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Fast Escape in Incompressible Vector Fields

Stefan Steinerberger

FAST ESCAPE IN INCOMPRESSIBLE VECTOR FIELDS

STEFAN STEINERBERGER

Abstract. Swimmers caught in a rip current flowing away from the shore are advised to swim orthogonally to the current to escape it. We describe a mathematical principle in a similar spirit. More precisely, we consider flows γ in the plane induced by incompressible vector fields $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ satisfying $c_1 < \|v\| < c_2$. The length ℓ a flow curve $\dot{\gamma}(t) = \mathbf{v}(\gamma(t))$ before γ leaves a disk of radius 1 around the initial position can be as long as $\ell \sim c_2/c_1$. The same is true for the orthogonal flow $\mathbf{v}^{\perp} = (-\mathbf{v}_2, \mathbf{v}_1)$. We show that a combination does strictly better: there always exists a curve flowing first along the orthogonal vector field $\mathbf{v}^{\perp}$ and then along $\mathbf{v}$ with total length at most $\sqrt{\pi c_2/c_1}$. Moreover, if the escape length of $\mathbf{v}$ is uniformly $\sim c_2/c_1$, then the escape length of $\mathbf{v}^{\perp}$ is uniformly ~ 1 (allowing for a fast escape from the current).

1. Introduction

1.1. **Introduction.** We study an interesting property of solutions of autonomous systems in the plane when the underlying vector field $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ is incompressible. Suppose you find yourself blindfolded in a the middle of the ocean and want to get to a point at a distance at least 1 from your starting point. Being blindfolded, simply swimming straight or using external landmarks for navigation is not possible: you can only use the local currents of the ocean. If swimming with the stream does take a very long time to get you to distance 1 from your starting point, then swimming *orthogonally* to the stream is guaranteed to be very effective and get you out very quickly.

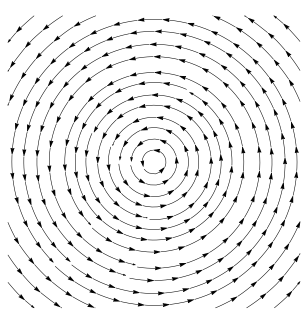

Figure 1. Flow of $(-y, x)$.

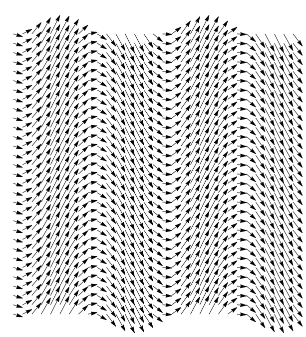

Figure 2. Flow of $(1, \cos(4x))$.

Put differently, a complicated global geometry of the stream lines of $\mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2)$ forces the flow along the orthogonal vector field $\mathbf{v}^{\perp} = (-\mathbf{v}_2, \mathbf{v}_1)$ to be rather simple. Needless to say, such a statement fails dramatically for general vector fields. We will now give a precise formulation: suppose we start in $\gamma(0) \in \mathbb{R}^2$ and are exposed to a smooth vector field $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ that is incompressible

$$\frac{\partial}{\partial x}\mathbf{v}_1 + \frac{\partial}{\partial y}\mathbf{v}_2 = 0.$$

It is entirely possible that following the vector field will lead us in circles, the easiest example being the incompressible vector field

$$\mathbf{v}(x, y) = (-y, x).$$

This is illustrated in Fig.1 where following the vector field will never allow an escape from the unit disk even if one starts outside of the origin (in which the vector field vanishes): to prevent this, we additionally assume throughout the paper that

$$0 < c_1 < \|\mathbf{v}\| < c_2 < \infty.$$

The Poincaré-Bendixson theorem (see [1, 8, 9]) then implies that the flow escapes: if the flow curve of such a vector field were contained in a compact subset of $\mathbb{R}^2$, then it is either periodic or has to approach a limit cycle but any limit cycle encloses at least one critical point [5, Corollary 1.8.5.] which violates the assumption $\|\mathbf{v}\| > c_1$. Our statements will be geometric statements about the length of the flow curves as curves in the plane and not about the value of time t in $\gamma(t)$. Such statements are invariant under dilating the vector field with a constant scalar

$$(\mathbf{v}_1, \mathbf{v}_2) \to (\lambda \mathbf{v}_1, \lambda \mathbf{v}_2) \qquad \text{for some } \lambda > 0$$

which corresponds to rescaling time $t \to t/\lambda$. Invariance under this scaling implies that our results about the geometry of curves will only depend on the ratio c_2/c_1.

2. Main results

Explicit examples, which are given in the next section, show that the length of the flow induced by either $\mathbf{v}$ and $\mathbf{v}^{\perp}$ might require a total length $\ell \sim c_2/c_1$ to escape the unit disk, where

$$\ell = \inf_{t>0} \left\{ \int_0^t \|\gamma'(s)\| ds : \|\gamma(t) - \gamma(0)\| = 1 \right\}.$$

We assume $\mathbf{v} : \mathbb{R}^2 \to \mathbb{R}^2$ is a smooth, divergence-free vector field satisfying $c_1 < \|\mathbf{v}\| < c_2$ for some real $0 < c_1 < c_2 < \infty$. $\mathbf{v}^{\perp}$ denotes the vector field $(-\mathbf{v}_2, \mathbf{v}_1)$. We write $e^{t\mathbf{v}}$ to denote the flow induced by the vector field $\mathbf{v}$ and $e^{t\mathbf{v}^{\perp}}$ for the flow induced by $\mathbf{v}^{\perp}$. We abuse notation and write the length of the flow line as

$$|e^{t\mathbf{v}} \circ e^{s\mathbf{v}^{\perp}} \gamma(0)| := \int_0^s \left\| \frac{\partial}{\partial u} e^{u\mathbf{v}^{\perp}} \gamma(0) \right\| du + \int_0^t \left\| \frac{\partial}{\partial u} e^{u\mathbf{v}} \left(e^{s\mathbf{v}^{\perp}} \gamma(0) \right) \right\| du$$

which is understood as the length of curve given in $\mathbb{R}^2$ by concatenation of the flows

$$\gamma(0) \underbrace{\longrightarrow}_{\mathbf{v}^{\perp}} e^{s\mathbf{v}^{\perp}} \gamma(0) \underbrace{\longrightarrow}_{\mathbf{v}} e^{t\mathbf{v}} \circ e^{s\mathbf{v}^{\perp}} \gamma(0).$$

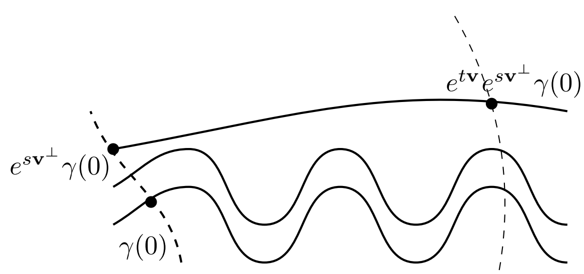

Figure 3. The statement illustrated: starting in $\gamma(0)$, the flow curves of $\mathbf{v}$ (thick) are long; flowing along $\mathbf{v}^{\perp}$ (dashed) it is possible to reach a short flow line of $\mathbf{v}$.

Theorem 1 (Fast escape)**.** *Let $\boldsymbol{v} : \mathbb{R}^2 \to \mathbb{R}^2$ be incompressible and let $\gamma(t)$ denote the induced flow. For all $\gamma(0) \in \mathbb{R}^2$ there exist $s, t \geq 0$ such that $\|(e^{t\boldsymbol{v}} \circ e^{s\boldsymbol{v}^{\perp}})\gamma(0) - \gamma(0)\| = 1$ and*

$$|e^{t\boldsymbol{v}} \circ e^{s\boldsymbol{v}^{\perp}} \gamma(0)| \leq \sqrt{4\pi} \sqrt{\frac{c_2}{c_1}}.$$

There is an aspect to the Theorem not covered by Figure 5. It is certainly possible that *all* flow lines of $\mathbf{v}$ are equally inefficient (see the next section for an example): in that case, the regularizing flow $\mathbf{v}^{\perp}$ must necessarily be efficient itself because the statement allows for $s, t = 0$ (obviously not both at the same time). We make this precise.

Theorem 2. *Let $\boldsymbol{v}: 2\mathbb{D} \to \mathbb{R}^2$ be incompressible and $c_1 \leq \|\boldsymbol{v}\| \leq c_2$. If, for all $\gamma(0) \in \mathbb{D}$,*

$$\inf_{t \in \mathbb{R}} \left\{ |e^{t\boldsymbol{v}}\gamma(0)| : \|e^{t\boldsymbol{v}}\gamma(0) - \gamma(0)\| = 1 \right\} \geq \mathcal{L},$$

then

$$\inf_{t \in \mathbb{R}} \left\{ |e^{t\boldsymbol{v}^{\perp}}(0,0)| : \|e^{t\boldsymbol{v}^{\perp}}(0,0)\| = 1 \right\} \leq \pi \frac{c_2/c_1}{\mathcal{L}}.$$

It is not a difficult to modify the proof to obtain a version of the statement, where we restrict $t > 0$ in both the assumption and the conclusion at the cost of a larger constant.

We are not aware of any directly related results. However, any life guard knows about rip currents: these are strong outward flows that are highly localized but powerful enough to carry swimmers away from the shore. The common advice is to swim orthogonally to the flow to escape the (usually quite narrow) region where they occur.

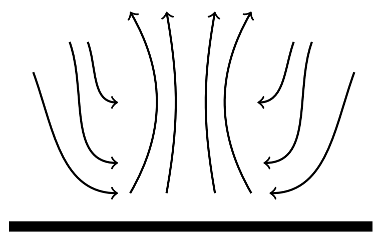

FIGURE 4. A rip current close to a beach

There is also an old problem of Bellman [2, 3, 4] (see also the survey of Finch & Wetzel [7])

> A hiker is lost in a forest whose shape and dimensions are precisely known to him.
> What is the best path for him to follow to escape from the forest?

'Best' can be understood as minimizing the worst case length or the expected length. Our problem is somehow dual: in Bellman's problem, the location is unknown but movement is unrestricted while in our problem the location is perfectly understood (at the center of a disk of radius 1) while movement is restricted and the way we can move at certain points is unknown (but, since its generated by an incompressible vectorfield, it does have some regularity) – this turns our problem into one of Bellman-type where the unknown geometry is that of of incompressible vector fields.

Clearly, these result suggests a large number of variations – an elementary variation we have been unable to find in the literature is the following quantitative Poincaré-Bendixson theorem.

Theorem 3 (Quantitative Poincaré-Bendixson)**.** *Let $\boldsymbol{v}: 2\mathbb{D} \to \mathbb{R}^2$ be a smooth vector field satisfying $c_1 < \|v\| < c_2$ and let $\gamma(t)$ be the flow induced by $\boldsymbol{v}$ acting on $\gamma(0) = (0,0)$. Then*

$$\inf_{t>0} \left\{ \int_0^t \|\gamma'(s)\| ds : \|\gamma(t)\| = 1 \right\} \leq \pi \frac{c_2}{c_1} + \frac{1}{c_1} \int_{\mathbb{D}} |\mathrm{curl}\, \boldsymbol{v}| \ dxdy.$$

The proof is not difficult and very close in spirit to the proof of the classical Dulac criterion showing the nonexistence of closed orbits. The statement seems to capture the intuitive notion that in order for the flow to spend a long time in the domain, one requires a large amount of rotation. Note that the integral term vanishes completely for irrotational vector fields. In that case, there is a very easy proof: an irrotational $\mathbf{v}$ can be written as a gradient flow $\mathbf{v} = \nabla A$. Since $\|\mathbf{v}\| \leq c_2$

$$\sup_{x \in \mathbb{D}} A(x) - A(0) \leq c_2.$$

Since $\mathbf{v}$ is the gradient flow along which A it increases at least as quickly as c_1 times the arclength measure, we can conclude that the total length satisfies

$$c_1 \inf_{t>0} \left\{ \int_0^t \|\gamma'(s)\| ds : \|\gamma(t)\| = 1 \right\} \leq c_2$$

which is stronger than the Theorem by a factor of π.

Open problems. Even in two dimensions many natural questions remain. It is not clear to us how one would construct extremal configurations giving sharp constants. Our Theorem guarantees the existence of an efficient route which first flows along $\mathbf{v}^{\perp}$ and then along $\mathbf{v}$. Suppose we allow for changing the vector field twice (i.e. first $\mathbf{v}^{\perp}$, then $\mathbf{v}$ and then once more $\mathbf{v}^{\perp}$): can one reduce the length to $\sim (c_2/c_1)^{1/3}$? An even more naive question is whether the length can be reduced to $\sim (c_2/c_1)^{1/k}$ if one is allowed to switch $k-1$ times. There is a natural toy model for the case '$k=\infty$' which also seems nontrivial (see the last Section). We do not know what type of results to expect in higher dimensions because the orthogonal flow is no longer uniquely defined.

3. Explicit constructions

3.1. **Incompressible flows.** Consider (see Fig. 2) the incompressible vector field

$$\mathbf{v}_N(x,y) = \left(1, \frac{N}{2}\cos(Nx)\right) \qquad \text{satisfying} \qquad 1 \leq \|\mathbf{v}\| \leq \sqrt{\frac{N^2}{4}+1}.$$

Note that $c_2/c_1 \sim N$. The initial condition $\gamma(0) = (0,0)$ gives rise to the flow $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$

$$\gamma(t) = \left(t, \frac{1}{2}\sin(Nt)\right).$$

The distance of $\gamma(t)$ to the origin is bounded by

$$\|\gamma(t)\| = \sqrt{t^2 + \frac{1}{4}\sin^2(Nt)} \leq \sqrt{t^2+\frac{1}{4}}.$$

Therefore, in oder to have $\|\gamma(t)\| \geq 1$, we certainly need $t \geq \sqrt{3}/2$. The length of the curve up to that point is at least

$$\int_0^{\sqrt{3}/2} \|\gamma'(t)\| dt = \int_0^{\sqrt{3}/2} \sqrt{1+\frac{N^2}{4}\cos^2(Nt)}dt \geq \frac{N}{2}\int_0^{\sqrt{3}/2} |\cos(Nt)| dt \geq \frac{N}{8}.$$

This means that $\gamma(t)$ travels a total length of $\ell \sim N \sim c_2/c_1$ before leaving a disk of radius 1.

3.2. **Irrotational flows.** The next natural question is whether it might be advantageous to immediately follow the orthogonal flow $\mathbf{v}^{\perp} = (-\mathbf{v}_2, \mathbf{v}_1)$. It turns out that here, too, the curve can be as long as $\ell \sim c_2/c_1$: in two dimensions, we can use

$$\operatorname{curl} \mathbf{v}^{\perp} = \operatorname{curl}(-\mathbf{v}_2, \mathbf{v}_1) = \frac{\partial \mathbf{v}_1}{\partial x} + \frac{\partial \mathbf{v}_2}{\partial y} = \operatorname{div} \mathbf{v} = 0.$$

Hence, $\mathbf{v}^{\perp}$ is irrotational and can therefore be written as $\mathbf{v}^{\perp} = \nabla A$ for some scalar potential $A: \mathbb{R}^2 \rightarrow \mathbb{R}$. It suffices to construct an A with $c_1 \leq \|\nabla A\| \leq c_2$ such that its gradient flow at a point moves in a zigzag line. We construct the scalar potential A explicitely: given $N \in \mathbb{N}$ we define the curve $\gamma: \mathbb{R} \rightarrow \mathbb{R}^2$ via

$$\gamma(t) = (\sin(Nt), t).$$

Our (merely Lipschitz-continuous) candidate for A is

$$\tilde{A}(x,y) = Ny - 2N \inf_{t \in \mathbb{R}} \|(x,y) - \gamma(t)\|.$$

We remark that following γ for $0 \leq t \leq 1$ gives rise to a curve of length $\sim N$. It is not difficult to see that, whenever defined, we have $0.1 \leq \|\nabla \tilde{A}\| \leq 10N$ and that, by the usual density arguments, $\tilde{A}$ can be approximated in the C^1-norm by a smooth function A having all necessary properties. This yields an irrotational vector field $\mathbf{v}^{\perp} = \nabla A$ where the escape length is of order $N \sim c_2/c_1$.

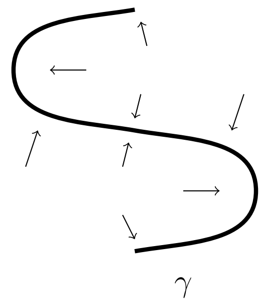

FIGURE 5. A part of the curve γ and the local gradients of $\tilde{A}$.

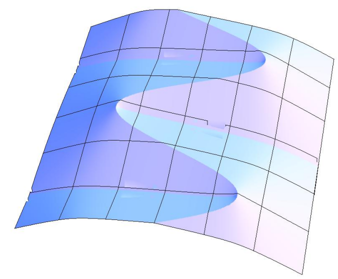

FIGURE 6. A plot of $\tilde{A}$.

4. PROOFS

4.1. **Proof of the Theorem 1.** The proof combines basic facts about incompressible vector fields in the plane with the coarea formula. In two dimensions we have

$$\operatorname{div} \mathbf{v} = \operatorname{curl} \mathbf{v}^{\perp} \quad \text{and thus} \quad \operatorname{curl} \mathbf{v}^{\perp} = 0 \quad \text{for incompressible } \mathbf{v}.$$

Therefore there exists a scalar function $A : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\mathbf{v}^{\perp} = \nabla A.$$

Trivially,

$$\langle \mathbf{v}, \nabla A \rangle = \langle \mathbf{v}, \mathbf{v}^{\perp} \rangle = 0.$$

Altogether, we have the existence of a function $A : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathbf{v}^{\perp}$ is the gradient flow with respect to A while $\mathbf{v}$ flows along the level sets of the function A. We know that the function A cannot change too much on a set of large measure inside the unit disk $\mathbb{D}$ because

$$\int_{\mathbb{D}} |\nabla A| = \int_{\mathbb{D}} \|\mathbf{v}^{\perp}\| = \int_{\mathbb{D}} \|\mathbf{v}\| \leq c_2 \pi.$$

We may assume w.l.o.g. (after possibly adding a constant) that $A(\gamma(0)) = 0$. We now use the coarea formula (see e.g. Federer [6])

$$\int_{-\infty}^{\infty} \mathcal{H}^1(A^{-1}(t)) dt = \int_{\mathbb{D}} |\nabla A|,$$

where $\mathcal{H}^1$ is the one-dimensional Hausdorff measure. Hence, for every parameter $z > 0$, we have

$$z \inf_{0<t<z} \mathcal{H}^1(A^{-1}(t)) \leq \int_{-z}^{z} \mathcal{H}^1(A^{-1}(t)) dt \leq \int_{-\infty}^{\infty} \mathcal{H}^1(A^{-1}(t)) dt = \int_{\mathbb{D}} |\nabla A| \leq c_2 \pi.$$

Setting $z = \sqrt{\pi c_1 c_2}$, this implies the existence of a $0 < t_0 \leq \sqrt{c_1 c_2 \pi}$ with

$$\mathcal{H}^1(A^{-1}(t_0)) \leq \sqrt{\frac{c_2}{c_1}} \pi.$$

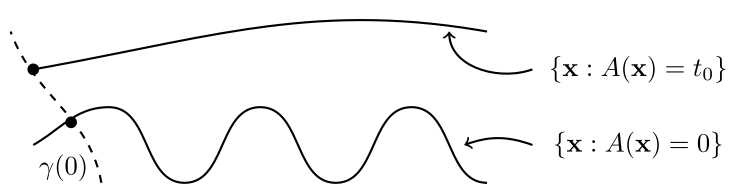

FIGURE 7. There exists a flow line of length at most $\sqrt{c_2 \pi / c_1}$ described as level set $A(\mathbf{x}) = t_0$ for some $|t_0| \leq \sqrt{\pi c_1 c_2}/2$. It remains to estimate the length of the dashed line.

Let $s_0 > 0$ be the smallest number for which

$$A\left(e^{s_0 \mathbf{v}^{\perp}\gamma(0)}\right) = t_0.$$

A priori it might not clear that such a s_0 exists: it follows from the fact that $\mathbf{v}^{\perp}$ is the gradient flow for A and $\|\mathbf{v}^{\perp}\| \geq c_1$. More precisely, using the fundamental theorem of calculus and the chain rule, we have

$$A\left(e^{s_0\mathbf{v}^{\perp}}\gamma(0)\right) = \int_0^{s_0} \left(A\left(e^{s\mathbf{v}^{\perp}}\gamma(0)\right)\right)' ds = \int_0^{s_0} \left\langle (\nabla A)(e^{s\mathbf{v}^{\perp}}\gamma(0)), (e^{s\mathbf{v}^{\perp}}\gamma(0))' \right\rangle ds.$$

However, since $\mathbf{v}^{\perp}$ is the gradient flow of A, the scalar product simplifies for every s to

$$\left\langle \nabla A\left(e^{s\mathbf{v}^{\perp}}\gamma(0)\right), (e^{s\mathbf{v}^{\perp}}\gamma(0))' \right\rangle = \left\| \nabla A\left(e^{s\mathbf{v}^{\perp}}\gamma(0)\right)\right\| \|(e^{s\mathbf{v}^{\perp}}\gamma(0))'\| \geq c_1 \|(e^{s\mathbf{v}^{\perp}}\gamma(0))'\|$$

and therefore

$$A\left(e^{s_0\mathbf{v}^{\perp}}\gamma(0)\right) = \int_0^{s_0} \left\langle (\nabla A)(e^{s\mathbf{v}^{\perp}}\gamma(0)), (e^{s\mathbf{v}^{\perp}}\gamma(0))' \right\rangle ds \geq c_1 \int_0^{s_0} \|(e^{s\mathbf{v}^{\perp}}\gamma(0))'\| ds = c_1 |e^{s_0\mathbf{v}^{\perp}}\gamma(0)|.$$

This implies that the length from $\gamma(0)$ along $\mathbf{v}^{\perp}$ until it hits $A^{-1}(t_0)$ is at most

$$\frac{t_0}{c_1} \leq \frac{\sqrt{c_1 c_2 \pi}}{c_1} = \sqrt{\pi}\sqrt{\frac{c_2}{c_1}}.$$

Summarizing, the curve from $(0,0)$ along $\mathbf{v}^{\perp}$ until it intersects $A^{-1}(t_0)$ has length at most $\sqrt{c_2\pi/c_1}$ and the level set $\{\mathbf{x} : A(\mathbf{x}) = t_0\}$ has length at most $\sqrt{c_2\pi/c_1}$ and this yields the statement. □

4.2. **Proof of Theorem 2.** Theorem 2 can be shown by a variant of the argument above. Let us first consider the curve

$$\gamma = \left\{ e^{t\mathbf{v}^{\perp}}(0,0)| : \|e^{t\mathbf{v}^{\perp}}(0,0)\| \leq 1 \right\}.$$

We want to give an upper bound on the length γ. Repeating the argument from the proof above, we can again define a potential $A : 2\mathbb{D} \rightarrow \mathbb{R}$ via $\nabla A = \mathbf{v}^{\perp}$. Since $\mathbf{v}^{\perp}$ is the gradient flow of A, we see that

$$\sup_{x\in\gamma} A(x) - \inf_{x\in\gamma} A(x) \geq c_1 |\gamma|.$$

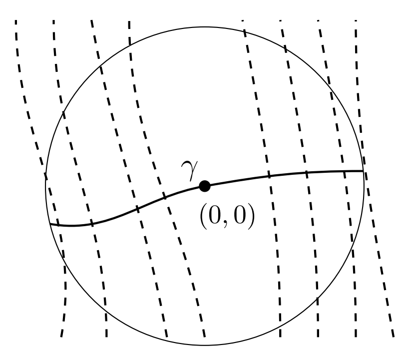

FIGURE 8. Definition of Ω: starting flows along $\mathbf{v}$ at every point in γ until they leave a unit disk around their starting point.

We repeat our previous argument on the domain $\Omega \subset 2\mathbb{D}$

$$\Omega = \bigcup_{x\in\gamma} \left\{ e^{t\mathbf{v}}x : \|e^{t\mathbf{v}}x - x\| \leq 1 \right\}.$$

Since A is constant along the flow of $\mathbf{v}$, we have

$$\sup_{x\in\Omega} A(x) - \inf_{x\in\Omega} A(x) \geq c_1 |\gamma|.$$

The coarea formula applied to $A : \Omega \to \mathbb{R}$ implies

$$\int_{\inf_{x\in\Omega} A(x)}^{\sup_{x\in\Omega} A(x)} \mathcal{H}^1(A^{-1}(t))dt = \int_{\Omega} |\nabla A| = \int_{\Omega} \|\mathbf{v}\| \leq \int_{2\mathbb{D}} \|\mathbf{v}\| \leq 4\pi c_2.$$

By assumption, we have for all $t \in A(\Omega)$

$$\mathcal{H}^1(A^{-1}(t)) \geq 2\mathcal{L},$$

where the constant is because flows both forward and backward in time are allowed. Combining all these statements gives

$$c_1|\gamma|2\mathcal{L} \leq \left(\sup_{x\in\Omega} A(x) - \inf_{x\in\Omega} A(x)\right) 2\mathcal{L} \leq \int_{\inf_{x\in\Omega} A(x)}^{\sup_{x\in\Omega} A(x)} \mathcal{H}^1(A^{-1}(t))dt \leq 4\pi c_2$$

which implies the statement

$$|\gamma| \leq 2\pi \frac{c_2/c_1}{\mathcal{L}}.$$

This is the total length of γ, which leaves Ω in two points; we take the shorter path. □

4.3. **Proof of Theorem 3.**

Proof of Theorem 3. The Poincaré-Bendixson theorem coupled with the condition $\|\mathbf{v}\| \geq c_1$ implies that the length of the flow until it exits the unit disk has length $\ell < \infty$. The same reasoning implies that if we reverse the flow of time by replacing $\mathbf{v}$ with $-\mathbf{v}$, the new escape length ℓ_- is also finite. We consider γ on the time of existence $[-\ell_-, \ell]$. Furthermore, the flow has no self-intersections and $\gamma : [-\ell_-, \ell] \to \mathbb{R}^2$ is injective and thus splits the unit ball into two connected components Ω_1, Ω_2. There exists $i \in \{1, 2\}$ such that

$$\mathcal{H}^1(\partial\Omega_i) \leq \ell^- + \ell + \pi$$

and we assume w.l.o.g. that $i = 1$.

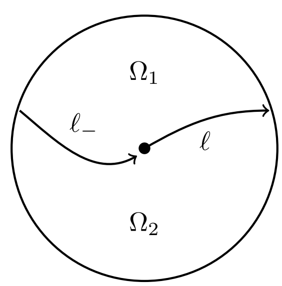

FIGURE 9. Ω_1 is the half of the unit disk with $\Omega_1 \cap \partial\mathbb{D}$ having length at most π.

Now we consider the orthogonal vector field

$$\mathbf{v}^\perp(x, y) = (-v_2(x, y), v_1(x, y))$$

along γ. Trivially, $\|\mathbf{v}^\perp\| \geq c_1$. We split the boundary of Ω_1 into the part contained in the interior of the open unit disk $\mathbb{D}$ and the the part contained in the boundary of the open unit disk $\mathbb{D}$, i.e.

$$\partial\Omega_1 = (\partial\Omega_1 \cap \mathbb{D}) \cup (\partial\Omega_1 \cap \partial\mathbb{D}).$$

On $\partial\Omega_1 \cap \mathbb{D}$ the vector field $\mathbf{v}^\perp$ is a multiple of the normal at the boundary and the map

$$n : \partial\Omega_1 \cap \mathbb{D} \to \mathbb{R}$$
$$x \to \left\langle \nu(x), \mathbf{v}^\perp \right\rangle$$

is a continuous function satisfying $|n(x)| \geq c_1$. Therefore

$$\left| \int_{\partial\Omega_1 \cap \mathbb{D}} \left\langle \mathbf{v}^\perp, \nu(x) \right\rangle d\sigma \right| \geq c_1(\ell^- + \ell)$$

because $\left\langle \nu(x), \mathbf{v}^{\perp} \right\rangle$ cannot change sign on $\partial \Omega_1 \cap \mathbb{D}$. On the other part of the boundary, we have

$$\left| \int_{\partial \Omega_1 \cap \partial \mathbb{D}} \left\langle \mathbf{v}^{\perp}, \nu(x) \right\rangle d\sigma \right| \leq \int_{\partial \Omega_1 \cap \partial \mathbb{D}} \|\mathbf{v}^{\perp}\| d\sigma \leq c_2 \pi.$$

The divergence theorem implies

$$\int_{\Omega_1} \operatorname{div} \mathbf{v}^{\perp} dxdy = \int_{\partial \Omega_1} \left\langle \mathbf{v}^{\perp}, \nu(x) \right\rangle d\sigma = \int_{\partial \Omega_1 \cap \mathbb{D}} \left\langle \mathbf{v}^{\perp}, \nu(x) \right\rangle d\sigma + \int_{\partial \Omega_1 \cap \partial \mathbb{D}} \left\langle \mathbf{v}^{\perp}, \nu(x) \right\rangle d\sigma$$

and thus

$$\begin{aligned} c_1(\ell^- + \ell) &\leq \left| \int_{\partial \Omega_1 \cap \mathbb{D}} \left\langle \mathbf{v}^{\perp}, \nu(x) \right\rangle d\sigma \right| \\ &\leq \left| \int_{\Omega_1} \operatorname{div} \mathbf{v}^{\perp} dxdy \right| + \left| \int_{\partial \Omega_1 \cap \partial \mathbb{D}} \left\langle \mathbf{v}^{\perp}, \nu(x) \right\rangle d\sigma \right| \\ &\leq \left| \int_{\Omega_1} \operatorname{div} \mathbf{v}^{\perp} dxdy \right| + c_2 \pi = \left| \int_{\Omega_1} \operatorname{curl} \mathbf{v} dxdy \right| + c_2 \pi. \end{aligned}$$

□

5. A Toy Problem

There are many other natural questions related to these statements: suppose $\mathbf{v} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a nonvanishing vector field and we are allowed to move along either $\mathbf{v}$ or $\mathbf{v}^{\perp}$ (forwards or backwards) and change as many times as we wish between the two. How long is the shortest such path starting in $(0, 0)$ and leaving the unit disk and what is the vector field maximizing the length of escape? Trivially, such a path has length at least 1.

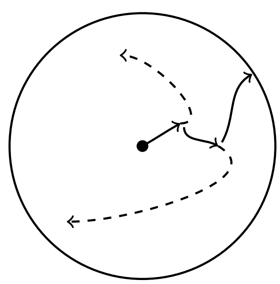

FIGURE 10. Switching between $\mathbf{v}$ and $\mathbf{v}^{\perp}$ to find the fastest escape route.

Proposition. *If $\boldsymbol{v} : \mathbb{D} \rightarrow \mathbb{R}^2$ is smooth, then for any point $\boldsymbol{x} \in \partial \mathbb{D}$ and any $\varepsilon > 0$ there exists an escape path from 0 to $\boldsymbol{x}$ with length at most $\sqrt{2} + \varepsilon$. There are examples, where the escape path to any point on the boundary has length at least*

$$\frac{1 + \sqrt{4 - \sqrt{7}}}{2} \sim 1.08...$$

The upper bound $\sqrt{2}$ is trivially sharp (for example, take $\mathbf{v} = (1, 0)$ and $\mathbf{x} = (1/\sqrt{2}, 1/\sqrt{2})$) but answers the wrong question since we are trying to find the most efficient escape to *some* point on the boundary. The lower bound can certainly be improved.

Sketch of a proof. We start with the easy upper bound. Since $\mathbf{v}$ is differentiable, it is not difficult to see that for every $\mathbf{x} \in \partial \mathbb{D}$ there is an admissible path from $(0, 0)$ to $\mathbf{x}$ of length at most $\sqrt{2}$: for every two unit vectors $|\mathbf{n}| = 1 = |\mathbf{v}|$ we have

$$\max \left(\langle \mathbf{n}, \mathbf{v} \rangle, \langle \mathbf{n}, -\mathbf{v} \rangle, \left\langle \mathbf{n}, \mathbf{v}^{\perp} \right\rangle, \left\langle \mathbf{n}, -\mathbf{v}^{\perp} \right\rangle \right) \geq \frac{1}{\sqrt{2}}.$$

Therefore we can at very point $\mathbf{y}$ pick the one direction among $\mathbf{v}(\mathbf{y}), -\mathbf{v}(\mathbf{y}), \mathbf{v}^{\perp}(\mathbf{y}), -\mathbf{v}^{\perp}(\mathbf{y})$ which has the biggest scalar product with $\mathbf{x} - \mathbf{y}$ and follow that vector field with additional corrections coming from the orthogonal vector field (see Fig. 11). Clearly, the worst case is if the maximum

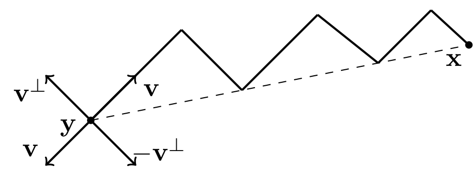

FIGURE 11. The local construction.

inner product is precisely $1/\sqrt{2}$ in which case the constructed path is a factor $\sqrt{2}$ longer than required. It is possible to attain use this construction to get a path of length $\sqrt{2}+\varepsilon$ for every $\varepsilon > 0$ by using smoothness and making the step size in the construction sufficiently small (depending on ε). The trivial lower bound 1 can be improved as well: we define the vector field to be

$$\mathbf{v}(x,y) = \begin{cases} (1,0) & \text{if } x^2+y^2 \leq \frac{1}{4} \\ (1,1) & \text{otherwise.} \end{cases}$$

An explicit computation yields that the length is at least

$$\frac{1+\sqrt{4-\sqrt{7}}}{2} \sim 1.08\dots$$

□

In this example for the lower bound some points on the boundary were easier to reach with a short path than others: this immediately suggests that the construction is not optimal and that one can do better. One way of doing better leads to a curious problem in probability theory: let us decompose the unit disk $\mathbb{D}$ into a large number of annuli

$$\mathbb{D} = \bigcup_{k=1}^{n} B\left(0, \frac{k}{n}\right) \setminus B\left(0, \frac{k-1}{n}\right)$$

and choose the value of $\mathbf{v}$ within each annulus to be in a fixed but randomly chosen direction (following a uniform distribution of the angle θ in $[0, 2\pi]$). This yields a random vector field $\mathbf{v}$. It seems natural to conjecture that, as $n \to \infty$, the shortest path from $(0,0)$ to any point $\mathbf{x}$ on

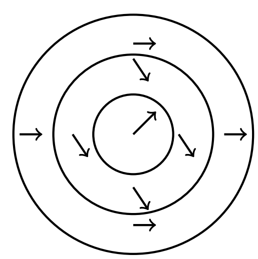

FIGURE 12. A random vector field constant in each annulus.

the boundary should be roughly of the same length (with some error tending to 0 as $n \to \infty$) and that this length should be given by

$$\left(\frac{4}{\pi}\int_0^{\frac{\pi}{4}} \cos\theta \, d\theta\right)^{-1} = \frac{\pi}{\sqrt{8}} \sim 1.1107...$$

A random vector field should not be able to create special 'prefered' directions.

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Department of Mathematics, Yale University, 10 Hillhouse Avenue, New Haven, 06511 CT, USA
E-mail address: stefan.steinerberger@yale.edu