On a Differential Equation Arising in Plant Vascular Biology

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February 10, 2006*

Abstract

The purpose of this report is to elaborate on and provide proofs for the technical claims in [5]. In particular, we show Result 1 below and provide and explanation for the simulations of Schema 1 involving the creation of new c-vascular strands.

1 Introduction

Consider an abstract biological system consisting of two types of cells: ground cells and c-vascular cells. Suppose they all lay on a 2D surface where a group of ground cells is completely surrounded by c-vascular cells — such a collection will be referred to as an areole. Assume that each cell produces a hormone at a regular (constant) rate K. The substance is allowed to diffuse though the interfaces of the ground cells with constant D but the c-vascular ones drain it and may be assumed to be sinks for the hormone. Mathematically, the dynamical behavior of the concentration c over the region is given by $c_t = D\nabla^2 c + K$. With time, c_t tends toward zero, so after a long enough interval, the equation (a kind of Poisson equation)

$$\nabla^2 c = -\frac{K}{D} \tag{1}$$

approximates well the state of the dynamical system (here $\nabla^2 c = c_{xx} + c_{yy}$). The distribution of concentration over cells in an areole will be shown to obey the following

Result 1. *Consider an areole and suppose that P is a ground cell which is furthest from a c-vascular boundary. Let Q be a c-vascular cell which is closest to P and denote by L the distance between P and Q. Then*

(a) $c(P)$ is proportional to L^2/D;

(b) the change in c at the interface of Q nearest to P is proportional to L/D.

(c) The difference of c is largest at an interface of the c-vascular boundary, larger than for any ground cell, and is proportional to L/D.

It is possible to create new c-vascular strands by converting some of the ground cells inside an areole to c-vascular. This conversion consists of increasing the diffusion constants.

Schema 1. *Let D_I be the diffusion constant across an interface I and Δc be the concentration difference through I. Then increase D_I to a higher value when $\Delta c > \alpha L_0/D_I$. ($\alpha$ is a constant of proportionality.) Alternatively, the flux $\phi = D_I \Delta c = \alpha L_0$ may be employed.*

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Assuming the validity of Result 1, one sees that the above schema will result in the sequential creation of c-vascular strands. According to part (c), the first cells to exceed a threshold on Δc in a growing areole will be those at the c-vascular boundary of the areole. The change in diffusion efficiency will increase a cell's Δc to the highest value at the interface with the next ground cell of highest relative c. Effectively, this results in a progression along the gradient of concentration ∇c over the areole. Our simulations mimic this behavior by displaying integral curves (in green) started at a local maximum of $|\nabla c|$ on the boundary and following ∇c.

In this report we present a proof of a theorem which allows us to conclude the above. We first introduce the technical definitions and results in Section 2 and then state the technical result in Section 3. Section 4 outlines the ideas of the proof and Section 5 contains the details.

2 Background

A collection of ground cells surrounded by c-vascular cells is called an *areole*. Our technical result will assume that an areole is a discretization of a continuous portion of $\mathbb{R}^2$ which we call a shape.

Definition 1. A *shape* is any subset $\Omega \in \mathbb{R}^2$ which is the closure of a bounded open set and has a boundary $\partial\Omega$ consisting of finitely many smooth curves.

A point $Q \in \partial\Omega$ is *concave* if for any line ℓ locally tangent to Q there is an open ball $B_\varepsilon(Q)$ such that $B_\varepsilon(Q) \cap \ell \cap (\Omega - \partial\Omega) = B_\varepsilon(Q) \cap \ell - \{Q\}$ (i.e. the line segment is inside Ω). If $B_\varepsilon(Q) \cap \ell \cap \Omega \subset \partial\Omega$, then Q is a *convex* point. The boundary has concave (convex) curvature[1] at concave (convex) points.

Definition 2. Let Ω be a shape and $P \in \mathbb{R}^2$. The *Euclidean distance function* on Ω, denoted $\mathcal{E}_\Omega$, is

$$\mathcal{E}_\Omega(P) = \inf_{Q \in \partial\Omega} \|P - Q\|_2$$

The *boundary support* of P, denoted $\text{bsupp}(P; \partial\Omega)$, is

$$\text{bsupp}(P; \partial\Omega) = \{Q \in \partial\Omega : \ \|P - Q\| = \mathcal{E}_\Omega(P)\}.$$

The *medial axis* of Ω, denoted $\mathbf{MA}(\Omega)$, is the set of points P which have two or more closest points on the boundary, i.e.

$$\mathbf{MA}(\Omega) = \{P \in \Omega : |\text{bsupp}(P; \partial\Omega)| \geq 2\}.$$

where $|\text{bsupp}(P; \partial\Omega)|$ denotes the cardinality of the set. Note that $\mathbf{MA}(\Omega)$ does not have to be restricted to the shape Ω and is well-defined on all of $\mathbb{R}^2$. Hence, there is an interior medial axis and an exterior one. Here, we will only be concerned with the interior one.

Theorem 3. *Let Ω be a shape and $P \in \Omega$. Suppose $|\text{bsupp}(P; \partial\Omega)| = 1$ and pick the unique $Q \in \text{bsupp}(P; \partial\Omega)$. Then,*

(a) $P \notin \partial\Omega$ implies

$$\nabla \mathcal{E}_\Omega(P) = \frac{Q - P}{\|Q - P\|}$$

where $Q - P$ is the vector from $Q \in \partial\Omega$ to P.

(b) If $\partial\Omega$ is C^k at Q, then $\nabla\mathcal{E}_\Omega$ is C^k at P.

Proof. Part (a) is due to Federer [6, 4.8(3)] and (b) is a consequence of the more general result by Krantz and Parks [8] (see also Mather [9]). □

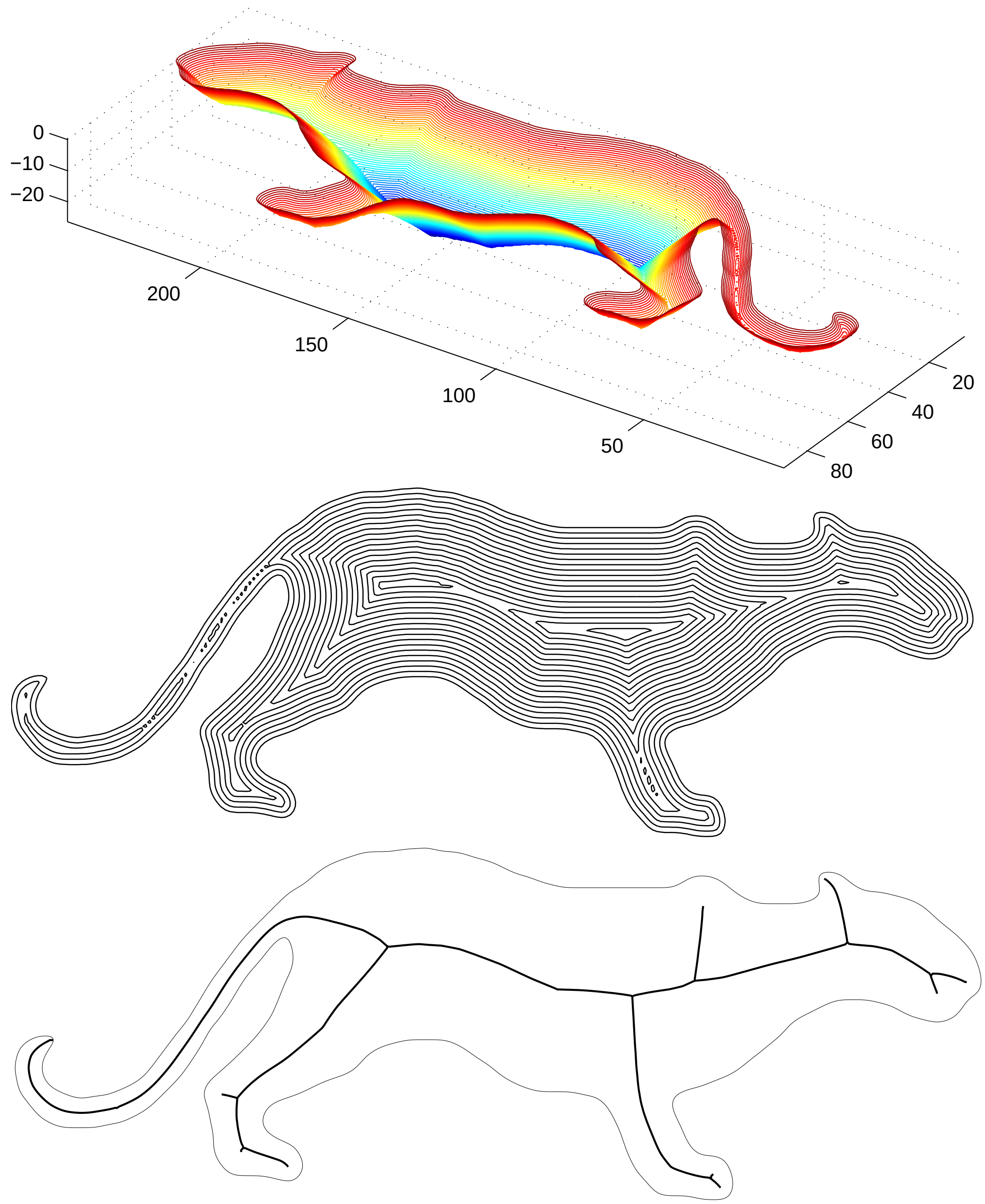

Figure 1: Examples of distance map and medial axis (see Siddiqi, Bouix, Tannenbaum and Zucker [11]). TOP: negative distance map $-\mathcal{E}_\Omega$, CENTER: level sets of $\mathcal{E}$, BOTTOM: medial axis computed as in Dimitrov, Damon and Siddiqi [4].

Corollary 4. *$\mathcal{E}_\Omega(P)$ is smooth at $P \in \Omega - \mathbf{MA}(\Omega)$.*

Proof. Immediate from Theorem 3(b) and the definition of shape. □

Theorem 5. *Let Ω be a shape. Then*

(i) *$\mathbf{MA}(\Omega)$ has no interior, i.e. it is thin.*

(ii) *$\mathbf{MA}(\Omega)$ consists of a finite number of connected piece-wise smooth curves.*

(iii) *if $P \in \mathbf{MA}(\Omega)$, $Q \in \operatorname{bsupp}(P; \partial\Omega)$ and C is the center of curvature for $\partial\Omega$ at Q, then $\|P - Q\| \le \|C - Q\|$ whenever $\|C - P\| \le \|C - Q\|$.*

Proof. Part (i) is shown in Matheron [10] and in Calabi and Hartnett [2]; (ii) is treated in detail by Choi *et al.* [3]. Part (iii) asserts that if a medial axis point is inside the circle of curvature of a point in its boundary support, then it cannot be further than the center of curvature. □

Theorem 6. *Let Ω be a shape. There is a unique c on Ω such that $c = 0$ on $\partial\Omega$ and $\nabla^2 c = -K/D$.*

Proof. See Theorem 4.3 of Gilbarg and Trudinger [7] for a more general statement and proof. □

Theorem 7 (Divergence)**.** *Let $\partial B_\varepsilon(P)$ be a circle or radius ε centered at $P \in \mathbb{R}^2$, $\mathcal{N}$ the inner normals. Then*

$$\nabla^2 c(P) = \lim_{\varepsilon \to 0} \int_{\partial B_\varepsilon(P)} \langle \nabla c, N \rangle \, ds.$$

Proof. See Warner [12, p. 151]. □

Definition 8. The Θ-notation for asymptotic behavior of a function is defined as:

$$\Theta(g(n)) = \{ f(n) : \exists c_1, c_2, n_0 \text{ positive } s.t.\ \forall n > n_0, 0 \le c_1 g(n) \le f(n) \le c_2 f(n) \}.$$

3 Statement of Result

Result 1 is based on the following theorem.

Theorem 9. *Let Ω be a shape and $c : \Omega \to \mathbb{R}$ the unique function satisfying $c(x, y) = 0$ on $(x, y) \in \partial\Omega$ and $\nabla^2 c = -K/D$. Suppose $P \in \Omega$ is such that $\mathcal{E}_\Omega(P) = L = \sup_\Omega \mathcal{E}_\Omega$ and $Q \in \operatorname{bsupp}(P; \partial\Omega)$. Suppose the smallest concave curvature radius is pL with $p > 0$. Then,*

(a) $c(Q) \in \Theta(L^2)$,

(b) $\frac{K}{2D} L \le |\nabla c| \le \frac{K}{D} L \frac{2p+1}{p}$,

(c) $\sup_{\partial\Omega} |\nabla c| > \sup_{\Omega - \partial\Omega} |\nabla c|$

If an areole is regarded as a discretization of a shape Ω, then the discrete approximation behaves as stated in Theorem 9. Observe that ∇c at $\partial\Omega$ is perpendicular to the boundary because $c = 0$ there; hence, $\nabla c(Q)$ points in the direction of P according to Theorem 3.

[1] We allow infinite curvature.

4 Organization of the Proof

Parts (a) and (b) of Theorem 9 follow from Lemma 18. The idea of the proof is to find appropriate bounding functions, one from below v and another u from above, that sandwich the unique solution c and that take the same values at the boundary. Thus, $v \leq c \leq u$ everywhere and $|\nabla v| \leq |\nabla c| \leq |\nabla u|$ on points where $v = c = u$, i.e. the boundary. Since the value of c must be the same at the boundary, its gradient there must be perpendicular to the boundary which gives the direction as claimed in Result 1(b). Lemma 13 and Lemma 17 give the lower bound and upper bound constructions and Lemma 18 collects them.

Part (c) of Theorem 9 is necessary to show that the c-vascular strand creation process is well defined. Result 1(c) is the non-technical version of Lemma 19. The proof is based on the idea that the boundary may be seen as evolving by considering level sets of c, i.e. points γ_{c_0} where $c(x, y) = c_0$. The gradient must be perpendicular to this level set and the solution of Eq. 1 inside it follows the same constraints as the shapes on which the problem is defined. We may move the level set curve γ_{c_0} so that the point on γ_{c_0} which is on the gradient curve initiated at the point Q of maximal gradient on γ_0 touches Q for small enough c_0. Knowing that the solution on the smaller shape must be strictly smaller than on the original shape shows that the maximum gradient must be strictly decreasing as the curve evolves. This is true for all curves, including the evolved ones (i.e. γ_{c_1} for $c_1 > c_0$), so the gradient in the interior of the shape must be lower than the maximum on the boundary.

5 The Proof

We begin with a Lemma 10 that will be used (indirectly) in most of the proofs that follow. It states that a discretization of the dynamic process will always move the concentration values in the same direction (up or down) if this direction is locally the same for all discrete points. This will be used to prove the next result, Lemma 11, which states that the equilibrium solution over a shape completely contained in another one will be bounded above by the solution over the bigger shape. This holds even if the initializing function is not smooth.

Lemma 10. *Let* $c_t = D\nabla^2 c + K$ *be approximated on a square lattice by* p_i *and its four neighbors* n_j *by* $\tilde{c}_t = \frac{D}{h^2}\left(\sum_j c(n_j) - 4c(p_i)\right) + K$ *where* h *is the lattice spacing. Suppose that* $\tilde{c}_t \leq (\geq) 0$ *everywhere on the domain of definition at time* t_0*. Then*

a) $c + \tau\tilde{c}_t$ *will also satisfy the inequality if* $0 < \tau \leq \frac{h^2}{4D}$*; and*

b) the discrete dynamics with such τ *make* c *decrease (increase) monotonically everywhere.*

Proof. Part b) follows directly from a). Let $\alpha = D/h^2$ and $\Lambda_p = \sum_{j=1,4} c(n_j) - 4c(p)$, so $\tilde{c}_t(t_0, p) =$

$\alpha\Lambda_p + K$. After the time step τ the approximation to c_t becomes:

$$\begin{aligned}
\tilde{c}_t(t_0+\tau) &= D\nabla^2(c+\tau\tilde{c}_t(t_0)) + K \\
&= \alpha\left(\sum_j \left(c(n_j)+\tau\tilde{c}_t(t_0,n_j)\right) - 4\left(c(p_i)+\tau\tilde{c}_t(t_0,p_i)\right)\right) + K \\
&= \alpha\left(\Lambda_{p_i} + \tau\left(\sum_{j=1}^{4}\tilde{c}_t(t_0,n_j) - 4\tilde{c}_t(t_0,p_i)\right)\right) + K \\
&= \alpha\left(\Lambda_{p_i} + \tau\left(\sum_{j=1}^{4}(\alpha\Lambda_{n_j}+K) - 4(\alpha\Lambda_{p_i}+K)\right)\right) + K \\
&= \alpha\left(\Lambda_{p_i} + \tau\alpha\left(\sum_{j=1}^{4}\Lambda_{n_j} - 4\Lambda_{p_i}\right)\right) + K \\
&= \alpha\Lambda_{p_i}(1-4\tau\alpha) + \alpha\tau\alpha\sum_{j=1}^{4}\Lambda_{n_j} + K
\end{aligned}$$

Now, all of the $\alpha\Lambda_{n_j} + K \leq 0$ and $\alpha\Lambda_{p_i} + K \leq 0$ by assumption. Also, $0 < 4\tau\alpha \leq 1$ so choosing the largest Λ_{n_j} an replace for the other three bounds the value above (since $\Lambda_{n_j} < 0$) and shows that τ is used in a linear interpolation between two non-positive numbers. This finishes the claim. □

Lemma 11. *Let $u_t = D\nabla^2 u + K = 0$ over Ω with $u = 0$ on $\partial\Omega$. If $\Omega' \subset \Omega$ and $D\nabla^2 c + K = 0$ on Ω' with $c = 0$ on $\partial\Omega'$ then $c \leq u$ on Ω'. If $\Omega \not\subset \Omega'$, then $c < u$ everywhere on $\Omega' - \partial\Omega'$.*

Proof. If $u = 0$ over Ω, then $u_t = K > 0$ and, by Lemma 10b, $u > 0$ in the interior $\Omega - \partial\Omega$ after any non-zero time step. Thus, at equilibrium, u will satisfy the dynamics everywhere on Ω' except possibly on $\partial\Omega'$ where it may need to be lower because it the boundary conditions. If a boundary point is lowered in the discretization of the problem, any neighbor will see its $\nabla^2 u$ decrease and strictly decrease if the neighbor is not moved. This holds at any time step and will affect all points after sufficiently long time because they are all connected. If $\Omega \not\subset \Omega'$, then u is not a solution ($u > 0$ somewhere on $\partial\Omega'$) and the dynamics on Ω' will strictly monotonically lower it everywhere. □

Now we show what the solution looks like in one dimension, Lemma 12, and then we turn to two special shapes: the circle (Lemma 13) and the open doughnut (Lemma 14). These shapes will be instrumental in providing the lower and upper bounds needed later on.

Lemma 12. *Suppose the domain is the segment $[0, L]$, $c_{1D}(0) = 0$ and $c_{1D}{}'(L) = 0$. Then the solution to Eq. 1 is*

$$c_{1D}(r) = \frac{K}{D}\left(-\frac{r^2}{2} + rL\right)$$

Proof. By inspection since the solution is unique: $\partial^2/(\partial r)^2[c_{1D}(r)] = -K/D$. □

Lemma 13 (Disc)**.** *Let the shape be a circle of radius L centered at the origin. Suppose $c(x, y) = 0$ on the boundary where $x^2 + y^2 = L^2$. Then the solution to Eq. 1 is*

$$c(x,y) = \frac{1}{2}c_{1D}\left(L - \sqrt{x^2+y^2}\right)$$

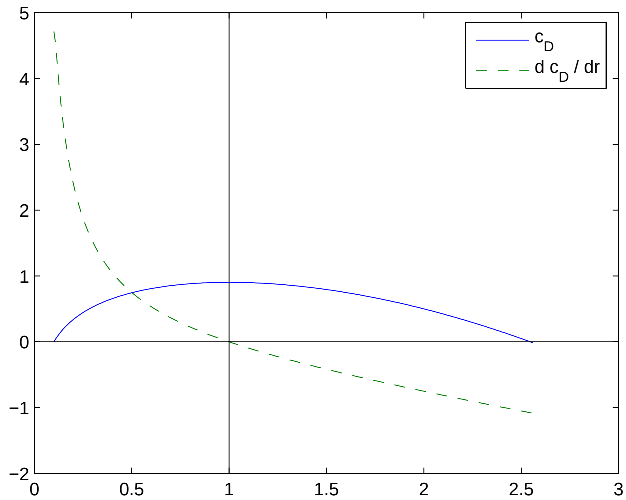

Figure 2: A plot of $c_D(r; 0.1, 0.9)$ and $\frac{dc_D}{dr}$. Notice that c_D is increasing from l to $l + L$.

and

$$|\nabla c(x, y)| = -\frac{1}{2}\frac{K}{D}\sqrt{x^2 + y^2}$$

Proof. By inspection since the solution is unique. We see that $c_{xx} = c_{yy} = -\frac{1}{2}K/D$, so Eq. 1 is satisfied. □

Lemma 14 (Open Doughnut)**.** *Let $0 \le l < l + L$ be the radii of two circles centered at the origin. Suppose $c(x, y) = 0$ on the boundary $x^2 + y^2 = l^2$ and $\nabla c(x, y) = 0$ for $x^2 + y^2 = (l + L)^2$. Then the solution to Eq. 1 is*

$$c(x, y) = c_D\left(\sqrt{x^2 + y^2}\right)$$

where

$$c_D(r) = c_D(r; l, L) = \frac{K}{D}\left(\frac{1}{4}\left(l^2 - r^2\right) + \frac{1}{2}\ln\left(\frac{r}{l}\right)(l + L)^2\right) \tag{2}$$

Further, $c(x, y) \ge 0$ for $l^2 \le x^2 + y^2 \le (l + L)^2$.

Proof. By inspection since the solution is unique. Since $c = 0$ at the inner boundary and ∇c points radially toward the outer boundary, the values of c are increasing radially in $l^2 \le x^2 + y^2 \le (l + L)^2$. □

The next two results (Lemma 15 and Lemma 16) are technical assertions used in the proof of the Upper Bound Lemma (Lemma 17). This is the last result needed to prove Lemma 18 and, therefore, parts (a) and (b) of Theorem 9.

Lemma 15. *Let Ω be a shape $P \in \Omega$, and $Q \in \mathrm{bsupp}(P; \partial\Omega)$. Let Σ be the circle of curvature of Ω at Q. Then $\mathcal{E}_\Omega(P') = \mathcal{E}_\Sigma(P') + O(\varepsilon^3)$ for $\|P - P'\| = \varepsilon$, and over a circular region R of radius ε centered at P*

$$\lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \int_{\partial R} \left\langle \frac{\partial}{\partial r} c_D(\mathcal{E}_\Omega)\nabla\mathcal{E}_\Omega, \mathcal{N}\right\rangle ds = \lim_{\varepsilon \to 0} \frac{1}{\pi\varepsilon^2} \int_{\partial R} \left\langle \frac{\partial}{\partial r} c_D(\mathcal{E}_\Sigma)\nabla\mathcal{E}_\Sigma, \mathcal{N}\right\rangle ds$$

Proof. Write the second order approximation $\mathcal{E}_\Omega = \mathcal{E}_\Sigma + O(\varepsilon^3)$ and $\nabla\mathcal{E}_\Omega = \nabla\mathcal{E}_\Sigma + O(\varepsilon^2)$. In a circular neighborhood R, the limit becomes:

$$\lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon^2}\int_0^{2\pi\varepsilon}\left\langle\frac{\partial}{\partial r}c_D(\mathcal{E}_\Omega)\nabla\mathcal{E}_\Omega,\mathcal{N}\right\rangle ds = \lim_{\varepsilon\to 0}\int_0^{2\pi}\frac{1}{\pi\varepsilon}\left\langle\frac{\partial}{\partial r}c_D(\mathcal{E}_\Omega)\nabla\mathcal{E}_\Omega,\mathcal{N}\right\rangle ds$$

Now, $\frac{\partial}{\partial r}c_D(\mathcal{E}_\Sigma + O(\varepsilon^3)) = \frac{\partial}{\partial r}c_D(\mathcal{E}_\Sigma) + O(\varepsilon^3)$ by inspection of $\frac{\partial}{\partial r}c_D(r)$, which gives:

$$\begin{aligned}\tfrac{1}{\pi\varepsilon}\tfrac{\partial}{\partial r}c_D(\mathcal{E}_\Omega)\nabla\mathcal{E}_\Omega &= \tfrac{1}{\pi\varepsilon}\tfrac{\partial}{\partial r}c_D(\mathcal{E}_\Sigma + O(\varepsilon^3))\left(\nabla\mathcal{E}_\Sigma + O(\varepsilon^2)\right)\\ &= \tfrac{1}{\pi\varepsilon}\left(\tfrac{\partial}{\partial r}c_D(\mathcal{E}_\Sigma)\nabla\mathcal{E}_\Sigma + \tfrac{\partial}{\partial r}c_D(\mathcal{E}_\Sigma)O(\varepsilon^2) + O(\varepsilon^3)\nabla\mathcal{E}_\Sigma + O(\varepsilon^5)\right)\\ &\to \lim_{\varepsilon\to 0}\frac{1}{\pi\varepsilon}\frac{\partial}{\partial r}c_D(\mathcal{E}_\Sigma)\nabla\mathcal{E}_\Sigma\end{aligned}$$

□

Lemma 16. *Suppose the shape Ω is the disc as in Lemma 13 with radius $l+L$ and c is the solution. Then,*

$$u(x,y) = c_D(l + L - \sqrt{x^2+y^2})$$

satisfies $\nabla^2 u < \nabla^2 c = -K/D$ for all points except the center.

Proof. Write $f = u - c$ where c is the solution for the disc from Lemma 13. Letting $R(x,y) = \sqrt{x^2+y^2}$

$$\begin{aligned}f(R(x,y)) &= c_D(l+L-R) - \tfrac{1}{2}c_{1D}(l+L-R)\\ &= \tfrac{K}{D}\left(\tfrac{l^2}{4} + \tfrac{1}{2}\left(\ln\left(\tfrac{l+L-R}{l}\right)(l+L)^2 - (l+L-R)(l+L)\right)\right)\end{aligned}$$

A direct calculation shows that

$$\nabla^2 f(x,y) = -\frac{1}{2}\frac{K}{D}\frac{l+L}{\sqrt{x^2+y^2}}$$

which demonstrates that $f(r) < 0$ everywhere and $f(0) = -\infty$ in the center of the disc. Therefore, $\nabla^2\, u\; =\; \nabla^2\, f\; +\; \nabla^2 c < -K/D$ because $\nabla^2 c = -K/D$. □

Lemma 17 (Upper Bound)**.** *Let the conditions of Theorem 9 hold. Define $u(x,y) = 2c_D(l + \mathcal{E}_\Omega(x,y))$ with $l = pL$. Then $u \geq c$.*

Proof. We shall show that any discretization $\tilde{u}$ with spacing $h < h_0$ (for some $h_0 > 0$) of u will satisfy $D\nabla^2\tilde{u} + K \leq 0$. Thus, Lemma 10 shows that a dynamical process initialized with u will decrease u everywhere with each time step and, by Theorem 6, it should converge to c. Hence, $c \leq u$.

First, we treat non-medial axis points. Let $P = (x,y) \in \Omega$ (not on the medial axis) and $Q \in \partial\Omega$ which is closest to P, i.e. $\|P - Q\| = \mathcal{E}(P)$. Suppose $\partial\Omega$ near Q is approximated by the circle of curvature at Q. Thus, Lemma 15 applies and, by the Divergence Theorem 7, $\nabla^2 u(P)$ is the same as if the boundary were a circle at Q. Following Theorem 5 MA2, there are three cases: (a) P outside the circle, (b) P inside, and (c) the circle has infinite radius – it is a line segment. Lemma 14 shows that $\nabla^2 u(P) = -2K/D < -K/D$ which takes care of (a), and (b) is covered by Lemma 16. If the boundary is locally a straight line, then

$$\nabla^2 u = \frac{\partial^2}{\partial r^2}2c_D(l+r) = -\frac{K}{D}\left(1 + (l+L)^2/r^2\right) \tag{3}$$

with $l < r < l + L$ and r is in the direction of the gradient. So, $\nabla^2 u(P) < -K/D$ for all $P \in \Omega - \mathbf{MA}(\Omega)$.

Now suppose that the of u on Ω are sampled on a discrete square lattice with spacing $h > 0$. There, $\nabla^2 u(p)$ is approximated by the formula for Λ_p in the proof of Lemma 10. The error of the approximation is $O(h^2)$ (see Abramowitz and Stegun [1]). Thus, from the above, h may be chosen so that $\Lambda_P^u < -K/D$ for $P \in \Omega$ further than h from $\mathbf{MA}(\Omega)$.

Let $\mathcal{E}_{\Sigma_P}$ to be the distance function from the circle of curvature at the boundary point corresponding to $P \notin \mathbf{MA}(\Omega)$. Hence, if $\|P' - P\| = h$, then $\mathcal{E}_{\Sigma_P}(P') = \mathcal{E}_\Omega(P') + \varepsilon_{P'}$ where $\varepsilon_{P'} = O(h^3)$. Set

$$\varepsilon_P = \sup_{\|P'-P\|=h} |\varepsilon_{P'}| \quad \text{and} \quad \varepsilon = \sup_{P \in \Omega - \mathbf{MA}(\Omega)} \varepsilon_{P'}$$

where $0 \le \varepsilon < h$ for small enough h. So, choose such an h and define

$$u_h(x, y) = 2c_D(l + \mathcal{E}_\Omega(x, y); l, L + 2h)$$

and notice that we may refine the grid (i.e. choose h smaller) and the above properties will still hold. Thus, refine h if necessary to make $\Lambda_P^{u_h} < -K/D$ on shape points further than h from the medial axis. Refine it further to $\Lambda_P^{u_h} < -K/D$ on an open doughnut with l inner radius and $L + 2h$ outer radius. Make sure that h is small enough so that $c_D(r + h) - 2c_D(r) + c_D(r - h) < -K/D$ (this is needed in the tangent line construction below), which is possible because of Eq. 3.

Now we show that this also makes $\Lambda_P^{u_h} < -K/D$ for points on the medial axis and those closer than h from it. Pick such a P and let consider $Q \in \text{bsupp}(P; \partial\Omega)$. If Q is concave, then approximate the boundary by its circle of curvature and look at $u_h^\Sigma = 2c_D(l + \mathcal{E}_\Sigma(x, y); l, L + 2h)$. A neighbor N of P used in $\Lambda_P^{u_h}$ satisfies $\mathcal{E}_\Sigma(N) \ge \mathcal{E}_\Omega(N)$ (because Q is concave and the difference is no more than ε. Hence, $u_h^\Sigma(N) \ge u_h(N)$ since c_D is increasing until $l + L + 2h$. Further, $\mathcal{E}_\Sigma(P) = \mathcal{E}_\Omega(P)$ because the circle of curvature touches $\partial\Omega$ at Q. Hence, $\Lambda_P^{u_h} < \Lambda_P^{u_h^\Sigma} < -K/D$ because P is not a medial axis point for Σ.

If, on the other hand, there is a concave $Q \in \text{bsupp}(P; \partial\Omega)$, then instead of the circle of curvature we may take the tangent line ℓ_P at Q define $\mathcal{E}_{\ell_P}$ exactly similarly to $\mathcal{E}_{\Sigma_P}$ above. Refine h so that any point P' for which $\|P - P'\| = h$ is closest to a point $Q' \in \ell_P$ that lies outside the Ω or on $\partial\Omega$.[2]. Thus, as before, $\mathcal{E}_{\ell_P}(N) \ge \mathcal{E}_\Omega(N)$ for any N near P, i.e. $\|N - P\| = h$. Therefore, $\Lambda_P^{u_h} < \Lambda_P^{u_h^\Sigma} < -K/D$.

Finally, Lemma 11 shows that $u_h > c$ from which we conclude that $u \ge c$ since $\lim_{h\to 0} u_h \to u$. □

Remark 1. The function $u(x, y)$ need not be smooth on Ω. In fact, it will fail to have fist derivatives on the medial axis of most shapes.

Lemma 18. *Let the conditions of Theorem 9 hold. Then $c = \Theta(L^2)$. Further, if P is the center of the largest inscribed circle and Q a point on the boundary of the shape and the circle, then $\nabla c(Q)$ points toward P and*

$$\frac{K}{2D} L \le |\nabla c| \le \frac{K}{D} L \frac{2p + 1}{p}.$$

Proof. The Disc Lemma 13 gives the lower bound function and the Upper Bound Lemma 17 the rest. The gradient points in the direction of the normal to the boundary because $c = 0$ on $\partial\Omega$. The magnitude follows from a simple calculation of $\partial/\partial r[c_D(r)]$ at $r = pL$ (see the Open Doughnut Lemma for the definition of $c_D(r)$). □

[2] This must be possible since Q is convex.

Finally, the following result completes the proof of Theorem 9.

Lemma 19 (Decreasing Gradient)**.** *Let c satisfy the Poisson equation (Eq. 1) on Ω and $c = 0$ on $\partial\Omega$. Then*

$$M = \sup_{\Omega} |\nabla c| = \sup_{\partial\Omega} |\nabla c|$$

and

$$|\nabla c(x, y)| < M, \quad (x, y) \in \Omega - \partial\Omega \ .$$

Proof. Let $\gamma_{c_0} = \{(x, y) \in \Omega : c(x, y) = c_0\}$. A number $0 < c_0 < \sup_{\Omega} c$ must exist since $c > 0$ on $\Omega - \partial\Omega$ by Lemma 10b. Let the shape Ω' be defined by $(x, y) \in \Omega$ such that $c(x, y) \geq c_0$. Hence, $\Omega' \subset \Omega$ and $\Omega \not\subset \Omega'$. The boundary $\partial\Omega' = \gamma_{c_0}$ is regular, so there is a unique v satisfying Eq. 1 on Ω' with $v = 0$ on $\partial\Omega'$. Thus, $v = c - c_0$ and $\nabla v = \nabla c$ on Ω'.

Thus, $\gamma_0 = \partial\Omega$ and γ_{c_0} is connected for small enough c_0 (because Ω is the closure of an open set). Further, if $c_0 < \varepsilon_0$ for some $\varepsilon_0 > 0$, then γ_{c_0} is a smooth curve because $\nabla_{\gamma_{c_0}} c = 0$ on γ_{c_0}, c is at least twice differentiable, and $\nabla c \neq 0$ when taken over Ω on points of γ_0. In fact, ∇c is perpendicular to the curve $\gamma_0 = \partial\Omega$. Let $Q \in \gamma_0$ be such that $\nabla c(Q) = \sup_{\partial\Omega} |\nabla c|$. Let β be the integral curve segment starting at $\beta(0) = Q$ with tangents in the direction of ∇c and such that $\beta(1) \in \gamma_{c_0}$. Since ∇c is perpendicular to γ_0, $\beta(1)$ will be the closest point to γ_0 from γ_{c_0} for small enough c_0. Thus, γ_{c_0} may be translated so that $\beta(1)$ touches Q ensuring that γ_{c_0} is completely contained in Ω; denote this translated curve by γ'_{c_0}.

The solution v' to Eq. 1 on γ'_{c_0} and its interior must be the translated v. By Lemma 11 $v' < c$ everywhere except on $\gamma_0 \cap \gamma'_{c_0}$ (e.g. at Q) where $v' = c$. Hence, $|\nabla v'(Q)| < |\nabla u(Q)|$. Therefore, $|\nabla u(Q)|$ is strictly decreasing in the direction of $\nabla u(Q)$.

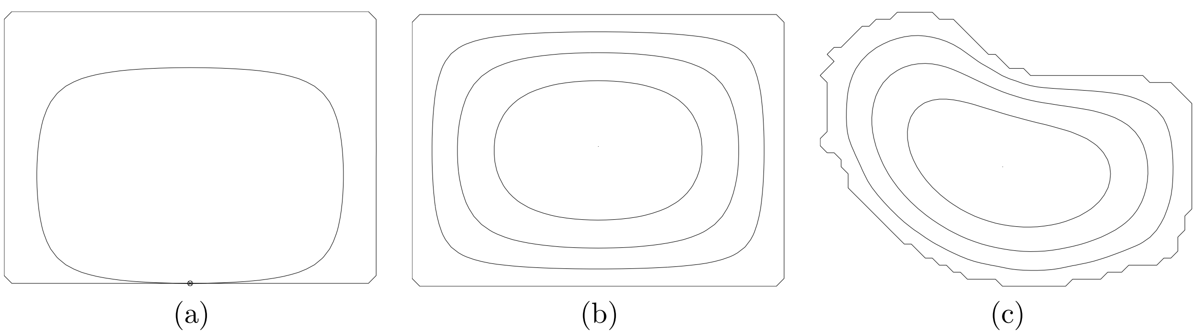

Figure 3: Level sets of c for (a) an areole from Berleth *et al.*(b) a rectangular shape.

□

Aknowledgements

We thank R. Beals for discussions.

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