

(c) $f \times g > f$ and g

for all f, g of the form $\sum_{i=1}^{n} a_i e^{b_i x}$ where $a_i \geq 2$ and $b_i \geq 2$.

Property II is as before. Indeed, since the interpretation just given holds for the distributive law, it follows that rules (1), (3a) and (3b) taken together must always halt!

In summary we present necessary conditions for a set of rewriting rules to halt for all expressions and all possible sequences of rule applications. The method depends on finding a interpretation in which the value of the expression is preserved under rule application, and in which the operators are monotonic (i.e. the value of an expression is greater than the value of its operands).

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