Department of Computer Science

YALE UNIVERSITY

New Haven, Connecticut 06520

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by

Martin H. Schultz

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BICUBIC INTERPOLATION OVER RIGHT TRIANGLES*

Martin H. Schultz
Department of Computer Science
Yale University

In this note we improve the error bound recently given by C. A. Hall in [2] for an interpolation scheme due to G. Birkhoff. This interpolation scheme is defined over right triangles using bicubic polynomials.

Consider the right triangle, Δ, with vertices $(0,0)$, $(h,0)$, and $(0,k)$ and the set $P \equiv \{p(x,y) \mid$ there exist real constants a_{ij}, $0 \le i+j \le 3$, such that $p(x,y) = \sum_{0 \le i+j \le 3} a_{ij} x^i y^j$ for all $(x,y) \in \Delta\}$ of all real bicubic polynomials on Δ. We define an interpolation mapping M from $C^2(\Delta)$ to P by

$$(1) \quad (D_x^i D_y^j Mf)(0,0) \equiv \left(\frac{\partial^{i+j}}{\partial x^i \partial y^j} Mf\right)(0,0) = D_x^i D_y^j f(0,0), \quad 0 \le i,\ j \le 1,$$

$$(2) \quad (D_x^i D_y^j Mf)(0,k) = D_x^i D_y^j f(0,k), \quad 0 \le i+j \le 1,$$

and

$$(3) \quad (D_x^i D_y^j Mf)(h,0) = D_x^i D_y^j f(h,0), \quad 0 \le i+j \le 1,$$

for all $f \in C^2(\Delta)$.

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Following [2] we have that the interpolation mapping M is well-defined.

Theorem 1. The interpolation mapping M is well-defined on $C^2(\Delta)$, i.e., Mf exists and is unique for all $f \in C^2(\Delta)$.

Before stating and proving our main result (Theorem 2) on an error bound, we consider two preliminary results.

Lemma 1. Let $f \in C^4[0,h]$ and $c(x)$ be the unique cubic polynomial such that if $e(x) \equiv c(x) - f(x)$ then $e(0) = e(h) = De(0) = De(h) = 0$. Then

$$(4) \qquad \|e\|_{L^\infty[0,h]} \leq \frac{1}{384} h^4 \|D^4 f\|_{L^\infty[0,h]} ,$$

$$(5) \qquad \|De\|_{L^\infty[0,h]} \leq \frac{\sqrt{3}}{216} h^3 \|D^4 f\|_{L^\infty[0,h]} ,$$

and

$$(6) \qquad \|De\|_{L^1[0,h]} \leq \frac{1}{72} h^4 \|D^4 f\|_{L^\infty[0,h]} .$$

Proof. For a proof of (4) see any standard reference on interpolation theory and for a proof of (5) see [1]. The proof of (6) is as follows.

By Rolle's Theorem and the interpolation conditions, there exists a point $\xi \in (0,h)$ such that $De(\xi) = 0$. Define a new function

$$F(z) \equiv De(z) - \alpha z(h - z)(z - \xi)$$

for all $z \in [0,h]$, where α is a real constant to be chosen.

Given a fixed $x \in (0,h)$ such that $x \neq \xi$, choose α such that $F(x) = 0$, i.e., $\alpha \equiv De(x)/[x(h-x)(x-\xi)]$. Then $F(0) = F(h) = F(\xi) = F(x) = 0$ and by Rolle's Theorem there exists $\Theta \in [0,h]$ such that $D^3F(\Theta) = 0$.

Computing D^3F, we find that $\alpha = \frac{1}{6} D^4f(\Theta)$ and hence $De(x) = \frac{1}{6} x\,(h-x)(x-\xi)\,D^4f(\Theta)$. Thus,

$$\|De\|_{L^1[0,h]} \leq \frac{1}{6} \|D^4f\|_{L^\infty[0,h]} \max_{\xi\in[0,h]} \left[\int_0^\xi x(h-x)(\xi-x)dx + \int_\xi^x x(h-x)(x-\xi)dx \right]$$

$$\equiv \frac{1}{6} \|D^4f\|_{L^\infty[0,h]} \max_{\xi\in[0,h]} \Phi(\xi) .$$

Since $D^2\Phi(\xi) > 0$ for all $\xi \in (0,h)$, the maximum of $\Phi(\xi)$ occurs for $\xi = 0$ and/or $\xi = h$. Thus $\Phi(\xi) \leq \frac{1}{12} h^4$ and (6) follows immediately. Q.E.D.

<u>Lemma 2</u>. Let $f \in C^3[0,h]$ and $q(x)$ be the unique quadratic polynomial such that if $e(x) \equiv q(x) - f(x)$ then $e(0) = e(h) = De(0) = 0$. Then

$$(7) \qquad \|De\|_{L^\infty[0,h]} \leq \frac{1}{2} h^2 \|D^3f\|_{L^\infty[0,h]}$$

and

(8) $$\|De\|_{L^1[0,h]} \leq \frac{1}{6} h^3 \|D^3 f\|_{L^\infty[0,h]} \quad .$$

Proof. By Rolle's Theorem and the interpolation conditions, there exists a point $\xi \in (0,h)$ such that $De(\xi) = 0$. Define a new function $F(z) \equiv De(z) - \alpha z(z - \xi)$ for all $z \in [0,h]$, where α is a real constant to be chosen.

Given a fixed $x \in (0,h)$ such that $x \neq \xi$, choose α such that $F(x) = 0$, i.e., $\alpha \equiv De(x)/[x(x - \xi)]$. Then $F(0) = F(\xi) = F(x) = 0$ and by Rolle's Theorem there exists $\Theta \in [0,h]$ such that $D^2F(\Theta) = 0$.

Computing D^2F, we find that $\alpha = \frac{1}{2} D^3 f(\Theta)$ and hence $De(x) = \frac{1}{2} x(x - \xi) D^3 f(\Theta)$. Thus

$$\|De\|_{L^\infty[0,h]} \leq \frac{1}{2} \|D^3 f\|_{L^\infty[0,h]} \max_{x\in[0,h]} \max_{\xi\in[0,h]} x|x - \xi|$$

$$\leq \frac{1}{2} h^2 \|D^3 f\|_{L^\infty[0,h]} \quad ,$$

which proves (7).

Moreover,

$$\|De\|_{L^1[0,h]} \le \frac{1}{2}\|D^3f\|_{L^\infty[0,h]} \left[\int_0^\xi x(\xi - x)dx + \int_\xi^h x(x-\xi)dx \right]$$

$$\le \frac{1}{2}\|D^3f\|_{L^\infty[0,h]} \max_{\xi\in[0,h]} \left[\frac{1}{3}\xi^3 + \frac{1}{3}h^3 - \frac{1}{2}h^2\xi \right]$$

$$\le \frac{h^3}{6}\|D^3f\|_{L^\infty[0,h]} ,$$

which proves (8). Q.E.D.

We proceed now to our main result, which improves Theorem 7 of [2]. Throughout the remainder of this note, we will let $\|\cdot\|$ denote $\|\cdot\|_{L^\infty(\Delta)}$.

Theorem 2. If $f \in C^4(\Delta)$, then $e(x,y) \equiv Mf(x,y) - f(x,y)$ satisfies

(9) $$\|D_xD_ye\| \le \frac{1}{2}h^2\|D_x^3 D_yf\| + \frac{1}{2}k^2\|D_y^3 D_xf\| + hk\|D_x^2 D_y^2 f\| ,$$

(10) $$\|D_xe\| \le \frac{8}{81}h^3\|D_x^4 f\| + \frac{1}{2}h^2k\|D_x^3 D_yf\| + \frac{1}{6}k^3\|D_y^3 D_xf\| + \frac{1}{2}hk^2\|D_x^2 D_y^2 f\| ,$$

(11) $$\|D_ye\| \le \frac{8}{81}k^3\|D_y^4 f\| + \frac{1}{2}k^2h\|D_y^3 D_xf\| + \frac{1}{6}h^3\|D_x^3 D_yf\| + \frac{1}{2}kh^2\|D_x^2 D_y^2 f\| ,$$

and

(12) $$\|e\| \le \frac{1}{2}\left(\frac{1}{384} + \frac{1}{72}\right)\left(h^4\|D_x^4 f\| + k^4\|D_y^4 f\|\right) + \frac{1}{6}h^3k\|D_x^3 D_yf\| + \frac{1}{6}k^3h\|D_y^3 D_xf\| + \frac{1}{4}h^2k^2\|D_x^2 D_y^2 f\| .$$

Proof. If (c,d) is any point in Δ,

$$\Delta_{xy} D_x D_y e(c,d) \equiv D_x D_y e(c,d) - D_x D_y e(c,0) - D_x D_y e(0,d) + D_x D_y e(0,0)$$

$$= \int_0^c \int_0^d D_x^2 D_y^2 e(x,y)dy\, dx$$

$$= \int_0^c \int_0^d D_x^2 D_y^2 f(x,y)dy\, dx$$

and hence

$$(13) \qquad |\Delta_{xy} D_x D_y e(c,d)| \leq cd \, \|D_x^2 D_y^2 f\| .$$

Using (13) and (7) we obtain

$$(14) \quad |D_x D_y e(c,d)| \leq |D_x D_y e(c,0)| + |D_x D_y e(0,d)| + |D_x D_y e(0,0)| + cd \, \|D_x^2 D_y^2 f\|$$

$$\leq \frac{1}{2} h^2 \|D_x^3 D_y f\| + \frac{1}{2} k^2 \|D_y^3 D_x f\| + hk \|D_x^2 D_y^2 f\| ,$$

which proves (9).

Moreover,

$$(15) \qquad |D_x e(c,d)| \leq |D_x e(c,0)| + \int_0^d |D_y D_x e(c,y)| dy .$$

Using (5), (14), and (8) to bound the right-hand side of (15), we have

$$|D_x e(c,d)| \le \frac{8}{81} h^3 \|D_x^3 f\| + \int_0^d |D_x D_y e(c,0)| dy + \int_0^d |D_x D_y e(0,y)| dy$$

$$+ \int_0^d hy \|D_x^2 D_y^2 f\| dy$$

$$\le \frac{8}{81} h^3 \|D_x^4 f\| + \frac{1}{2} h^2 k \|D_x^3 D_y f\| + \frac{1}{6} k^3 \|D_y^3 D_x f\|$$

$$+ \frac{1}{2} hk^2 \|D_x^2 D_y^2 f\| ,$$

which proves (10). Inequality (11) follows by symmetry.

Finally,

$$(16) \quad |e(c,d)| \le |e(0,d)| + \int_0^c |D_x e(x,d)| dx$$

$$\le |e(0,d)| + \int_0^c |D_x e(c,0)| dx + \int_0^c \int_0^d |D_y D_x e(x,y)| dy\, dx.$$

From (14) and (16) we have

$$(17) \quad |e(c,d)| \le |e(0,d)| + \int_0^c |D_x e(c,0)| dx$$

$$+ \int_0^c \int_0^d [\, D_x D_y e(x,0) + D_x D_y e(0,y) + xy \|D_x^2 D_y^2 f\| \,] dy\, dx.$$

Using (4), (6), and (8) to bound the right-hand side of (17), we obtain

$$(18) \quad |e(c,d)| \le \frac{1}{384} k^4 \|D_y^4 f\| + \frac{1}{72} h^4 \|D_x^4 f\|_\infty + \frac{1}{6} h^3 k \|D_x^3 D_y f\|$$

$$+ \frac{1}{6} k^3 h \|D_y^3 D_x f\| + \frac{1}{4} h^2 k^2 \|D_x^2 D_y^2 f\| .$$

By symmetry, we also have

$$(19)\quad |e(c,d)| \le \frac{1}{384} h^4 \|D_x^4 f\| + \frac{1}{72} k^4 \|D_y^4 f\| + \frac{1}{6} h^3 k \|D_x^3 D_y f\| + \frac{1}{6} k^3 h \|D_y^3 D_x f\| + \frac{1}{4} h^2 k^2 \|D_x^2 D_y^2 f\| .$$

and (12) follows by adding (18) and (19) and dividing by 2.

Q.E.D.

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