The Laplace Transform is frequently encountered in mathematics, physics, engineering and other fields. However, the spectral properties of the Laplace Transform tend to complicate its numerical treatment; therefore, the closely related "Truncated" Laplace Transforms are often used in applications.
We have constructed efficient algorithms for the evaluation of the Singular Value Decomposition (SVD) of Truncated Laplace Transforms; in the current paper, we introduce algorithms for the evaluation of the right singular functions and singular values of Truncated Laplace Transforms. Algorithms for the computation of the left singular functions will be introduced separately in an upcoming paper.
The resulting algorithms are applicable to all environments likely to be encountered in applications, including the evaluation of singular functions corresponding to extremely small singular values (e.g. 10^{-1000}).

On the Analytical and Numerical Properties of the Truncated Laplace Transform I.

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1 Introduction

The Laplace Transform $\mathcal{L}$ is a linear mapping $L^2[0,\infty) \to L^2[0,\infty)$; for a function $f \in L^2[0,\infty)$, it is defined by the formula:

$$(\mathcal{L}(f))(\omega) = \int_0^\infty e^{-t\omega} f(t) \mathrm{d}t. \tag{1}$$

As is well-known, $\mathcal{L}$ has a continuous spectrum, and $\mathcal{L}^{-1}$ is not continuous (see, for example, [1]). These and related properties tend to complicate the numerical treatment of $\mathcal{L}$.

In addressing these problems, we find it useful to draw an analogy between the numerical treatment of the Laplace Transform, and the numerical treatment of the Fourier Transform $\mathcal{F}$; for a function $f \in L^1(\mathbb{R})$, the latter is defined by the formula:

$$(\mathcal{F}(f))(\omega) = \int_{-\infty}^\infty e^{-it\omega} f(t) \mathrm{d}t, \tag{2}$$

where $\omega \in \mathbb{R}$.

In various applications in mathematics and engineering, it is useful to define the "Truncated" Fourier Transform $\mathcal{F}_c : L^2[-1,1] \to L^2[-1,1]$; for a given $c > 0$, $\mathcal{F}_c$ of a function $f \in L^2[-1,1]$ is defined by the formula:

$$(\mathcal{F}_c(f))(\omega) = \int_{-1}^1 e^{-ict\omega} f(t) \mathrm{d}t. \tag{3}$$

The operator $\mathcal{F}_c$ has been analyzed extensively; one of most notable observations, made by Slepian et al. around 1960, was that the integral operator $\mathcal{F}_c$ commutes with a second order differential operator (see [2]). This property of $\mathcal{F}_c$ was used in analytical and numerical investigations of the eigendecomposition of this operator, for example in [2, 3, 4, 5, 6, 7, 8].

For $0 < a < b < \infty$, the linear mapping $\mathcal{L}_{a,b} : L^2[a,b] \to L^2[0,\infty)$ defined by the formula

$$(\mathcal{L}_{a,b}(f))(\omega) = \int_a^b e^{-t\omega} f(t) \mathrm{d}t, \tag{4}$$

will be referred to as the *Truncated Laplace Transform* of f; obviously, $\mathcal{L}_{a,b}$ is a compact operator (see, for example, [1]) .

The Singular Value Decomposition (SVD) of $\mathcal{L}_{a,b}$ has been analyzed, inter alia, in [1] and [9]; Bertero and Grünbaum observed that each of the symmetric operators $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ and $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ commutes with a differential operator (see [9]). Despite [9, 10, 1, 11, 12, 13, 14, 15], much more is known about the numerical and analytical properties of $\mathcal{F}_c$ than about the properties of $\mathcal{L}_{a,b}$.

We have constructed algorithms for the efficient evaluation of the of the SVD of $\mathcal{L}_{a,b}$. In this paper, we introduce algorithms for the efficient evaluation of the right singular functions and singular values of $\mathcal{L}_{a,b}$. The remaining algorithms, including the algorithm for the numerical evaluation of the left singular functions, will be discussed in upcoming papers along with additional analytical results.

The paper is organized as follows. Section 2 summarizes the various standard mathematical facts and simple derivations that are used later in the paper. Section 3 contains the derivation of various properties of the right singular functions of the Truncated Laplace Transform, which are used in the algorithms. Section 4 describes the algorithms for the evaluation of the right singular functions and singular values of the Truncated Laplace Transform. Section 5 contains numerical results obtained using the algorithms. Section 6 contains generalizations and conclusions.

2 Preliminaries

2.1 The Legendre Polynomials

In this subsection we summarize some of the properties of the the standard Legendre Polynomials, and restate these properties for shifted and normalized forms of the Legendre Polynomials.

We define the *Shifted Legendre Polynomial* of degree $k = 0, 1, ...$, which we will be denoting by P_k^*, by the formula

$$P_k^*(x) = P_k(2x - 1), \tag{5}$$

where P_k is the Legendre Polynomial of degree k; the standard definition of the Legendre Polynomials can be found, inter alia, in [16].

As is well-known, the Legendre Polynomials form an orthogonal basis in $L^2[-1, 1]$, but they are not normalized; it immediately follows that the Shifted Legendre Polynomials form an orthogonal basis in $L^2[0, 1]$ and that they are also not normalized. Therefore, we find it convenient to define the *Normalized Shifted Legendre Polynomial* of degree $k = 0, 1, ...$, which we will be denoting by $\overline{P_k^*}$, by the formula

$$\overline{P_k^*}(x) = P_k^*(x)\sqrt{2k + 1}; \tag{6}$$

the Normalized Shifted Legendre Polynomials $\overline{P_0^*}, \overline{P_1^*}, ...$ form an orthonormal basis in $L^2[0, 1]$.

The following well-known properties of the Legendre Polynomials can be found, inter alia, in [16], [17]:

$$\int_{-1}^{1} (P_k(x))^2 \mathrm{d}x = \frac{2}{2k + 1} \tag{7}$$

$$(k + 1)P_{k+1}(x) = (2k + 1)xP_k(x) - kP_{k-1}(x) \tag{8}$$

$$(1 - x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_k(x) = -kxP_k(x) + kP_{k-1}(x) \tag{9}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left((1 - x^2)\frac{\mathrm{d}}{\mathrm{d}x}P_k(x)\right) = -k(1 + k)P_k(x) \tag{10}$$

$$(2k + 1)P_k(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(P_{k+1}(x) - P_{k-1}(x)\right) \tag{11}$$

The following properties of the Shifted Legendre Polynomials are easily derived from the properties of the Legendre Polynomials by substituting (5) into (7-11).

$$\int_0^1 (P_k^*(x))^2 \mathrm{d}x = \frac{1}{2k+1} \tag{12}$$

$$xP_k^*(x) = \frac{1}{2}\left(\frac{kP_{k-1}^*(x)}{1+2k} + P_k^*(x) + \frac{(1+k)P_{k+1}^*(x)}{1+2k}\right) \tag{13}$$

$$x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}P_k^*(x) = \frac{k(1+k)}{2(1+2k)}\left(P_{k-1}^*(x) - P_{k+1}^*(x)\right) \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)\frac{\mathrm{d}}{\mathrm{d}x}P_k^*(x)\right) = -k(1+k)P_k^*(x) \tag{15}$$

2.2 The Legendre Functions of the second kind

As is well-known, the Legendre Polynomial $P_k(x)$ is not the only solution for the differential equation (10) in the interval $[-1,1]$; the other solution is the Legendre Function of the second kind $Q_k(x)$, defined by the formula

$$Q_k(x) = \frac{1}{2}\int_{-1}^1 (x-t)^{-1} P_k(t)\mathrm{d}t, \tag{16}$$

where P_k is the Legendre Polynomial.

Having defined the Shifted Legendre Polynomials, we find it convenient to similarly define the *Shifted Legendre Function of the second kind* of degree k, which we will be denoting by Q_k^*, by the formula

$$Q_k^*(x) = Q_k(2x-1). \tag{17}$$

The following identities can be found, for example, in [16], [17]:

$$Q_k(z) = (-1)^{k+1} Q_k(-z), \tag{18}$$

$$Q_k(z) = \int_0^\infty \frac{\mathrm{d}\varphi}{\left(z+\sqrt{z^2-1}\cosh(\varphi)\right)^{k+1}}. \tag{19}$$

By (6), (16), (18) and (17),

$$\int_0^1 (x+y)^{-1}\overline{P_k^*}(x)\mathrm{d}x = 2(-1)^k Q_k^*(y+1)\sqrt{2k+1} \tag{20}$$

for all $y > 0$.

2.3 Singular Value Decomposition (SVD) of integral operators

The Singular Value Decomposition (SVD) of integral operators and its key properties are summarized in the following theorem, which can be found, for example, in [18].

Theorem 2.1. *Suppose that the function $K : [c,d] \times [a,b] \to \mathbb{R}$ is square integrable, and $T : L^2[a,b] \to L^2[c,d]$ is defined by the formula*

$$(T(f))(x) = \int_a^b K(x,t) f(t) \mathrm{d}t. \tag{21}$$

Then, there exist two orthonormal sequences of functions $u_0, u_1, ...$, where $u_n : [a,b] \to \mathbb{R}$ and $v_0, v_1, ...$, where $v_n : [c,d] \to \mathbb{R}$, and a sequence $\alpha_0, \alpha_1, ... \in \mathbb{R}$, where $\alpha_0 \geq \alpha_1 \geq ... \geq 0$, such that

$$(T(f))(x) = \sum_{n=0}^{\infty} \alpha_n \left(\int_a^b u_n(t) f(t) \mathrm{d}t \right) v_n(x) \tag{22}$$

for any $f \in L^2[a,b]$. The sequence $\alpha_0, \alpha_1, ...$ is uniquely determined by K.

The functions $u_0, u_1, ...$ are referred to as the *right singular functions*, the functions $v_0, v_1, ...$ are referred to as the *left singular functions*, and the values $\alpha_0, \alpha_1, ...$ are referred to as the *singular values* of the operator T. Together, the right singular functions, the left singular functions and the singular values are referred to as the SVD of the operator T.

It immediately follows from Theorem 2.1 that

$$T(u_n) = \alpha_n v_n, \tag{23}$$

$$T^*(v_n) = \alpha_n u_n. \tag{24}$$

Observation 2.2. The right singular functions $u_0, u_1, ..$ of T are eigenfunctions of the operator $T^* \circ T$ and the left singular functions $v_0, v_1, ...$ are eigenfunctions of the operator $T \circ T^*$; the singular values $\alpha_0, \alpha_1, ...$ of T are the square roots of the eigenvalues of $T^* \circ T$ and $T \circ T^*$. In other words, for every $n = 0, 1, ...$,

$$((T^* \circ T)(u_n))(\tau) = \int_c^d \overline{K(x,\tau)} \left(\int_a^b K(x,t) u_n(t) \mathrm{d}t \right) \mathrm{d}x = \alpha_n^2 u_n(\tau) \tag{25}$$

and

$$((T \circ T^*)(v_n))(\xi) = \int_a^b K(\xi,t) \left(\int_c^d \overline{K(x,t)} v_n(x) \mathrm{d}x \right) \mathrm{d}t = \alpha_n^2 v_n(\xi) \tag{26}$$

Remark 2.3. The function K can be expressed using the singular functions as follows (see [18]),

$$K(x,t) = \sum_{n=0}^{\infty} v_n(x)\alpha_n u_n(t) \tag{27}$$

and it can be approximated by truncation of small singular values (also see [18]):

$$K(x,t) \simeq \sum_{n=0}^{p} v_n(x)\alpha_n u_n(t) \tag{28}$$

3 Analytical apparatus

3.1 Bounds on the Legendre functions of the second kind

For any $x > 1$, the function $Q_k(x)$ (defined in (16)) decays rapidly as k grows. More formally, for any $\delta > 0$ and $k = 0, 1, ...$, there is a uniform bound on $|Q_k(x)|$, where $x \geq 1+\delta$; the bound decreases superalgebraically as k grows. The following lemma provides an explicit bound.

Lemma 3.1. *Suppose that $\delta > 0$; then, for all $y \geq 0$,*

$$|Q_k(1+\delta+y)| < \left(\log\left(2\frac{1+\tilde{\delta}}{\tilde{\delta}}\right)+1\right)\left(\frac{1}{1+\tilde{\delta}}\right)^{k+1}, \tag{29}$$

where

$$\tilde{\delta} = \sqrt{(1+\delta)^2-1} \tag{30}$$

and Q_k is defined in (16).

Proof. By (16),

$$|Q_k(1+\delta+y)| = \int_0^\infty \frac{\mathrm{d}\varphi}{\left((1+\delta+y)+\cosh(\varphi)\sqrt{(1+\delta+y)^2-1}\right)^{k+1}}. \tag{31}$$

Since $(1+\delta+y) \geq (1+\delta)$,

$$|Q_k(1+\delta+y)| \leq \int_0^\infty \frac{\mathrm{d}\varphi}{\left((1+\delta)+\cosh(\varphi)\sqrt{(1+\delta)^2-1}\right)^{k+1}}. \tag{32}$$

Since $\delta > 0$, clearly $\tilde{\delta} > 0$ and therefore,

$$|Q_k(1+\delta+y)| < \int_0^\infty \frac{\mathrm{d}\varphi}{\left(1+\tilde{\delta}\cosh(\varphi)\right)^{k+1}}. \tag{33}$$

Introducing the notation

$$\nu = \log\left(2\frac{1+\tilde{\delta}}{\tilde{\delta}}\right), \tag{34}$$

we break the integral in (33) into integrals on the two intervals $[0,\nu)$ and $[\nu,\infty)$:

$$|Q_k(1+\delta+y)| < \int_0^\nu \frac{\mathrm{d}\varphi}{\left(1+\tilde{\delta}\cosh(\varphi)\right)^{k+1}} + \int_\nu^\infty \frac{\mathrm{d}\varphi}{\left(1+\tilde{\delta}\cosh(\varphi)\right)^{k+1}}. \tag{35}$$

Clearly,

$$\frac{1}{\left(1+\tilde{\delta}\cosh(\varphi)\right)^{k+1}} \le \frac{1}{\left(1+\tilde{\delta}\right)^{k+1}}, \tag{36}$$

and

$$\frac{1}{\left(1+\tilde{\delta}\cosh(\varphi)\right)^{k+1}} \le \frac{1}{\left(\tilde{\delta}\exp(\varphi)/2\right)^{k+1}}, \tag{37}$$

so that,

$$|Q_k(1+\delta+y)| < \frac{\nu}{\left(1+\tilde{\delta}\right)^{k+1}} + \int_{\nu}^{\infty} \frac{\mathrm{d}\varphi}{\left(\tilde{\delta}\exp(\varphi)/2\right)^{k+1}}. \tag{38}$$

Substituting (34) into (38), we obtain

$$|Q_k(1+\delta+y)| < \frac{1}{\left(1+\tilde{\delta}\right)^{k+1}} \left(\log\left(2\frac{1+\tilde{\delta}}{\tilde{\delta}}\right) + \frac{1}{k+1}\right), \tag{39}$$

and from it, we obtain (29).

□

Corollary 3.2. *By, (17) and (29),*

$$|Q_k^*(1+\delta/2+y)| < \left(\log\left(2\frac{1+\tilde{\delta}}{\tilde{\delta}}\right)+1\right)\left(\frac{1}{1+\tilde{\delta}}\right)^{k+1} \tag{40}$$

where

$$\tilde{\delta} = \sqrt{(1+\delta)^2-1}, \tag{41}$$

$\delta, y > 0$ *and* Q_k^* *is defined in (17).*

3.2 The Truncated Laplace Transform

Definition 3.3. *For any pair of real numbers* a, b, *such that* $0 < a < b < \infty$, *the* Truncated Laplace Transform $\mathcal{L}_{a,b}$ *is the linear mapping* $L^2[a,b] \to L^2[0,\infty)$, *defined by the formula*

$$\left(\mathcal{L}_{a,b}(f)\right)(\omega) = \int_a^b e^{-t\omega} f(t)\mathrm{d}t, \tag{42}$$

Obviously, the adjoint of $\mathcal{L}_{a,b}$ is

$$\left((\mathcal{L}_{a,b})^*(g)\right)(t) = \int_0^\infty e^{-t\omega} g(\omega)\mathrm{d}\omega. \tag{43}$$

The operators $\mathcal{L}_{a,b}$ and $(\mathcal{L}_{a,b})^*$ are compact, the range of $(\mathcal{L}_{a,b})^*$ is dense in $L^2[a,b]$ and the range of $\mathcal{L}_{a,b}$ is dense in $L^2[0,\infty)$ (see, for example, [1]).

3.3 The SVD of the Truncated Laplace Transform

By Theorem 2.1, there exist an orthonormal sequence of right singular functions $u_0, u_1, ... \in L^2[a,b]$, an orthonormal sequence of left singular functions $v_0, v_1, ... \in L^2[0,\infty)$ and a sequence of real numbers $\alpha_0, \alpha_1, ... \in \mathbb{R}$ such that

$$\left(\mathcal{L}_{a,b}(f)\right)(\omega) = \sum_{n=0}^{\infty} \alpha_n \left(\int_a^b u_n(t) f(t) \mathrm{d}t \right) v_n(\omega), \tag{44}$$

and for all $n = 0, 1, ...,$

$$\mathcal{L}_{a,b}(u_n) = \alpha_n v_n, \tag{45}$$

$$\left(\mathcal{L}_{a,b}\right)^*(v_n) = \alpha_n u_n, \tag{46}$$

and

$$\alpha_n \geq \alpha_{n+1} \geq 0. \tag{47}$$

Remark 3.4. The multiplicity of the singular values of $\mathcal{L}_{a,b}$ is one (see [9]); in other words, for all $n = 0, 1, ...$

$$\alpha_n > \alpha_{n+1}. \tag{48}$$

Remark 3.5. According to Observation 2.2, the right singular functions $u_0, u_1, ...$ of $\mathcal{L}_{a,b}$ are eigenfunctions of the integral operator $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b} : L^2[a,b] \to L^2[a,b]$ given by the formula

$$\left(\left(\left(\mathcal{L}_{a,b}\right)^* \circ \mathcal{L}_{a,b}\right)(f)\right)(t) = \int_0^{\infty} e^{-\omega t} \left(\int_a^b e^{-\omega s} f(s) \mathrm{d}s \right) \mathrm{d}\omega = \int_a^b \frac{1}{t+s} f(s) \mathrm{d}s, \tag{49}$$

and the corresponding eigenvalues of $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$ are $\alpha_0^2, \alpha_1^2, ...,$ where α_n is the singular value of $\mathcal{L}_{a,b}$ associated with the right singular function u_n. In other words,

$$\left(\left(\left(\mathcal{L}_{a,b}\right)^* \circ \mathcal{L}_{a,b}\right)(u_n)\right)(t) = \int_a^b \frac{1}{t+s} u_n(s) \mathrm{d}s = \alpha_n^2 u_n(t). \tag{50}$$

Similarly, the left singular functions v_n of $\mathcal{L}_{a,b}$ are eigenfunctions of the integral operator $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^* : L^2[0,\infty) \to L^2[0,\infty)$ given by the formula

$$\begin{aligned} &\left(\left(\mathcal{L}_{a,b} \circ \left(\mathcal{L}_{a,b}\right)^*\right)(g)\right)(\omega) = \\ &= \int_a^b e^{-\omega t} \left(\int_0^{\infty} e^{-\rho t} g(\rho) \mathrm{d}\rho \right) \mathrm{d}t = \int_0^{\infty} \frac{e^{-a(\omega+\rho)} - e^{-b(\omega+\rho)}}{\omega+\rho} g(\rho) \mathrm{d}\rho, \end{aligned} \tag{51}$$

and the corresponding eigenvalues $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$ are $\alpha_0^2, \alpha_1^2, ...$ In other words,

$$\left(\left(\mathcal{L}_{a,b} \circ \left(\mathcal{L}_{a,b}\right)^*\right)(v_n)\right)(\omega) = \int_0^{\infty} \frac{e^{-a(\omega+\rho)} - e^{-b(\omega+\rho)}}{\omega+\rho} v_n(\rho) \mathrm{d}\rho = \alpha_n^2 v_n(\omega). \tag{52}$$

3.4 The differential operators $\tilde{D}_t$ and $\hat{D}_\omega$ associated with the singular functions of $\mathcal{L}_{a,b}$

In this subsection we summarize several properties related to the differential operator $\tilde{D}_t$, defined by the formula

$$\left(\tilde{D}_t(f)\right)(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left((t^2-a^2)(b^2-t^2)\frac{\mathrm{d}}{\mathrm{d}t}f(t)\right) - 2(t^2-a^2)f(t), \tag{53}$$

where $f \in C^2[a,b]$; and properties related to the differential operator $\hat{D}_\omega$, defined by the formula

$$\begin{aligned}&\left(\hat{D}_\omega(f)\right)(\omega) = \\ &= -\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left(\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}f(\omega)\right) + (a^2+b^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}f(\omega)\right) + \left(-a^2b^2\omega^2+2a^2\right)f(\omega),\end{aligned} \tag{54}$$

where $f \in C^4[0,\infty) \cap L^2[0,\infty)$. For a derivation of these properties, see [9].

Theorem 3.6. *The differential operator $\tilde{D}_t$, defined in (53), commutes with the integral operator $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$, (specified in (49)) in $L^2[a,b]$. In other words,*

$$\tilde{D}_t \circ ((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}) = ((\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}) \circ \tilde{D}_t \tag{55}$$

Theorem 3.7. *The differential operator $\hat{D}_\omega$, defined in (54), commutes with the integral operator $\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^*$, (specified in (51)) in $L^2[0,\infty)$. In other words,*

$$\mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^* \circ \hat{D}_\omega = \hat{D}_\omega \circ \mathcal{L}_{a,b} \circ (\mathcal{L}_{a,b})^* \tag{56}$$

Theorem 3.8. *The right singular functions $u_0, u_1, \ldots$ (defined in (45)) of $\mathcal{L}_{a,b}$ (defined in (42)) are also the eigenfunctions of $\tilde{D}_t$.*

Theorem 3.9. *The left singular functions $v_0, v_1, \ldots$ (defined in (46)) of $\mathcal{L}_{a,b}$ (defined in (42)) are also the eigenfunctions of $\hat{D}_\omega$.*

We denote the eigenvalues of the differential operator $\tilde{D}_t$ by $\tilde{\chi}_0, \tilde{\chi}_1, \ldots$, and the eigenvalues of the differential operator $\hat{D}_\omega$ by $\chi_0^*, \chi_1^*, \ldots$. By Theorem 3.8, the singular function u_n is the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((t^2-a^2)(b^2-t^2)\frac{\mathrm{d}}{\mathrm{d}t}u_n(t)\right) - 2(t^2-a^2)u_n(t) = \tilde{\chi}_n u_n(t), \tag{57}$$

and by Theorem 3.9, the left singular function v_n is the solution to the differential equation

$$\begin{aligned}&-\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}\left(\omega^2\frac{\mathrm{d}^2}{\mathrm{d}\omega^2}v_k(\omega)\right) + (a^2+b^2)\frac{\mathrm{d}}{\mathrm{d}\omega}\left(\omega^2\frac{\mathrm{d}}{\mathrm{d}\omega}v_k(\omega)\right) + (-a^2b^2\omega^2+2a^2)v_k(\omega) = \\ &= \chi_k^* v_k(\omega).\end{aligned} \tag{58}$$

Remark 3.10. The singular values α_n (defined in (45)) of the integral operator $\mathcal{L}_{a,b}$ are known to decay exponentially as n grows; consequently, the direct numerical computation of the singular functions of $\mathcal{L}_{a,b}$ beyond the first few singular functions is impossible.

The differential operators $\tilde{D}_t$ and $\hat{D}_\omega$ are advantageous in the numerical treatment of the singular functions u_n and v_n because their eigenvalues increase with n, and because the differential operators can be treated using numerical tools developed for differential equations. Such tools are developed below in Sections 3.4, 3.5, 3.6, 3.7 and 3.8 and used to construct the SVD of the operator $\mathcal{L}_{a,b}$ in Section 4.

3.5 The operator T_γ and the function ψ_n

The right singular functions u_n (see (45)) of $\mathcal{L}_{a,b}$ (see (42)) are defined of the interval $[a, b]$; we find it convenient to shift this interval to the interval $[0, 1]$.

We introduce the operator $T_\gamma : [0, 1] \to [0, \infty)$, defined be the formula

$$(T_\gamma(f))\,(\tilde{\omega}) = \int_0^1 e^{-\tilde{\omega}\left(x+\frac{1}{\gamma-1}\right)} f(x)\mathrm{d}x. \tag{59}$$

This operator is related to the operator $\mathcal{L}_{a,b}$, where

$$\gamma = b/a \tag{60}$$

by a the changes of variables

$$x = \frac{t-a}{b-a}\ , \qquad t = a + (b-a)x, \tag{61}$$

and

$$\tilde{\omega} = \omega(b-a)\ , \qquad \omega = \frac{\tilde{\omega}}{b-a}. \tag{62}$$

We denote the singular values of T_γ by $\tilde{\alpha}_0, \tilde{\alpha}_1, ...$, the right singular functions of T_γ by $\psi_0, \psi_1, ...$ and the left singular functions of T_γ by $\tilde{v}_0, \tilde{v}_1, ...$.

Suppose now that $0 < a < b < \infty$. Then a simple calculation shows that for any $n = 0, 1, 2, ...$

$$\psi_n(x) = \sqrt{b-a}\ u_n(a + (b-a)x), \tag{63}$$

$$\tilde{v}_n(\tilde{\omega}) = \frac{1}{\sqrt{b-a}} v_n(\tilde{\omega}/(b-a)), \tag{64}$$

(up to the ambiguity in sign) and

$$\tilde{\alpha}_n = \alpha_n, \tag{65}$$

where α_n, u_n and v_n are a singular value, right singular function and left singular function of $\mathcal{L}_{a,b}$, and where $\tilde{\alpha}_n, \psi_n$ and $\tilde{v}_n$ are a singular value, right singular function and left singular function of T_γ and $\gamma = b/a$.

The operator $T_\gamma^* \circ T_\gamma$ is defined by the formula

$$\left(\left(T_\gamma^* \circ T_\gamma\right)(f)\right)(x) = \int_0^1 \frac{1}{x+y+\beta} f(y) \mathrm{d}y, \tag{66}$$

with $f \in L^2[0,1]$, and β is defined by the the formula:

$$\beta = \frac{2}{\gamma - 1} = \frac{2a}{b-a}. \tag{67}$$

By Observation 2.2, the right singular functions $\psi_0, \psi_1, ...$ of T_γ are the eigenfunctions of the integral operator $T_\gamma^* \circ T_\gamma$. Clearly, $T_\gamma^* \circ T_\gamma$ has the same eigenvalues as $(\mathcal{L}_{a,b})^* \circ \mathcal{L}_{a,b}$:

$$\left(\left(T_\gamma^* \circ T_\gamma\right)(\psi_n)\right)(x) = \int_0^1 \frac{1}{x+y+\beta} \psi_n(y) \mathrm{d}y = \alpha_n^2 \psi_n(x). \tag{68}$$

Similarly, by (53) and (61), $\psi_0, \psi_1, ...$ are the eigenfunctions of the differential operator D_x, defined by the formula

$$\left(D_x(f)\right)(x) = \frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)(\beta+x)(\beta+1+x)\frac{\mathrm{d}}{\mathrm{d}x} f(x)\right) - 2x(x+\beta) f(x). \tag{69}$$

In other words, ψ_n is the solution to the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1-x)(\beta+x)(\beta+1+x)\frac{\mathrm{d}}{\mathrm{d}x}\psi_n(x)\right) - 2x(x+\beta)\psi_n(x) - \chi_n \psi_n(x) = 0, \tag{70}$$

with $\chi_0, \chi_1, ...$ the eigenvalues of D_x.

Remark 3.11. A simple computation shows that the eigenvalues $\chi_0, \chi_1, ...$ of the operator D_x are related to the eigenvalues $\tilde{\chi}_0, \tilde{\chi}_1, ...$ of the operator $\tilde{D}_t$, defined in (53) by the formula

$$\tilde{\chi}_n = (b-a)^2 \chi_n. \tag{71}$$

Remark 3.12. The operator T_γ is determined by the single parameter γ; therefore, the singular value decomposition of the operator $\mathcal{L}_{a,\gamma a}$ is determined by γ, in the following sense. The sequence of singular values $\alpha_0, \alpha_1, ...$ of the truncated Laplace transform $\mathcal{L}_{a,\gamma a}$ depends only on γ, and is independent of the value of a. If $\gamma = \frac{b}{a} = \frac{\tilde{b}}{\tilde{a}}$, then the the sequences of right and left singular functions of $\mathcal{L}_{a,b}$ are identical to those of the truncated Laplace transform $\mathcal{L}_{\tilde{a},\tilde{b}}$ up to trivial scaling.

3.6 Expansion of ψ_n in the basis of Legendre Polynomials

Let f be a smooth function in $L^2[0,1]$, then f can be expressed in the basis of Normalized Shifted Legendre Polynomials $\overline{P_k^*}$ (defined in (6)); let $h = (h_0, h_1, ...)^\top$ be the vector where

$$h_k = \int_0^1 f(x)\overline{P_k^*}(x)\mathrm{d}x, \tag{72}$$

then clearly h is the vector of coefficients in the expansion,

$$f(x) = \sum_{k=0}^{\infty} h_k \overline{P_k^*}(x). \tag{73}$$

We introduce the notation $h^n = (h_0^n, h_1^n, ...)^\top$ for the vector of coefficients of the expansion of the function ψ_n (defined in (63)) in the basis of Normalized Shifted Legendre Polynomials; where the element h_k^n is defined by the formula

$$h_k^n = \int_0^1 \psi_n(x)\overline{P_k^*}(x)\mathrm{d}x, \tag{74}$$

so that

$$\psi_n(x) = \sum_{k=0}^{\infty} h_k^n \overline{P_k^*}(x). \tag{75}$$

3.7 Decay of the coefficients

Since the function ψ_n (defined in (63)) is a smooth solution of a differential equation (specified in (70)), we expect the coefficients h_k^n in the expansion of ψ_n to decay rapidly. In this subsection we provide an estimate for the actual decay.

Lemma 3.13. *Suppose that* $0 < \beta < \infty$. *Then,*

$$\int_0^1 \left(\int_0^1 \frac{1}{x+y+\beta}\overline{P_k^*}(x)\mathrm{d}x\right)^2 \mathrm{d}y \le \left(\frac{2\sqrt{2k+1}}{\left(1+\tilde{\beta}\right)^{k+1}}\left(\log\left(2\frac{1+\tilde{\beta}}{\tilde{\beta}}\right)+1\right)\right)^2, \tag{76}$$

where $\overline{P_k^*}$ *is defined in (6) and*

$$\tilde{\beta} = \sqrt{(1+(2\beta))^2 - 1} = 2\sqrt{\beta(1+\beta)}. \tag{77}$$

Proof. Based on (20),

$$\left|\int_0^1 (x+y+\beta)^{-1}\overline{P_k^*}(x)\mathrm{d}x\right| = 2Q_k^*(y+\beta+1)\sqrt{2n+1}, \tag{78}$$

where Q_k^* is defined in (17). So, by Corollary 3.2,

$$\left|\int_0^1 (x+y+\beta)^{-1}\overline{P_k^*}(x)\mathrm{d}x\right| < \frac{2\sqrt{2k+1}}{\left(1+\tilde{\beta}\right)^{k+1}}\left(\log\left(2\frac{1+\tilde{\beta}}{\tilde{\beta}}\right)+1\right). \tag{79}$$

By squaring (79) and integrating over y, we obtain (76). □

Lemma 3.14. *Suppose that h_k^n is the $k+1$-th coefficient in the expansion of ψ_n, specified in (74); then,*

$$|h_k^n| \le c_n\sqrt{2k+1}\left(\frac{1}{1+\tilde{\beta}}\right)^{k+1}, \tag{80}$$

where, c_n is defined by the formula

$$c_n = 2\alpha_n^{-2}\left(\log\left(2\frac{1+\tilde{\beta}}{\tilde{\beta}}\right)+1\right) = 2\alpha_n^{-2}\left(\log\left(2+\sqrt{\frac{\gamma-1}{2(\gamma+1)}}\right)+1\right), \tag{81}$$

and $\tilde{\beta}$ is defined by the formula

$$\tilde{\beta} = \sqrt{(1+(2\beta)^2)-1} = \sqrt{4\beta(1+\beta)} = 2\sqrt{2}\frac{\sqrt{\gamma+1}}{\gamma-1}. \tag{82}$$

The parameters α_n, β and γ in (81) and (82) are defined in (45), (67) and (60), respectively.

Proof. We substitute (68) into (74) and change the order of integration:

$$\begin{aligned} h_k^n &= \alpha_n^{-2}\int_0^1\int_0^1 \frac{1}{x+y+\beta}\overline{P_k^*}(x)\psi_n(y)\mathrm{d}x\mathrm{d}y = \\ &= \alpha_n^{-2}\int_0^1 \psi_n(y)\left(\int_0^1 \frac{1}{x+y+\beta}\overline{P_k^*}(x)\mathrm{d}x\right)\mathrm{d}y. \end{aligned} \tag{83}$$

By the Cauchy-Schwarz inequality,

$$|h_k^n| \le \alpha_n^{-2}\sqrt{\int_0^1 (\psi_n(y))^2\,\mathrm{d}y}\sqrt{\int_0^1\left(\int_0^1 \frac{1}{x+y+\beta}\overline{P_k^*}(x)\mathrm{d}x\right)^2\mathrm{d}y}. \tag{84}$$

Now, by (76),

$$|h_k^n| \le \alpha_n^{-2}\sqrt{1}\left(\frac{2\sqrt{2k+1}}{\left(1+\tilde{\beta}\right)^{k+1}}\left(\log\left(2\frac{1+\tilde{\beta}}{\tilde{\beta}}\right)+1\right)\right). \tag{85}$$

□

3.8 A matrix representation of the differential operator D_x in the basis of $\overline{P_k^*}$

The purpose of this subsection is to express the differential operator D_x in the basis of Normalized Shifted Legendre Polynomials $\overline{P_k^*}$ as the matrix M described in Lemma 3.15; Theorem 3.16 shows that the matrix M is in fact a five-diagonal matrix; and Corollary 3.17 provides the relation between the eigenvectors of M and the functions ψ_n defined in (63).

Lemma 3.15. *Let f be a smooth function with the expansion $h = (h_0, h_1, ...)^\top$ specified in (73):*

$$f(x) = \sum_{k=0}^{\infty} h_k \overline{P_k^*}(x). \tag{86}$$

Suppose that $\varphi = D_x(f)$, with the expansion $\eta = (H_0, H_1, ...)^\top$ such that

$$\varphi(x) = \sum_{k=0}^{\infty} \eta_k \overline{P_k^*}(x). \tag{87}$$

Then,

$$\eta = Mh, \tag{88}$$

where the matrix elements M_{jk} of M are specified via the formula

$$M_{jk} = \int_0^1 \overline{P_j^*}(x) \left(D_x(\overline{P_k^*})\right)(x)\mathrm{d}x, \tag{89}$$

with $0 \leq j, k < \infty$.

Proof. By the linearity of the differential operator D_x (defined in (69)),

$$\varphi(x) = \left(D_x(f)\right)(x) = \sum_{k=0}^{\infty} h_k \left(D_x(\overline{P_k^*})\right)(x). \tag{90}$$

Combining (86) and (90),

$$\sum_{k=0}^{\infty} \eta_k \overline{P_k^*}(x) = \sum_{k=0}^{\infty} h_k \left(D_x(\overline{P_k^*})\right)(x). \tag{91}$$

Now, by multiplying both sides of (91) by $\overline{P_j^*}$ and integrating, we have

$$\eta_j = \int_0^1 \left(\sum_{k=0}^{\infty} h_k \left(D_x(\overline{P_k^*})\right)(x)\right) \overline{P_j^*}(x)\mathrm{d}x, \tag{92}$$

and by linearity

$$\eta_j = \sum_{k=0}^{\infty} h_k \left(\int_0^1 \overline{P_j^*}(x) \left(D_x(\overline{P_k^*}) \right)(x) \mathrm{d}x \right). \tag{93}$$

□

Theorem 3.16. *For any $k \geq 0$,*

$$\begin{aligned}
&\left(D_x(\overline{P_k^*})\right)(x) = \\
&= -\tfrac{(k-1)^2 k^2}{4\sqrt{2k-3}(2k-1)\sqrt{2k+1}} \overline{P_{k-2}^*}(x) \\
&- \tfrac{k^3(1+\beta)}{\sqrt{2k-1}\sqrt{2k+1}} \overline{P_{k-1}^*}(x) \\
&- \tfrac{\left(-4-6\beta-2k\beta(2+3\beta)+k^2\left(7+12\beta+2\beta^2\right)+\left(2k^3+k^4\right)\left(7+16\beta+8\beta^2\right)\right)}{2(2k-1)(2k+3)} \overline{P_k^*}(x) \\
&- \tfrac{(k+1)^3(1+\beta)}{\sqrt{2k+1}\sqrt{2k+3}} \overline{P_{k+1}^*}(x) \\
&- \tfrac{(k+1)^2(k+2)^2}{4\sqrt{2k+1}(2k+3)\sqrt{2k+5}} \overline{P_{k+2}^*}(x),
\end{aligned} \tag{94}$$

where $\overline{P_k^*}$ is the Normalized Shifted Legendre Polynomial defined in (6) and $\beta = \frac{2a}{b-a}$ is defined in (67).

In other words, M is the five-diagonal matrix

$$\begin{aligned}
M_{k-2,k} &= -\tfrac{(k-1)^2 k^2}{4\sqrt{2k-3}(2k-1)\sqrt{2k+1}} \\
M_{k-1,k} &= -\tfrac{k^3(1+\beta)}{\sqrt{2k-1}\sqrt{2k+1}} \\
M_{k,k} &= -\tfrac{\left(-4-6\beta-2k\beta(2+3\beta)+k^2\left(7+12\beta+2\beta^2\right)+\left(2k^3+k^4\right)\left(7+16\beta+8\beta^2\right)\right)}{2(2k-1)(2k+3)} \\
M_{k+1,k} &= -\tfrac{(k+1)^3(1+\beta)}{\sqrt{2k+1}\sqrt{2k+3}} \\
M_{k+2,k} &= -\tfrac{(k+1)^2(k+2)^2}{4\sqrt{2k+1}(2k+3)\sqrt{2k+5}}.
\end{aligned} \tag{95}$$

Proof. By the definition of D_x in (69),

$$\begin{aligned}
&\left(D_x(P_k^*)\right)(x) = \\
&= \frac{d}{\mathrm{dx}} \left((\beta+x)(\beta+1+x)x(1-x) \frac{d}{\mathrm{dx}} P_k^*(x) \right) - 2x(x+\beta) P_k^*(x).
\end{aligned} \tag{96}$$

Using the chain rule,

$$
\begin{aligned}
&\left(D_x(P_k^*)\right)(x) = \\
=&\left(\frac{d}{\mathrm{dx}}(\beta+x)(\beta+1+x)\right)\left(x(1-x)\frac{d}{\mathrm{dx}}P_k^*(x)\right) \\
&+(\beta+x)(\beta+1+x)\frac{d}{\mathrm{dx}}\left(x(1-x)\frac{d}{\mathrm{dx}}P_k^*(x)\right)-2x(x+\beta)P_k^*(x) = \\
=&(1+2x+2\beta)\left(x(1-x)\frac{d}{\mathrm{dx}}P_k^*(x)\right) \\
&+\left(x^2+x(1+2\beta)+\beta+\beta^2\right)\frac{d}{\mathrm{dx}}\left(x(1-x)\frac{d}{\mathrm{dx}}P_k^*(x)\right) \\
&-2x(x+\beta)P_k^*(x).
\end{aligned}
\tag{97}
$$

Using identities (13), (14) and (15),

$$
\begin{aligned}
&\left(D_x(P_k^*)\right)(x) = \\
&-\frac{(-1+k)^2k^2P_{k-2}^*(x)}{4(-1+2k)(1+2k)} \\
&-\frac{k^3(1+\beta)P_{k-1}^*(x)}{1+2k} \\
&-\frac{\left(-4+7k^2+14k^3+7k^4-6\beta-4k\beta+12k^2\beta+32k^3\beta+16k^4\beta-6k\beta^2+2k^2\beta^2+16k^3\beta^2+8k^4\beta^2\right)}{2(-1+2k)(3+2k)}P_k^*(x) \\
&-\frac{(1+k)^3(1+\beta)P_{k+1}^*(x)}{1+2k} \\
&-\frac{(1+k)^2(2+k)^2P_{k+2}^*(x)}{4(1+2k)(3+2k)}.
\end{aligned}
\tag{98}
$$

Finally, substituting (6) into (98) gives (94). □

Corollary 3.17. *Suppose that $h^n = (h_0^n, h_1^n, ...)^\top$ is the vector of coefficients defined in (74), in the expansion of the function $\psi_n(x)$ defined in (63); then, h^n is the $n+1$-th eigenvector of M :*

$$
Mh^n = \chi_n h^n, \tag{99}
$$

where M is the five-diagonal matrix (95), χ_n are the eigenvalues of the differential operator D_x, and $k = 0, 1, 2....$

Proof. By (70), $\psi_n(x)$ is an eigenfunction of D_x, with the eigenvalue χ_n:

$$
\left(D_x(\psi_n)\right)(x) = \chi_n\psi_n(x), \tag{100}
$$

so that

$$\left(D_x\left(\sum_{k=0}^{\infty} h_k^n \overline{P_k^*}\right)\right)(x) = \left(D_x(\psi_n)\right)(x) = \chi_n \sum_{k=0}^{\infty} h_k^n \overline{P_k^*}(x). \tag{101}$$

Therefore, by Lemma 3.15, we obtain (99). □

3.9 A relation between u_n,u_m, and the ratio α_n/α_m

Lemma 3.18. *For any $n, m = 0, 1, ...,$*

$$\frac{\alpha_m^2}{\alpha_n^2} = \frac{\int_a^b u_n'(t)u_m(t)\mathrm{d}t}{\int_a^b u_n(t)u_m'(t)\mathrm{d}t}, \tag{102}$$

if the integrals are not 0; where u_n and u_m are right singular functions (defined in (45)) and α_n and α_m are the singular values (defined in (45)) of $\mathcal{L}_{a,b}$ (defined in (42)).

Similarly,

$$\frac{\alpha_m^2}{\alpha_n^2} = \frac{\int_0^1 \psi_n'(x)\psi_m(x)\mathrm{d}x}{\int_0^1 \psi_n(x)\psi_m'(x)\mathrm{d}x} \tag{103}$$

if the integrals are not 0; where ψ_n and ψ_m are defined in (63).

Proof. We recall from (46) that

$$u_n(t) = \frac{1}{\alpha_n}\left(\mathcal{L}^*(v_n)\right)(t) = \frac{1}{\alpha_n}\int_0^\infty e^{-\omega t} v_n(\omega)\mathrm{d}\omega. \tag{104}$$

Therefore, the derivative of $u_n(t)$ is

$$u_n'(t) = \frac{1}{\alpha_n}\int_0^\infty (-\omega)e^{-\omega t} v_n(\omega)\mathrm{d}\omega. \tag{105}$$

We multiply both sides of the expression by $u_m(t)$, integrate both sides, and change the order of integration:

$$\int_a^b u_n'(t)u_m(t)\mathrm{d}t = \frac{1}{\alpha_n}\int_a^b \left(\int_0^\infty (-\omega)e^{-\omega t} v_n(\omega)\mathrm{d}\omega\right) u_m(t)\mathrm{d}t. \tag{106}$$

By rearranging the result, we obtain

$$\int_a^b u_n'(t)u_m(t)\mathrm{d}t = \frac{\alpha_m}{\alpha_n}\int_0^\infty (-\omega)v_n(\omega)v_m(\omega)\mathrm{d}\omega. \tag{107}$$

m and n are clearly interchangeable, so that

$$\int_0^\infty (-\omega)v_n(\omega)v_m(\omega)\mathrm{d}\omega = \frac{\alpha_m}{\alpha_n}\int_a^b u_m'(t)u_n(t)\mathrm{d}t. \tag{108}$$

By substituting (108) into (107), we obtain (102). Substituting (63) into (102) we obtain (103).
□

4 Algorithms

4.1 Computing the right singular function u_n

In this section we introduce an algorithm for the numerical evaluation of $u_n(t)$, the $n+1$-th right singular function (defined in (45)) of $\mathcal{L}_{a,b}$ (the operator defined in (42)).

Step 1: Compute h^n, the $n+1$-th eigenvector of the matrix M, defined in (95).

Step 2: Compute the function $\psi_n(x)$ from h^n, using the expansion specified in (75).

Step 3: Obtain $u_n(t)$ from $\psi_n(x)$ using (63).

Remark 4.1. For computations in precision ϵ, the vector $h^n = (h_0^n, h_1^n, ...)^\top$ is truncated at K, such that $|h_k^n| \ll \epsilon$ for all $k > K$. By lemma 3.13 the coefficients h_k^n decay rapidly as k grows; the actual position of the last significant coefficient, larger in magnitude than ϵ, is given in Figure 6 and Table 4 in Section 5, for several combinations of γ and n.

4.2 Computing the singular value α_n

In this subsection we present an algorithm for computing the $N+1$ first singular values $\alpha_0, \alpha_1, ...\alpha_N$ (defined in (45)) of $\mathcal{L}_{a,b}$ (the operator defined in (42)).

Step 1: Compute the first singular value α_0, for example, by the formula

$$\alpha_0 = \sqrt{\frac{\int_a^b \frac{1}{t+s} u_0(s) \mathrm{d}s}{u_0(t)}}, \tag{109}$$

derived from (50), where $t \in [a, b]$ and $u_0(t)$ is computed using the algorithm from the previous section.

Step 2: For every $n > 0$, compute α_n from α_{n-1} using the relation in Lemma 3.18, and the functions $u_n(t)$ and $u_{n-1}(t)$ computed using the algorithm in the previous section.

5 Numerical results

In this section we present results of several numerical experiments. The algorithms for computing the right singular functions u_n and singular values α_n of $\mathcal{L}_{a,b}$ (the operator defined in (42)) were implemented in FORTRAN 77, using double precision arithmetic, and compiled using GFORTRAN.

In Figures 1, 2 and 3 we present examples of right singular functions of the operator $\mathcal{L}_{a,b}$, where $a = 1$ and $b = 1.1$, $b = 10$ and $b = 100000$ respectively.

In Figure 4 and Table 1 we present the singular values α_n of the operator $\mathcal{L}_{a,b}$, for several ratios $\gamma = b/a$; α_n depends only on γ and n (see remark 3.12). In table 2 we present several singular values smaller than 10^{-1000}; the Fujitsu compiler with quadruple precision was used in this experiment.

In Figure 5 and Table 3 we present the eigenvalues of the matrix M defined in (95).

In Figure 6 and Table 4 we present for several combinations of γ and n the position of the last significant coefficient h_k^n in the expansion defined in (75), that is larger in magnitude than $\epsilon = 10^{-16}$. In numerical computations, the vectors are truncated around that point (see Remark 4.1).

In figure 5 we present the CPU time required for the computation of the expansion of the 101-st right singular function u_{100} of $\mathcal{L}_{1,\gamma}$, for varying γ; The experiment was performed on a ThinkPad X230 laptop with Intel Core i7-3520 CPU and 16GB RAM.

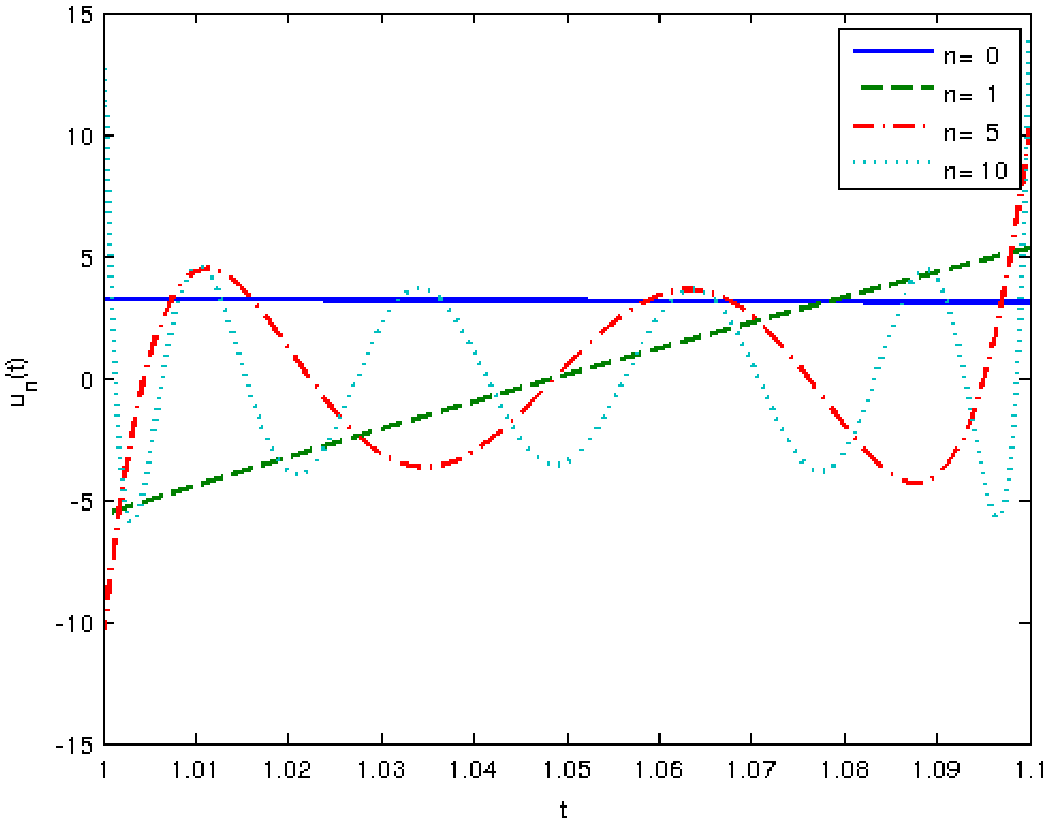

Figure 1: Right Singular functions of $\mathcal{L}_{1,1.1}$.

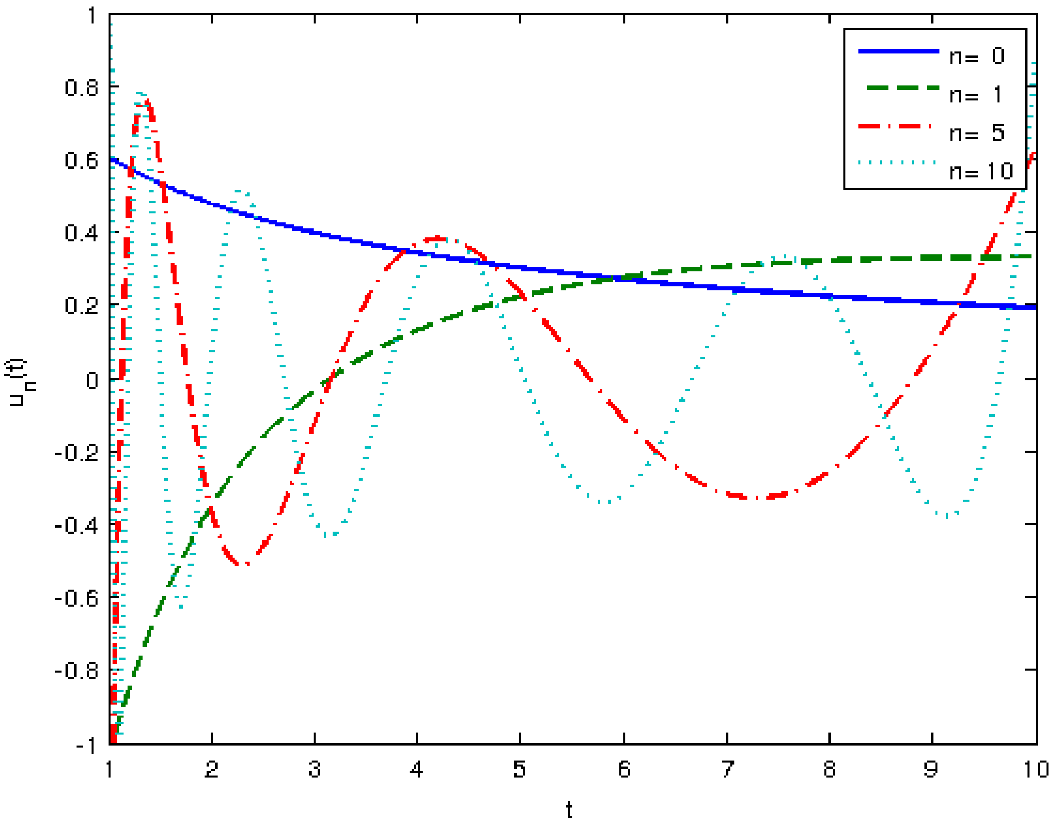

Figure 2: Right Singular functions of $\mathcal{L}_{1,10}$.

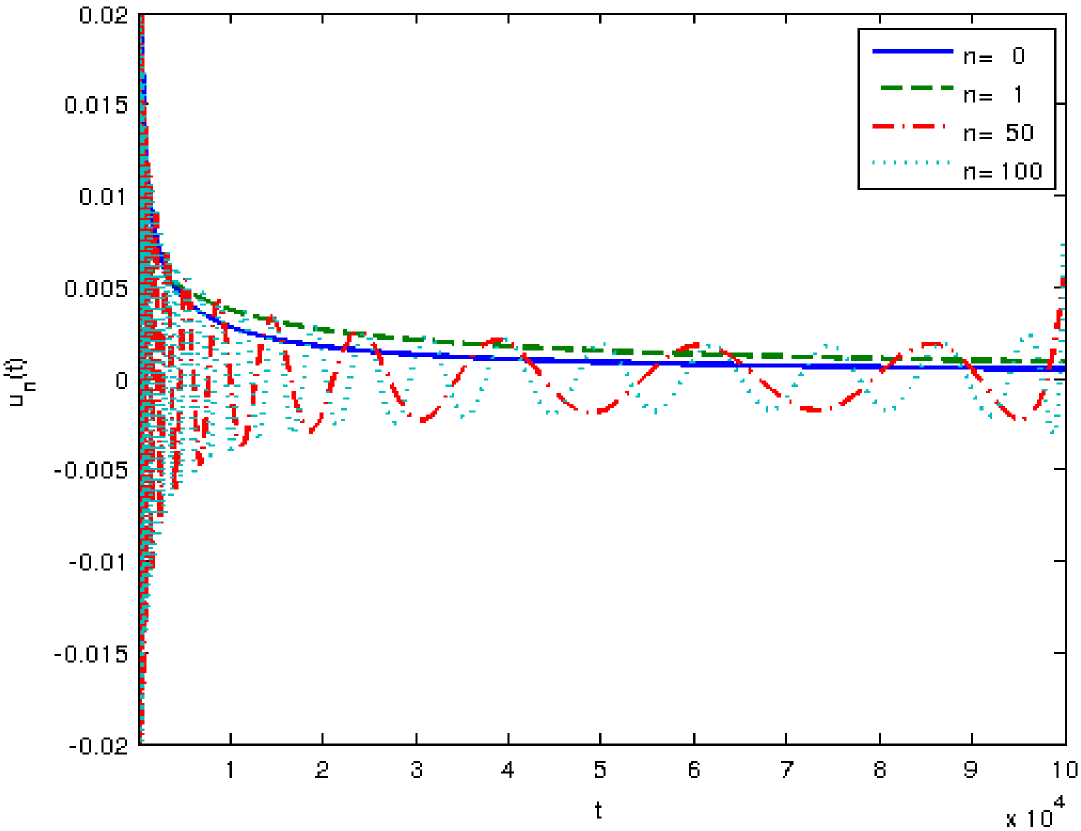

Figure 3: Right Singular functions of $\mathcal{L}_{1,100000}$.

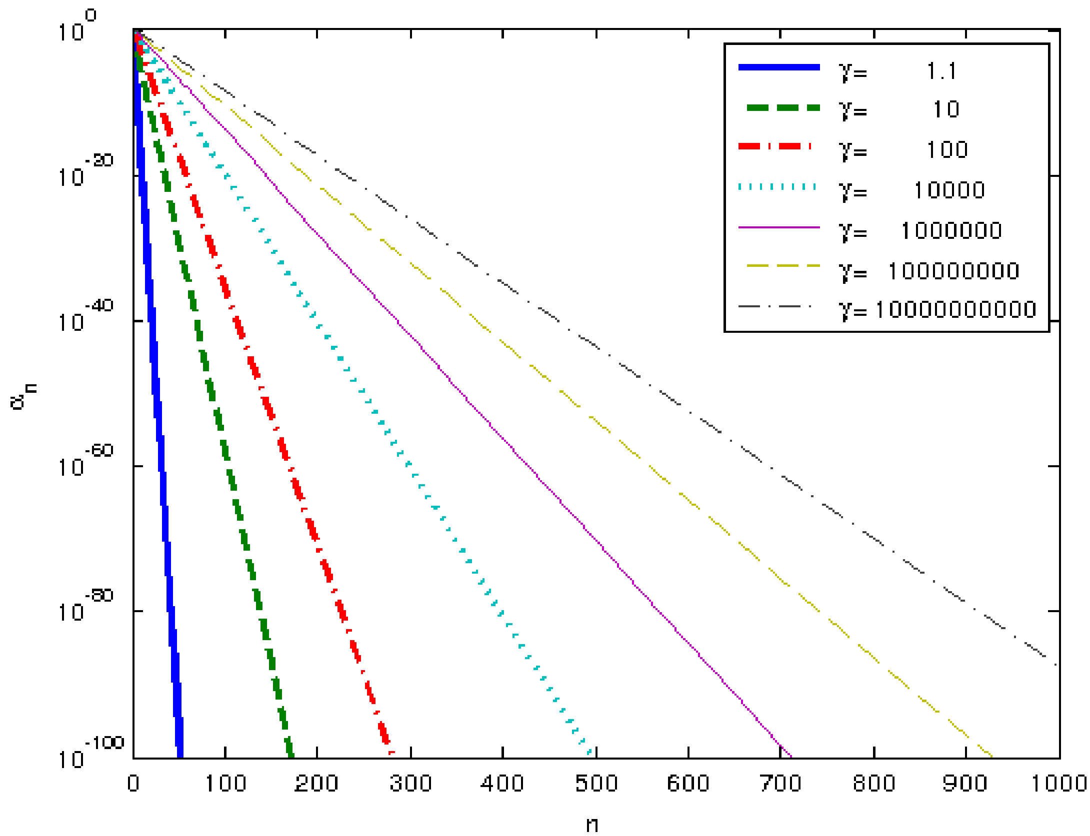

Figure 4: Singular values α_n of $\mathcal{L}_{a,b}$, with $\gamma = b/a$.

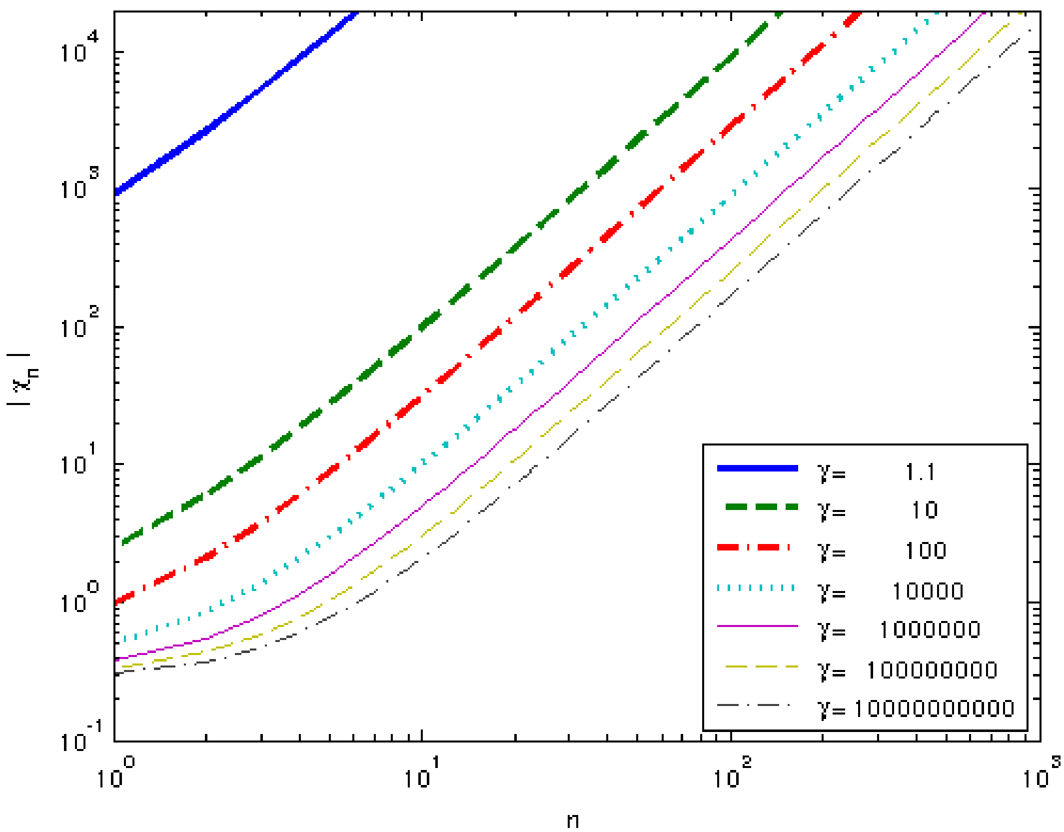

Figure 5: Magnitude of the eigenvalues of the matrix M defined in (95), with $\gamma = \frac{2+\beta}{\beta}$.

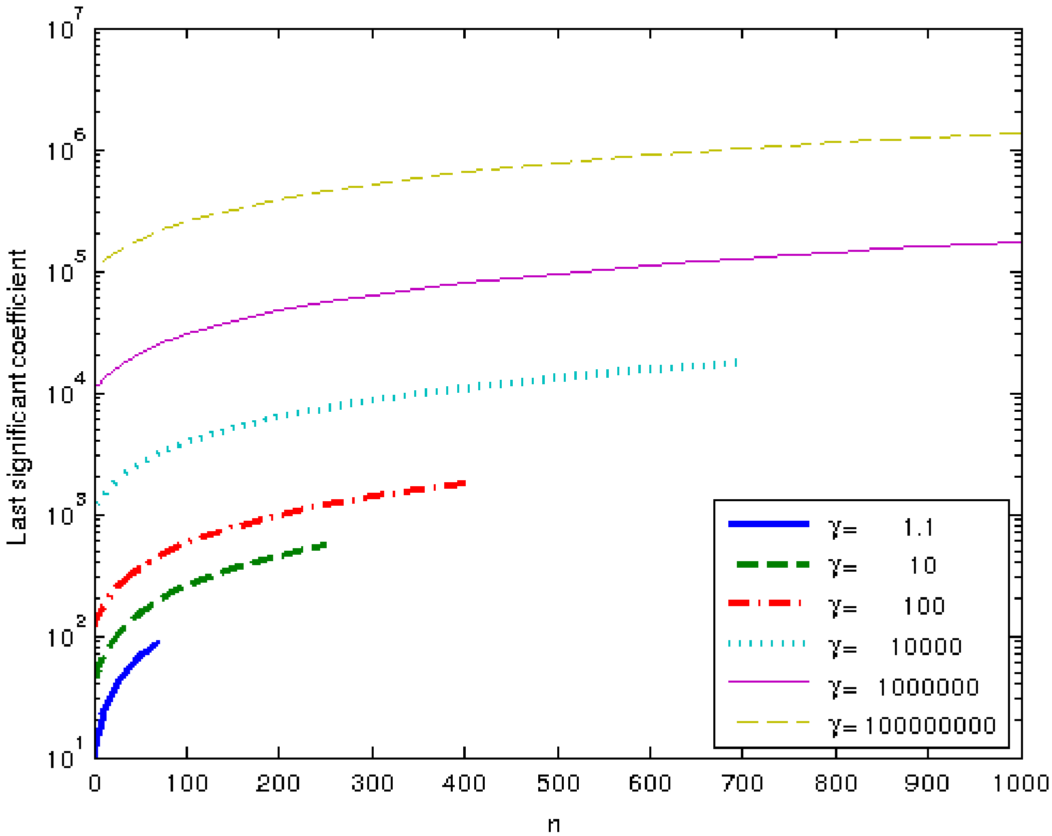

Figure 6: The position of the last significant coefficient larger in magnitude than 10^{-16}.

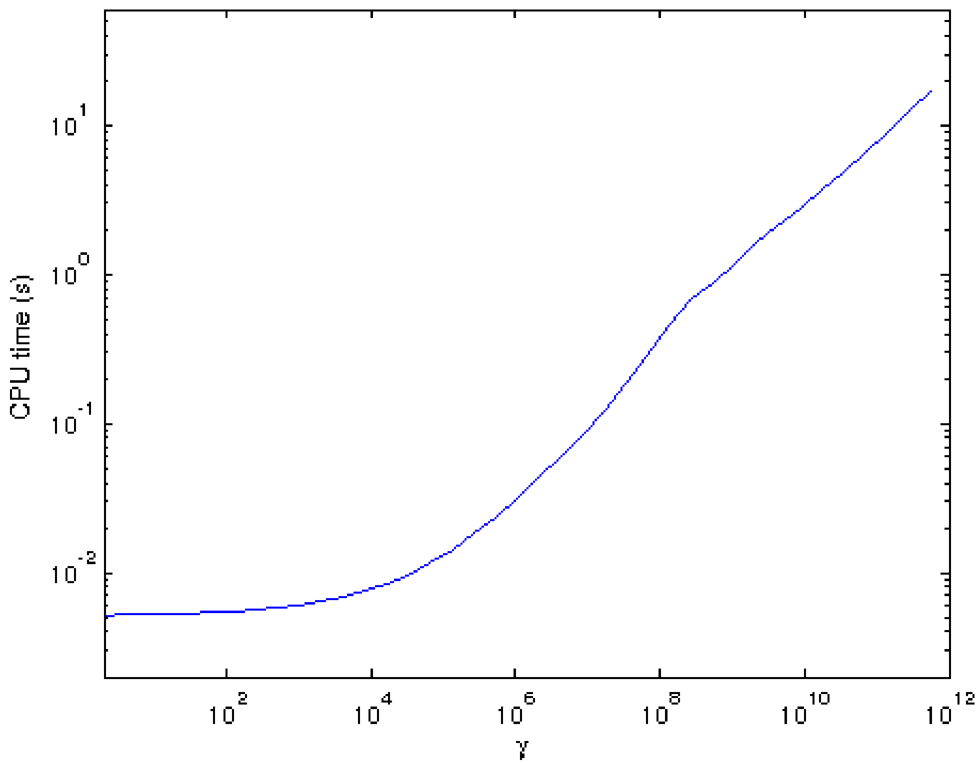

Figure 7: CPU time required for computing the expansion of the 101-st right singular function of $\mathcal{L}_{1,\gamma}$, as a function of γ. The experiment was performed on a ThinkPad X230 laptop with an Intel Core i7-3520 CPU and 16GB RAM.

Table 1: Singular values α_n of $\mathcal{L}_{a,b}$

n	γ =1.1E+00	γ =1.0E+01	γ =1.0E+02	γ =1.0E+04	γ =1.0E+06	γ =1.0E+08	γ =1.0E+10
0	2.18280E − 01	1.02356E + 00	1.31941E + 00	1.55687E + 00	1.64778E + 00	1.69163E + 00	1.71595E + 00
1	3.00227E − 03	3.09878E − 01	6.68211E − 01	1.12288E + 00	1.35702E + 00	1.48763E + 00	1.56644E + 00
2	3.69344E − 05	8.39567E − 02	3.04070E − 01	7.39927E − 01	1.04024E + 00	1.23673E + 00	1.36792E + 00
3	4.46186E − 07	2.23263E − 02	1.35394E − 01	4.73173E − 01	7.71417E − 01	9.95863E − 01	1.16064E + 00
4	5.35677E − 09	5.90020E − 03	5.98904E − 02	2.99697E − 01	5.64351E − 01	7.89136E − 01	9.68344E − 01
10	1.55445E − 20	1.94760E − 06	4.34546E − 04	1.86336E − 02	8.19585E − 02	1.81020E − 01	2.96456E − 01
20	8.99018E − 40	3.00805E − 12	1.15751E − 07	1.77967E − 04	3.20877E − 03	1.50761E − 02	3.95113E − 02
30	5.17974E − 59	4.62827E − 18	3.07157E − 11	1.69324E − 06	1.25143E − 04	1.25067E − 03	5.24436E − 03
40	2.98139E − 78	7.11415E − 24	8.14269E − 15	1.60942E − 08	4.87574E − 06	1.03648E − 04	6.95389E − 04
50	1.71536E − 97	1.09308E − 29	2.15775E − 18	1.52914E − 10	1.89890E − 07	8.58629E − 06	9.21696E − 05
60	9.86744E − 117	1.67918E − 35	5.71671E − 22	1.45257E − 12	7.39392E − 09	7.11151E − 07	1.22140E − 05
70	5.67549E − 136	2.57922E − 41	1.51440E − 25	1.37967E − 14	2.87871E − 10	5.88936E − 08	1.61838E − 06
80		3.96141E − 47	4.01148E − 29	1.31033E − 16	1.12070E − 11	4.87688E − 09	2.14423E − 07
90		6.08399E − 53	1.06254E − 32	1.24442E − 18	4.36274E − 13	4.03827E − 10	2.84079E − 08
100		9.34359E − 59	2.81432E − 36	1.18179E − 20	1.69830E − 14	3.34375E − 11	3.76350E − 09
150		7.98028E − 88	3.66768E − 54	9.12588E − 31	1.51765E − 21	1.30108E − 16	1.53547E − 13
200		6.81449E − 117	4.77880E − 72	7.04566E − 41	1.35593E − 28	5.06159E − 22	6.26325E − 18
250		5.81852E − 146	6.22602E − 90	5.43916E − 51	1.21135E − 35	1.96895E − 27	2.55460E − 22
300			8.11118E − 108	4.19880E − 61	1.08214E − 42	7.65882E − 33	1.04190E − 26
350			1.05669E − 125	3.24121E − 71	9.66687E − 50	2.97907E − 38	4.24935E − 31
400			1.37659E − 143	2.50198E − 81	8.63540E − 57	1.15876E − 43	1.73305E − 35
450				1.93132E − 91	7.71391E − 64	4.50712E − 49	7.06798E − 40
500				1.49081E − 101	6.89071E − 71	1.75309E − 54	2.88254E − 44
550				1.15077E − 111	6.15533E − 78	6.81876E − 60	1.17559E − 48
600				8.88291E − 122	5.49840E − 85	2.65220E − 65	4.79437E − 53
650				6.85676E − 132	4.91158E − 92	1.03159E − 70	1.95528E − 57
700				5.29275E − 142	4.38737E − 99	4.01240E − 76	7.97413E − 62
750					3.91910E − 106	1.56063E − 81	3.25205E − 66
800					3.50081E − 113	6.07013E − 87	1.32627E − 70
850					3.12716E − 120	2.36099E − 92	5.40884E − 75
900					2.79338E − 127	9.18312E − 98	2.20585E − 79
950					2.49523E − 134	3.57179E − 103	8.99599E − 84
1000					2.22890E − 141	1.38925E − 108	3.66878E − 88

Table 2: Examples of singular values α_n smaller than 10^{-1000}

γ	n	α_n
$1.1E+0$	520	$8.70727E-1002$
$1.0E+1$	1721	$3.66934E-1001$
$1.0E+2$	2797	$5.29961E-1001$
$1.0E+3$	3872	$5.71146E-1001$
$1.0E+4$	4946	$9.44191E-1001$
$1.0E+5$	6021	$8.89748E-1001$

Table 3: Eigenvalues of M defined in (95)

n	γ =1.1E+00	γ =1.0E+01	γ =1.0E+02	γ =1.0E+04	γ =1.0E+06	γ =1.0E+08	γ =1.0E+10
0	$-2.04999E+01$	$-6.76941E-01$	$-4.02795E-01$	$-3.07320E-01$	$-2.81024E-01$	$-2.69512E-01$	$-2.63417E-01$
1	$-9.01499E+02$	$-2.46573E+00$	$-9.74898E-01$	$-4.97667E-01$	$-3.78822E-01$	$-3.29667E-01$	$-3.04334E-01$
2	$-2.66350E+03$	$-6.03308E+00$	$-2.09466E+00$	$-8.49745E-01$	$-5.52513E-01$	$-4.34000E-01$	$-3.74332E-01$
3	$-5.30650E+03$	$-1.13857E+01$	$-3.77691E+00$	$-1.37597E+00$	$-8.08518E-01$	$-5.85716E-01$	$-4.75075E-01$
4	$-8.83050E+03$	$-1.85228E+01$	$-6.02070E+00$	$-2.07849E+00$	$-1.14945E+00$	$-7.86707E-01$	$-6.07788E-01$
10	$-4.84755E+04$	$-9.88159E+01$	$-3.12661E+01$	$-9.98975E+00$	$-4.99171E+00$	$-3.04921E+00$	$-2.09685E+00$
20	$-1.85030E+05$	$-3.75381E+02$	$-1.18224E+02$	$-3.72425E+01$	$-1.82310E+01$	$-1.08477E+01$	$-7.23038E+00$
30	$-4.09685E+05$	$-8.30376E+02$	$-2.61283E+02$	$-8.20779E+01$	$-4.00121E+01$	$-2.36778E+01$	$-1.56764E+01$
40	$-7.22440E+05$	$-1.46380E+03$	$-4.60443E+02$	$-1.44496E+02$	$-7.03349E+01$	$-4.15395E+01$	$-2.74348E+01$
50	$-1.12329E+06$	$-2.27565E+03$	$-7.15706E+02$	$-2.24496E+02$	$-1.09199E+02$	$-6.44326E+01$	$-4.25053E+01$
60	$-1.61225E+06$	$-3.26593E+03$	$-1.02707E+03$	$-3.22079E+02$	$-1.56605E+02$	$-9.23571E+01$	$-6.08881E+01$
70	$-2.18930E+06$	$-4.43465E+03$	$-1.39454E+03$	$-4.37244E+02$	$-2.12553E+02$	$-1.25313E+02$	$-8.25830E+01$
80		$-5.78179E+03$	$-1.81810E+03$	$-5.69992E+02$	$-2.77042E+02$	$-1.63301E+02$	$-1.07590E+02$
90		$-7.30736E+03$	$-2.29777E+03$	$-7.20323E+02$	$-3.50073E+02$	$-2.06320E+02$	$-1.35910E+02$
100		$-9.01136E+03$	$-2.83354E+03$	$-8.88236E+02$	$-4.31645E+02$	$-2.54370E+02$	$-1.67541E+02$
150		$-2.02078E+04$	$-6.35392E+03$	$-1.99154E+03$	$-9.67632E+02$	$-5.70094E+02$	$-3.75382E+02$
200		$-3.58650E+04$	$-1.12768E+04$	$-3.53440E+03$	$-1.71716E+03$	$-1.01160E+03$	$-6.66029E+02$
250		$-5.59829E+04$	$-1.76023E+04$	$-5.51683E+03$	$-2.68023E+03$	$-1.57890E+03$	$-1.03948E+03$
300			$-2.53303E+04$	$-7.93882E+03$	$-3.85684E+03$	$-2.27198E+03$	$-1.49574E+03$
350			$-3.44608E+04$	$-1.08004E+04$	$-5.24698E+03$	$-3.09085E+03$	$-2.03480E+03$
400			$-4.49939E+04$	$-1.41015E+04$	$-6.85067E+03$	$-4.03551E+03$	$-2.65667E+03$
450				$-1.78422E+04$	$-8.66790E+03$	$-5.10595E+03$	$-3.36134E+03$
500				$-2.20224E+04$	$-1.06987E+04$	$-6.30218E+03$	$-4.14882E+03$
550				$-2.66422E+04$	$-1.29430E+04$	$-7.62419E+03$	$-5.01910E+03$
600				$-3.17016E+04$	$-1.54008E+04$	$-9.07199E+03$	$-5.97219E+03$
650				$-3.72005E+04$	$-1.80722E+04$	$-1.06456E+04$	$-7.00808E+03$
700				$-4.31390E+04$	$-2.09572E+04$	$-1.23450E+04$	$-8.12678E+03$
750					$-2.40556E+04$	$-1.41701E+04$	$-9.32829E+03$
800					$-2.73676E+04$	$-1.61211E+04$	$-1.06126E+04$
850					$-3.08932E+04$	$-1.81978E+04$	$-1.19797E+04$
900					$-3.46323E+04$	$-2.04003E+04$	$-1.34296E+04$
950					$-3.85849E+04$	$-2.27286E+04$	$-1.49624E+04$
1000					$-4.27511E+04$	$-2.51827E+04$	$-1.65779E+04$

Table 4: The position of the last significant coefficient larger in magnitude than 10^{-16}.

n	γ =1.1E+00	γ =1.0E+01	γ =1.0E+02	γ =1.0E+04	γ =1.0E+06	γ =1.0E+08	γ =1.0E+10
0	9	38	116	1086	10278	97533	925806
1	10	41	122	1128	10620	100569	954053
2	12	44	128	1166	10903	102926	974824
3	13	46	134	1201	11167	105066	993178
4	14	49	140	1236	11420	107097	1010375
10	22	64	172	1429	12823	118236	1103518
20	33	88	222	1727	14971	135166	1243905
30	45	111	269	2009	17007	151171	1376260
40	56	132	314	2282	18972	166617	1503887
50	67	154	359	2548	20888	181665	1628164
60	77	174	403	2809	22766	196407	1749863
70	88	195	446	3066	24613	210901	1869483
80		216	489	3320	26435	225190	1987372
90		236	531	3571	28236	239306	2103793
100		256	574	3820	30018	253271	2218937
150		355	780	5037	38725	321401	2780169
200		453	983	6225	47195	387561	3324466
250		550	1183	7393	55511	452415	3857438
300			1381	8548	63715	516323	4382164
350			1578	9692	71835	579509	4900570
400			1774	10829	79888	642122	5413933
450				11959	87887	704265	5923168
500				13083	95840	766016	6428937
550				14203	103754	827434	6931756
600				15319	111636	888563	7432015
650				16431	119488	949440	7930051
700				17540	127314	1010094	8426106
750					135118	1070551	8920402
800					142902	1130829	9413114
850					150667	1190947	9904395
900					158415	1250918	10394374
950					166148	1310756	10883162
1000					173867	1370471	11370862

6 Conclusions and generalizations

In this paper we have introduced effective algorithms for the evaluation of the right singular functions and singular values of the Truncated Laplace Transform $\mathcal{L}_{a,b}$.

As is evident from Remark 2.3 and the more detailed discussion in [18], the right singular functions of $\mathcal{L}_{a,b}$ are an efficient basis for representing decaying exponentials on the interval $[a, b]$.

An algorithm for the computation of the left singular functions of $\mathcal{L}_{a,b}$, which is the remaining component in the computation of the SVD, will be presented in a future paper. Additional asymptotic properties of the Truncated Laplace Transform and of the associated differential operators will also be discussed in a future paper.

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