We introduce an algorithm for the evaluation of the Incomplete Gamma Function, $P(m,x)$, for all $m,x>0$. For small m, a classical recursive scheme is used to evaluate $P(m,x)$, whereas a newly derived asymptotic expansion is used to evaluate $P(m,x)$ in the large m regime. The number of operations required for evaluation is $O(1)$ for all x and m. Nearly full double and extended precision accuracies are achieved in their respective environments. The performance of the scheme is illustrated via several numerical examples.

An Algorithm for the Evaluation of the Incomplete Gamma Function

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Technical Report YALEU/DCS/TR-1534
June 20, 2017

◇ This author's work was supported in part under ONR N00014-14-1-0797, AFOSR FA9550-16-0175, NIH 1R01HG008383
⊖ This author's work was supported in part under ONR N00014-14-1-0797, NSF DMS-1300120

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Keywords: *Incomplete Gamma Function, Special Functions, Numerical Evaluation*

An Algorithm for the Evaluation of the Incomplete Gamma Function

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Contents

1 Introduction

Properties and evaluation of special functions is one of the most developed areas of numerical analysis. For some (such as Bessel functions), the theory has been fairly complete for many decades; others (such as Prolate Spheroidal Wave Functions) are still an active area of research. In this respect, the Incomplete Gamma Function occupies an intermediate position. Its mathematical properies appear to be well understood, but the relevant numerical techniques leave much to be desired, at least in certain regimes. The purpose of this paper is to introduce a numerical scheme (or rather a class of numerical schemes) for the evaluation of the Incomplete Gamma Functions that produces (more or less) full double precisoion accuracy whenever the calculations are performed in double precision, is sufficiently fast to be compatible with standard schemes for the evaluation of other special functions, and produces roughly extended precision accuracy when implemented in extended precision (though in this regime, the algorithm loses much of its efficiency). The algorithm is based on the combination of an identity concerning the Incomplete Gamma Function (see [1] formula 6.5.22), with an asymptotic expansion that appears to be new (see [4], [5], [2], [7] , [8] below); its performance is illustrated with several numerical examples, in both double and extended precision (see Section 6).

The structure of this paper is as follows. In Section 2, we introduce notation and summarize a number of elementary mathematical results to be used throughout the remainder of the paper. Section 3 describes a technique for the evaluation of $P(m, x)$ via direct summation. Section 4 describes an asymptotic expansion for the evaluation of $P(m, x)$. Section 5 contains a description of an algorithm to evaluate $P(m, x)$ for all $m, x > 0$. Section 6 contains the results of numerical experiments with the algorithm for $P(m, x)$ described in Section 5.

2 Preliminaries

In accordance with standard practice, we will be denoting by $\gamma(m, x)$ the Incomplete Gamma Function,

$$\gamma(m+1, x) = \int_0^x t^m e^{-t} dt. \tag{1}$$

We will be denoting by $\overline{\gamma}(m, x)$ a scaled version of the Incomplete Gamma Function,

$$\begin{aligned} \overline{\gamma}(m+1, x) &= \int_0^x \frac{t^m e^{-t}}{m^m e^{-m}} dt \\ &= \int_{-m}^{x-m} e^{\phi(s)} ds \end{aligned} \tag{2}$$

where

$$e^{\phi(s)} = \frac{(m+s)^m e^{-(m+s)}}{m^m e^{-m}}. \tag{3}$$

We will be denoting by $P(m, x)$ (see [1]), the Incomplete Gamma Function scaled by the Complete Gamma Function. That is,

$$P(m, x) = \frac{\gamma(m, x)}{\Gamma(m)} = \frac{(m-1)^{(m-1)} e^{-(m-1)}}{\Gamma(m)} \overline{\gamma}(m, x). \tag{4}$$

We define f_m to be the function on $\mathbb{C}$ defined by the formula,

$$f_m(z) = \exp\big(m\log(1+z/m) - z + z^2/2m\big) \tag{5}$$

and observe that

$$e^{\phi(s)} = f_m(s)e^{s^2/2m}. \tag{6}$$

Consistent with standard practice, we denote by $\Gamma(m)$ the Complete Gamma Function,

$$\Gamma(m+1) = \int_0^\infty t^m e^{-t} dt. \tag{7}$$

We will be denoting by $\overline{\Gamma}(m)$, the scaled version of $\Gamma(m)$,

$$\overline{\Gamma}(m+1) = \int_0^\infty \frac{t^m e^{-t}}{m^m e^{-m}} dt. \tag{8}$$

The following lemma, Stirling's Approximation, is a classical asymptotic expansion for the Gamma Function, $\Gamma(m)$ (see (7)). It can be found in, [6], Formula 6.1.37, for example. Proofs for bounds on the error terms for Stirling's Appromxation that are provided in Lemma 2.1 can be found in, for example, [9].

Lemma 2.1. *[Stirling's Approximation] For all $m > 0$,*

$$\begin{aligned}\Gamma(m) \sim& e^{-m} m^{m-1/2}(2\pi)^{1/2}\left[1 + \frac{1}{12m} + \frac{1}{288m^2} - \frac{139}{51840m^3} + ...\right] \\ =& e^{-m} m^{m-1/2}(2\pi)^{1/2}\sum_{k=0}^{\infty} g_k(m),\end{aligned} \tag{9}$$

where formulas for $g_k(m)$ can be found in [10]. Further, for $K > 2$, if

$$\Gamma(m) = e^{-m} m^{m-1/2}(2\pi)^{1/2}\left(\sum_{k=0}^{K-1} g_k(m) + R_K(m)\right) \tag{10}$$

then

$$|R_K(m)| < \frac{\Gamma(K)}{(K+1)m^K}. \tag{11}$$

Additionally,

$$\begin{aligned}\ln(\Gamma(m)) &\sim (m-1/2)\ln(m) - m + 1/2\ln(2\pi) + \frac{1}{12m} - \frac{1}{360m^3} + \frac{1}{1260m^5} + ... \\ &= (m-1/2)\ln(m) - m + 1/2\ln(2\pi) + \sum_{k=0}^{\infty} h_k(m)\end{aligned} \tag{12}$$

where $ln(x)$ is the natural logarithm, and formulas for $h_k(m)$ can be found in, for example, [1]. Further, suppose

$$\ln(\Gamma(m)) = (m-1/2)\ln(m) - m + 1/2\ln(2\pi) + \sum_{k=0}^{K-1} h_k(m) + R_K(m). \tag{13}$$

Then,

$$|R_K(m)| \leq \frac{|B_{2K}|}{(2K-1)m^{2K-1}} \tag{14}$$

where B_n is n^{th} Bernoulli number (see [1]). In addition, $|R_K(m)|$ is smaller in magnitude than the first neglected term (see [9]).

The following bound for $\Gamma(m)$ will be used throughout the remainder of the paper.

Observation 2.1. *Straightforward application of Stirling's Approximation (see (9)) shows that for $m > 1$,*

$$\Gamma(m+1) > m^{m+1/2}e^{-m} > 0. \tag{15}$$

It follows immediately from (15) that for $m > 1$,

$$0 < \frac{m^m e^{-m}}{\Gamma(m+1)} < m^{-1/2} \tag{16}$$

The following lemma will help in the proof of Lemma 2.3. It is a well known inequality. A proof can be found in, for example, Section VII of [3].

Lemma 2.2. *For all $x, \sigma > 0$,*

$$\frac{1}{(2\pi\sigma^2)^{1/2}} \int_{-\infty}^{-\sigma x} e^{-t^2/2\sigma^2} dt < \frac{1}{x} \frac{e^{-x^2/2}}{(2\pi)^{1/2}}. \tag{17}$$

2.1 Mathematical Apparatus

In this subsection, the main analytical tools are Lemma 2.8 and Corollary 2.1. They will be used in Sections 2.2 and 2.3. All other apparatus will be used in support of Lemma 2.8 and Corollary 2.1 or in Sections 2.2, 2.3, 3, and 4. We use the following lemma in the proof of Lemma 2.9.

Lemma 2.3. *For all $\alpha, m > 0$,*

$$\int_0^{m-\alpha m^{1/2}} m^{-1/2} e^{-(t-m)^2/2m+1} dt < \frac{1}{\alpha} \frac{e^{-\alpha^2/2+1}}{(2\pi)^{1/2}}. \tag{18}$$

Proof. Clearly,

$$\begin{aligned} \int_0^{m-\alpha m^{1/2}} m^{-1/2} e^{-(t-m)^2/2m+1} dt &= e m^{-1/2} \int_0^{m-\alpha m^{1/2}} e^{-(t-m)^2/2m} dt \\ &< e m^{-1/2} \int_{-\infty}^{m-\alpha m^{1/2}} e^{-(t-m)^2/2m} dt \\ &= e m^{-1/2} \int_{-\infty}^{-\alpha m^{1/2}} e^{-t^2/2m} dt. \end{aligned} \tag{19}$$

Applying Lemma 2.2 to (19),

$$e m^{-1/2} \int_{-\infty}^{-\alpha m^{1/2}} e^{-(t)^2/2m} dt < \frac{1}{\alpha} e^{-\alpha^2/2+1} \tag{20}$$

(18) follows immediately from the combination of (19) and (20). □

The following lemma will be used in the proof of Lemma 2.7.

Lemma 2.4. *Let $\{a_n\}$ be a non-negative, monotonically decreasing sequence in $\mathbb{R}$. Then,*

$$\left|\sum_{n=1}^{\infty}(-1)^{n+1}a_n\right| \leq a_1. \tag{21}$$

Proof. Clearly, since $\{a_n\}$ is monotonically decreasing, for all $i \in \mathbb{N}$,

$$a_i - a_{i+1} \geq 0. \tag{22}$$

therefore,

$$\sum_{n=1}^{\infty}(-1)^{n+1}a_n = \sum_{n=1}^{\infty}(a_n - a_{n+1}) \geq 0. \tag{23}$$

Combining (22) and (23) yields,

$$0 \leq \sum_{n=1}^{\infty}(-1)^{n+1}a_n = a_1 - \sum_{n=2}^{\infty}(a_n - a_{n+1}) \leq a_1. \tag{24}$$

(21) follows from (24). □

Lemma 2.5 and Lemma 2.4 will be used in Section 5.

Lemma 2.5. *For all $n \in \mathbb{N}$ and $m > 0$,*

$$\int t^{2n+1}e^{-t^2/2m}dt = 2e^{-t^2/2m}(-2mx^{2n+2} - \sum_{k=1}^{n+1}(2m)^{k+1}(n+1)(n)...(n-k+2)x^{2(n+1-k)}) \tag{25}$$

Proof. By the change of variables $x = t^2$,

$$\int_a^b t^{2n+1}e^{-t^2/2m}dt = 2\int_{a^2}^{b^2} x^{n+1}e^{-x/2m}dx. \tag{26}$$

From Formula 2.432.2 in [6] we know

$$\int x^n e^{-x/2m}dx = e^{-x/2m}\left(-2mx^n - \sum_{k=1}^{n}(2m)^{k+1}n(n-1)...(n-k+1)x^{n-k}\right). \tag{27}$$

Combining (26) and (27) yields (25). □

Lemma 2.6. *For all $n \in \mathbb{N}$ and $m > 0$*

$$\int t^{2n+2}e^{-t^2/2m}dt = \prod_{i=0}^{n}\alpha_i(2m)^{1/2}\frac{\sqrt{\pi}}{2}erf\left(\frac{t}{\sqrt{2m}}\right) - \sum_{i=0}^{n-1}\left(\beta_i\prod_{j=0}^{n-i}\alpha_{n-j}\right) - \beta_n, \tag{28}$$

where

$$\alpha_i = (2i+1)m \tag{29}$$

and

$$\beta_i = mt^{2i+1}e^{-t^2/2m} \tag{30}$$

and in accordance with standard practice,

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{31}$$

Proof. Integrating by parts,

$$\int t^{2n} e^{-t^2/2m} dt = \frac{t^{2n+1}}{2n+1} e^{-t^2/2m} + \frac{1}{m(2n+1)} \int t^{2n+2} e^{-t^2/2m} dt. \tag{32}$$

Now, rearranging the terms of (32),

$$\int t^{2n+2} e^{-t^2/2m} dt = (2n+1)m \int t^{2n} e^{-t^2/2m} dt - mt^{2n+1} e^{-t^2/2m}. \tag{33}$$

Repeated application of identity (33) yields,

$$\int t^{2n+2} e^{-t^2/2m} dt = \prod_{i=0}^{n} \alpha_i \int e^{-t^2/2m} dt - \sum_{i=0}^{n-1} \left(\beta_i \prod_{j=0}^{n-i} \alpha_{n-j} \right) - \beta_n, \tag{34}$$

where

$$\alpha_i = (2i+1)m \tag{35}$$

and

$$\beta_i = mt^{2i+1} e^{-t^2/2m}. \tag{36}$$

Through a straightforward change of variables, we obtain the identity

$$\int_a^b e^{-t^2/2m} dt = (2m)^{1/2} \frac{\sqrt{\pi}}{2} \left(\mathrm{erf}\left(\frac{b}{\sqrt{2m}} \right) - \mathrm{erf}\left(\frac{a}{\sqrt{2m}} \right) \right), \tag{37}$$

where, in accordance with standard practice,

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{38}$$

(28) follows directly from the combination of (34) and (37). □

The following bound will help in the proof of Lemma 2.8.

Lemma 2.7. *For all $m > 0$ and $\alpha \in (-m^{1/2}, m^{1/2})$,*

$$-\sum_{j=3}^{\infty} \frac{(\alpha/m^{1/2})^j}{j} \leq 1. \tag{39}$$

Proof. We will consider two cases.

Case 1: $\alpha \in [0, m^{1/2})$. Clearly, since $m > 0$,

$$\alpha/m^{1/2} \geq 0. \tag{40}$$

It follows immediately that

$$-\sum_{j=3}^{\infty} \frac{(\alpha/m^{1/2})^j}{j} \leq 0 \tag{41}$$

for all $\alpha \in [0, m^{1/2})$.
Case 2: $\alpha \in (-m^{1/2}, 0)$. Clearly, for all $\alpha \in (-m^{1/2}, 0)$,

$$-\sum_{j=3}^{\infty} \frac{(\alpha/m^{1/2})^j}{j} = \sum_{j=3}^{\infty} (-1)^{j+1} \frac{|\alpha/m^{j/2}|}{j}. \tag{42}$$

Furthermore, the sequence

$$\left\{ \frac{|\alpha/m^{j/2}|}{j} \right\} \tag{43}$$

is a non-negative, monotonically decreasing sequence in j. Therefore, according to Lemma 2.4,

$$\sum_{j=3}^{\infty} (-1)^{j+1} \frac{|\alpha/m^{j/2}|}{j} < \frac{|\alpha^3|}{3m^{3/2}} < \frac{1}{3}. \tag{44}$$

Combining (42) and (44) yields

$$-\sum_{j=3}^{\infty} \frac{(\alpha/m^{1/2})^j}{j} < \frac{1}{3}. \tag{45}$$

for all $\alpha \in (-m^{1/2}, 0)$. Combining (41) and (45) yields (39). □

The following lemma will be used in the proof of Corollary 2.1.

Lemma 2.8. *For all $m > 1$ and $\alpha \in (-m^{1/2}, m^{1/2})$,*

$$\frac{(m - \alpha m^{1/2})^m e^{-(m - \alpha m^{1/2})}}{\Gamma(m+1)} < m^{-1/2} e^{-\alpha^2/2 + 1} \tag{46}$$

where $\Gamma(m)$ is defined in equation (7).

Proof. It follows immediately from Observation 2.1 that for all $m > 1$,

$$\begin{aligned} \frac{(m - \alpha m^{1/2})^m e^{-(m - \alpha m^{1/2})}}{\Gamma(m+1)} &< \frac{(m - \alpha m^{1/2})^m e^{-(m - \alpha m^{1/2})}}{m^{m+1/2} e^{-m}} \\ &= m^{-1/2} \left(\frac{m - \alpha m^{1/2}}{m} \right)^m \frac{e^{-(m - \alpha m^{1/2})}}{e^{-m}}. \end{aligned} \tag{47}$$

Clearly,

$$\begin{aligned} m^{-1/2} \left(\frac{m - \alpha m^{1/2}}{m} \right)^m \frac{e^{-(m - \alpha m^{1/2})}}{e^{-m}} &= m^{-1/2} \left(1 - \frac{\alpha}{m^{1/2}} \right)^m e^{\alpha m^{1/2}} \\ &= m^{-1/2} \exp\left(m \log\left(1 - \frac{\alpha}{m^{1/2}} \right) \right) e^{\alpha m^{1/2}}. \end{aligned} \tag{48}$$

Expanding

$$\log(1 - \frac{\alpha}{m^{1/2}}) \tag{49}$$

into Taylor series yields

$$\begin{aligned} m^{-1/2} \exp\left(m \log\left(1 - \frac{\alpha}{m^{1/2}}\right)\right) e^{\alpha m^{1/2}} &= m^{-1/2} \exp\left(-m \sum_{j=1}^{\infty} \frac{(\alpha/m^{1/2})^j}{j}\right) e^{\alpha m^{1/2}} \\ &= m^{-1/2} e^{-\alpha^2/2} \exp\left(-\sum_{j=3}^{\infty} \frac{(\alpha/m^{1/2})^j}{j}\right) \end{aligned} \tag{50}$$

for $\alpha \in (-m^{1/2}, m^{1/2})$. According to Lemma 2.7,

$$e^{-\alpha^2/2} \exp\left(-\sum_{j=3}^{\infty} \frac{(\alpha/m^{1/2})^j}{j}\right) < e^{-\alpha^2/2+1} \tag{51}$$

for all $\alpha \in (-m^{1/2}, m^{1/2})$. Combining (48), (50), and (51) yields (46). □

The following inequality will be used in the proofs of Lemmas 2.9 and 2.10.

Corollary 2.1. *For all $m > 0$ and $t \in (0, 2m)$,*

$$\frac{t^m e^{-t}}{\Gamma(m+1)} < m^{-1/2} e^{-\frac{(t-m)^2}{2m}+1}. \tag{52}$$

Proof. Obviously, for all $m > 0$,

$$t = m - (\frac{-t+m}{m^{1/2}}) m^{1/2}. \tag{53}$$

Combining Observation 2.1 and (53) we have

$$\frac{t^m e^{-t}}{\Gamma(m+1)} = \frac{\left(m - (\frac{-t+m}{m^{1/2}}) m^{1/2}\right)^m e^{-m - (\frac{-t+m}{m^{1/2}}) m^{1/2}}}{\Gamma(m+1)}. \tag{54}$$

Combining Lemma 2.8 with (54) yields,

$$\begin{aligned} \frac{\left(m - (\frac{-t+m}{m^{1/2}}) m^{1/2}\right)^m e^{-m - (\frac{-t+m}{m^{1/2}}) m^{1/2}}}{\Gamma(m+1)} &< m^{-1/2} e^{-\frac{1}{2}\left(\frac{-t+m}{m^{1/2}}\right)^2 + 1} \\ &= m^{-1/2} e^{\frac{-(t-m)^2}{2m}+1} \end{aligned} \tag{55}$$

for $\frac{-t+m}{m^{1/2}} \in (-m^{1/2}, m^{1/2})$ or, equivalently, $t \in (0, 2m)$. □

2.2 $P(m,x)$ for Small x

The principal purpose of this subsection is to introduce Lemma 2.9, which shows that for sufficiently small x, the function $P(m,x)$ is well approximated by 0.

Lemma 2.9. *For all $m > 1$ and $\alpha \in (0, m^{1/2})$,*

$$P(m+1, m-\alpha m^{1/2}) < \frac{1}{\alpha} e^{-\alpha^2/2+1} \tag{56}$$

with $P(m,x)$ defined in (4).

Proof. Using (2) and applying Corollary 2.1 and Lemma 2.3, we have

$$\begin{aligned} P(m+1, m-\alpha m^{1/2}) &= \int_0^{m-\alpha m^{1/2}} \frac{t^m e^{-t}}{\Gamma(m+1)} dt \\ &< \int_0^{m-\alpha m^{1/2}} m^{-1/2} e^{-\frac{(t-m)^2}{2m}+1} dt \\ &< \frac{1}{\alpha} e^{-\alpha^2/2+1}. \end{aligned} \tag{57}$$

□

Remark 2.1. *Suppose $m > 1$. By observing that $P(m,x)$ is non-negative for all $x, m > 0$, and applying Lemma 2.9 with $\alpha = m^{1/6}$, we obtain the bound*

$$|P(m+1, m-m^{2/3})| < m^{-1/6} e^{\frac{-m^{1/3}}{2}+1} \tag{58}$$

for all $m > 1$ where $P(m,x)$ is defined in (4).

2.3 $P(m,x)$ for large x

The main purpose of this subsection is to introduce Lemma 2.12, which shows that for sufficiently large x, the function $P(m,x)$ is well approximated by $\Gamma(m)$. In the following lemma, we provide a bound to be used in the proof of Lemma 2.12.

Lemma 2.10. *For all $m > 1$ and $\alpha \in (0, m^{1/2})$,*

$$|P(m+1, 2m) - P(m+1, m+\alpha m^{1/2})| < m^{1/2} e^{-\alpha^2/2+1}, \tag{59}$$

where $P(m,x)$ is defined in (4).

Proof. Clearly, by (4) and applying Corollary 2.1,

$$\begin{aligned} |P(m+1,2m) - P(m+1, m+\alpha m^{1/2})| &= \int_{m+\alpha m^{1/2}}^{2m} \frac{t^m e^{-t}}{\Gamma(m+1)} dt \\ &< \int_{m+\alpha m^{1/2}}^{2m} m^{-1/2} e^{-\frac{(t-m)^2}{2m}+1} dt. \end{aligned} \tag{60}$$

Since,

$$\exp\left(-\frac{(t-m)^2}{2m}+1\right) = \exp\left(-\frac{(t-2(t-m)-m)^2}{2m}+1\right), \tag{61}$$

shifting the integration bounds of (60) and applying Lemma 2.9, we obtain

$$\int_{m+\alpha m^{1/2}}^{2m} m^{-1/2} e^{-\frac{(t-m)^2}{2m}+1} dt = \int_{0}^{m-\alpha m^{1/2}} m^{-1/2} e^{-\frac{(t-m)^2}{2m}+1} dt < m^{1/2} e^{-\alpha^2/2+1}. \tag{62}$$

Combining (60) and (62) yields (59). □

In the following lemma, we provide a bound to be used in the proof of Lemma 2.12.

Lemma 2.11. *For all $m > 1$,*

$$|1 - P(m+1, 2m)| < 10 m^{-1/2} e^{-m/5}, \tag{63}$$

where $P(m, x)$ is defined in (4).

Proof. Obviously, by (2) and (8),

$$\begin{aligned} |1 - P(m+1, 2m)| &= \int_{2m}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt \\ &= \int_{2m}^{\infty} \frac{t^m e^{-9t/10}}{\Gamma(m+1)} e^{-t/10} dt \\ &= \int_{2m}^{\infty} \psi_m(t) e^{-t/10} dt, \end{aligned} \tag{64}$$

where, by Observation 2.1,

$$\psi_m(t) = \frac{t^m e^{-9t/10}}{\Gamma(m+1)} < m^{-1/2} \frac{t^m e^{-9t/10}}{m^m e^{-m}}. \tag{65}$$

We now provide a bound for $\psi_m(t)$ on the interval $t \in (2m, \infty)$. Straightforward differentiation shows that for all $m > 1$ and $t \in (2m, \infty)$, the function $\psi_m(t)$ is decreasing as a function of t. Therefore, using (65), for $t \in (2m, \infty)$,

$$\psi_m(t) \le \psi_m(2m) < m^{-1/2} \frac{(2m)^m e^{-9m/5}}{m^m e^{-m}} = m^{-1/2} 2^m e^{-4m/5} < m^{-1/2}. \tag{66}$$

Therefore, combining (64) and (66) yields,

$$\int_{2m}^{\infty} \frac{t^m e^{-9t/10}}{\Gamma(m+1)} e^{-t/10} dt < \int_{2m}^{\infty} m^{-1/2} e^{-t/10} dt = 10 m^{-1/2} e^{-m/5}. \tag{67}$$

Combining (64) and (67) yields (63). □

The following lemma shows that for sufficiently large x, the function $P(m, x)$ is well-approximated by 1.

Lemma 2.12. *For all $m > 1$ and $\alpha \in (0, m^{1/2})$,*

$$\left| 1 - P(m+1, m + \alpha m^{1/2}) \right| < m^{1/2} e^{-\alpha^2/2+1} + 10 m^{-1/2} e^{-m/5}, \tag{68}$$

where $P(m, x)$ is defined in (4).

Proof. Obviously, by (2) and (4),

$$\begin{aligned} \left|1 - P(m+1, m+\alpha m^{1/2})\right| &= \int_{m+\alpha m^{1/2}}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt \\ &= \int_{m+\alpha m^{1/2}}^{2m} \frac{t^m e^{-t}}{\Gamma(m+1)} dt + \int_{2m}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt. \end{aligned} \tag{69}$$

According to Lemma 2.10,

$$\int_{m+\alpha m^{1/2}}^{2m} \frac{t^m e^{-t}}{\Gamma(m+1)} dt < m^{1/2} e^{-\alpha^2/2+1}. \tag{70}$$

According to Lemma 2.11,

$$\int_{2m}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt < 10 m^{-1/2} e^{-m/5}. \tag{71}$$

Combining (69), (70), and (71) yields (68). □

Corollary 2.2. *For all $m > 1$ and $x > m$,*

$$|1 - P(m+1, x)| < m^{1/2} e^{-\frac{(x-m)^2}{2m}+1} + 10 m^{-1/2} e^{-m/5}, \tag{72}$$

where $P(m, x)$ is defined in (2).

Proof. We consider two cases.

Case 1. Suppose $x \in (m, 2m)$. Obviously, by (2),

$$\begin{aligned} |1 - P(m+1, x)| &= \int_{x}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt \\ &= \int_{x}^{2m} \frac{t^m e^{-t}}{\Gamma(m+1)} dt + \int_{2m}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt. \end{aligned} \tag{73}$$

Using the identity

$$x = m + \left(\frac{x-m}{m^{1/2}} \right) m^{1/2} \tag{74}$$

and applying Lemma 2.10 to (73),

$$\int_{x}^{2m} \frac{t^m e^{-t}}{\Gamma(m+1)} dt = \int_{m+\frac{x-m}{m^{1/2}} m^{1/2}}^{2m} \frac{t^m e^{-t}}{\Gamma(m+1)} dt < m^{1/2} e^{-\frac{(x-m)^2}{2m}+1}. \tag{75}$$

Applying Lemma 2.11 to (73),

$$\int_{2m}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt < 10 m^{-1/2} e^{-m/5}. \tag{76}$$

Combining (73), (75), and (76) yields (72) for all $x \in (m, 2m)$.

Case 2. Suppose $x \geq 2m$. Obviously, by (2),

$$|1 - P(m+1, x)| = \int_{x}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt < \int_{2m}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt. \tag{77}$$

Applying Lemma 2.11 to (77),

$$\int_{2m}^{\infty} \frac{t^m e^{-t}}{\Gamma(m+1)} dt < 10 m^{-1/2} e^{-m/5}. \tag{78}$$

Combining (77) and (78) yields (72) for all $x \geq 2m$. □

3 Evaluation of $P(m,x)$ by Summation

The principal purpose of this section is Lemma 3.2, which provides a formula for evaluating $P(m,x)$ (see (2)), for all $m, x > 0$, to arbitrarily high precision. The following lemma will be used in the proof of Lemma 3.2.

Lemma 3.1. *For all $m > 1$, $k \geq 0$, $x \in (0,m)$,*

$$P(m+k+1,x) < \frac{(m+k)^{1/2}}{m+k-x} \exp\left(\frac{-(k+m-x)^2}{2k+2m} + 1 \right), \tag{79}$$

where $P(m,x)$ is defined in (4).

Proof. Clearly, since $m > 0$ and $k > 0$,

$$x = (m+k) - \alpha (m+k)^{1/2}, \tag{80}$$

where

$$\alpha = \frac{m+k-x}{(m+k)^{1/2}}. \tag{81}$$

Therefore, by (80), (81), and applying Lemma 2.9, we obtain,

$$\begin{aligned} P(m+k+1,x) &= P(m+k+1, (m+k) - \alpha(m+k)^{1/2}) \\ &< \frac{(m+k)^{1/2}}{m+k-x} \exp\left(\frac{-(k+m-x)^2}{2k+2m} + 1 \right). \end{aligned} \tag{82}$$

□

In the following lemma, we provide a formula for evaluating $P(m,x)$ and a bound on the error of the formula.

Lemma 3.2. *For all $m > 1$, $k \geq 1$, $x \in (0,m)$,*

$$P(m,x) = \sum_{i=0}^{k} \frac{x^{m+i} e^{-x}}{\Gamma(m+1+i)} + \rho_{k+1}(m,x) \tag{83}$$

where

$$|\rho_{k+1}(m,x)| < \frac{(m+k)^{1/2}}{m+k} \exp\left(\frac{-(k+m-x)^2}{2k+2m} + 1 \right) \tag{84}$$

Proof. By Formula 6.5.21 of [1],

$$P(m+1,x) = P(m,x) - \frac{x^m e^{-x}}{\Gamma(m+1)}. \tag{85}$$

Iteratively applying identity (85) k times yields

$$P(m,x) = P(m+k+1,x) + \sum_{i=0}^{k} \frac{x^{m+i}e^{-x}}{\Gamma(m+1+i)}. \tag{86}$$

According to Lemma 3.1,

$$0 < P(m+k+1,x) < \frac{(m+k)^{1/2}}{m+k-x} \exp\left(\frac{-(k+m-x)^2}{2k+2m}\right). \tag{87}$$

Combining (86) and (87) yields (84). □

Observation 3.1. *For all $m>1$, $\alpha \in (0, m^{1/2})$, and $x \in (m-\alpha m^{1/2}, m)$, applying Lemma 3.2 with $k \geq \lambda m^{1/2}$ where $\lambda \in (1, m^{1/2})$, we obtain the bound*

$$\begin{aligned}\rho_{k+1}(m,x) &< \frac{(m+\lambda m^{1/2})^{1/2}}{\lambda m^{1/2}} \exp\left(\frac{-(\lambda m^{1/2}-\alpha m^{1/2})^2}{2\lambda m^{1/2}+2m}\right) \\ &< \frac{(m+\lambda m^{1/2})^{1/2}}{\lambda m^{1/2}} \exp\left(\frac{-(\lambda-\alpha)^2}{4}\right).\end{aligned} \tag{88}$$

The following lemma will be used in Section 5.

Lemma 3.3. *For all $m>1$, $x>m$,*

$$P(m+k+1,x) = \left(1 - \sum_{i=0}^{k-1} \frac{x^{m+1+i}e^{-x}}{\Gamma(m+2+i)}\right) + \omega_k(m,x) \tag{89}$$

where

$$|\omega_k(m,x)| < m^{1/2}e^{-\frac{(x-m)^2}{2m}+1} + 10m^{-1/2}e^{-m/5}. \tag{90}$$

Proof. Lemma 3.3 follows immediately from the combination of (86) and Corollary 2.2. □

4 Evaluation of $P(m,x)$ by Asymptotic Expansion

In this section, we introduce an asymptotic expansion for the evaluation of $P(m,x)$ for sufficiently large m and $x \in (-m^{2/3}, m^{2/3})$.

We will denote by S_m the set of points $z \in \mathbb{C}$ such that $|z-x| \leq m^{2/3}$ for some $x \in (-m^{2/3}, m^{2/3})$. That is,

$$S_m = \{z \in \mathbb{C} : |z-x| < m^{2/3} \text{ for some } x \in (-m^{2/3}, m^{2/3})\}. \tag{91}$$

Observation 4.1. $|z| < 2m^{2/3}$ *for all $m>0$ and $z \in S_m$. In particular, if $m>100$, then for all $z \in S_m$, $|z| < 2m^{2/3} < m$.*

The following observations will be used in the proof of Lemma 4.2.

Observation 4.2. *Suppose we choose the branch cut for f_m to be the negative real axis with $x < -m$. Then, by (5), (91), and applying Observation 4.1, we observe that f_m is analytic on S_m, where f_m is defined in (5) and S_m is defined in (91).*

Observation 4.3. *It follows immediately from the combination of Observation 4.2 and Observation 4.1 that for all $m > 100$ and $\xi \in (-m^{2/3}, m^{2/3})$, the function f_m is analytic on the disk of radius $m^{2/3}$ centered at ξ, where f_m is defined in (5).*

The following lemma will be used in the proof of Lemma 4.2.

Lemma 4.1. *For all $m > 0$ and $z \in S_m$,*

$$|f_m(z)| < 15, \tag{92}$$

where f_m is defined in (5) and S_m is defined in (91).

Proof. Obviously, by (5), for all $m > 0$ and $z \in S_m$,

$$\begin{aligned} |f_m(z)| &= |\exp[m\log(1+z/m) - z + z^2/2m]| \\ &= \exp[\mathrm{Re}\{m\log(1+z/m) - z + z^2/2m\}]. \end{aligned} \tag{93}$$

Hence, expanding $\log(1+z)$ into Taylor series, we have

$$\begin{aligned} |f_m(z)| &= \exp[\mathrm{Re}\{m\sum_{k=1}^{\infty}(-1)^{k+1}(\frac{z}{m})^k - z + z^2/2m\}] \\ &= \exp[\mathrm{Re}\{\sum_{k=3}^{\infty}(-1)^{k+1}\frac{z^k}{km^{k-1}}\}]. \end{aligned} \tag{94}$$

Therefore, by (94), and the combination of Observation 4.1 and Lemma 2.4,

$$|f_m(z)| \le \exp[\mathrm{Re}\{\sum_{k=3}^{\infty}(-1)^{k+1}\frac{(2m^{2/3})^k}{km^{k-1}}\}] \le \exp\left(\frac{(2m^{2/3})^3}{3m^2}\right) = e^{8/3} < 15. \tag{95}$$

□

In Lemma 4.2, we provide a bound to be used in the proof of Theorem 4.1.

Lemma 4.2. *For all $m > 100$ and $\xi \in (-m^{2/3}, m^{2/3})$,*

$$\left|\frac{f_m^{(k)}(\xi)}{k!}\right| < \frac{15}{m^{2k/3}}, \tag{96}$$

where $f_m^{(k)}$ is the k^{th} derivative of f_m (see (5)).

Proof. Let $m > 100$ and $\xi \in (-m^{2/3}, m^{2/3})$. Let Γ_ξ be the positively oriented circular contour of radius $m^{2/3}$ centered at ξ. Then combining Observation 4.2 and the Cauchy Integral Formula and applying elementary integral transformations, we obtain

$$\begin{aligned} \left|\frac{f^{(k)}(\xi)}{k!}\right| &= \left|\frac{1}{2\pi i}\int_{\Gamma_\xi}\frac{f(z)}{(z-\xi)^{k+1}}dz\right| \\ &\le \frac{1}{2\pi}\int_{\Gamma_\xi}\frac{|f(z)|}{|(z-\xi)^{k+1}|}dz. \end{aligned} \tag{97}$$

By applying Lemma 4.1 to (97), we have

$$\begin{aligned} \left| \frac{f^{(k)}(\xi)}{k!} \right| &\leq \frac{1}{2\pi} \int_{\Gamma_\xi} \frac{15}{|(z-\xi)^{k+1}|} dz \\ &= \frac{15}{2\pi} \int_{\Gamma_\xi} \frac{1}{m^{2(k+1)/3}} dz. \end{aligned} \tag{98}$$

Now, combining (98) with the fact that Γ_ξ is of length $2\pi m^{2/3}$ yields,

$$\left| \frac{f^{(k)}(\xi)}{k!} \right| \leq 15 \frac{m^{2/3}}{m^{2(k+1)/3}} = \frac{15}{m^{2k/3}}. \tag{99}$$

□

The following observation will be used in the proof of Lemma 4.3.

Observation 4.4. *Expanding f_m into k-order Taylor series centered at 0, and using (2) and (6), we obtain*

$$\begin{aligned} \overline{\gamma}(m+1, x) &= \int_{-m}^{x-m} e^{-s^2/2m} f_m(s) ds. \\ &= \int_{-m}^{x-m} e^{-s^2/2m} (1 + f_m'(0)s + ... + \frac{f_m^{(k)}(0)}{k!} s^k + R_{k+1}) ds \end{aligned} \tag{100}$$

where $R_k(s)$ is the Taylor remainder term,

$$R_k(s) = \frac{f_m^{(k)}(\xi) s^k}{k!} \tag{101}$$

for some $\xi \in (0, x)$. The function $\overline{\gamma}(m, x)$ above is defined in (2), f_m is defined in (5), and $f_m^{(k)}$ is the k^{th} derivative of f_m.

The following lemma will be used in the proof of Theorem 4.1.

Lemma 4.3. *For all $m > 100$ and $x \in (-m^{2/3}, m^{2/3})$,*

$$\left| \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} R_k(s) ds \right| \leq 15 \frac{\Gamma(\frac{k+1}{2}) 2^{(k+1)/2}}{m^{k/6-1/2}}, \tag{102}$$

where R_k is defined in (101) and $\Gamma(k)$ is defined in (7).

Proof. Using (101) and applying elementary integral transformations to (102),

$$\begin{aligned} \left| \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} R_k(s) ds \right| &< \int_{-m^{2/3}}^{x-m} \left| e^{-s^2/2m} R_k(s) \right| ds \\ &= \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} \left| \frac{f^{(k)}(s) s^k}{k!} \right| ds. \end{aligned} \tag{103}$$

It follows immediately from applying Lemma 4.2 to (103) that

$$\begin{aligned}\left|\int_{-m^{2/3}}^{x-m} e^{-s^2/2m} R_k(s)ds\right| &< \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} \left|\frac{f^{(k)}(s)s^k}{k!}\right| ds \\ &< \frac{15}{m^{2k/3}} \int_{-m^{2/3}}^{x-m} e^{-s^2/2m}|s^k|ds \\ &\leq \frac{15}{m^{2k/3}} \int_{-\infty}^{\infty} e^{-s^2/2m}|s^k|ds.\end{aligned} \tag{104}$$

Combining forumulas 7.4.4 and 7.4.5 in [1], we obtain the identity,

$$\int_{-\infty}^{\infty} e^{-s^2/2m}|s^k|ds = \Gamma\left(\frac{k+1}{2}\right)(2m)^{(k+1)/2}, \tag{105}$$

and combining (104) and (105) yields,

$$\begin{aligned}\left|\int_{-m^{2/3}}^{x-m} e^{-s^2/2m} R_k(s)ds\right| &< \frac{15}{m^{2k/3}}\Gamma\left(\frac{k+1}{2}\right)(2m)^{(k+1)/2} \\ &= 15\frac{\Gamma(\frac{k+1}{2})2^{(k+1)/2}}{m^{k/6-1/2}}.\end{aligned} \tag{106}$$

□

The following theorem provides an asymptotic expansion for the evaluation of $P(m,x)$ where $P(m,x)$ is defined in (4).

Theorem 4.1. *For all $m > 100$ and $x \in (m - m^{2/3}, m + m^{2/3})$,*

$$P(m+1,x) \sim \frac{m^m e^{-m}}{\Gamma(m+1)} \sum_{i=0}^{\infty} \frac{f_m^{(i)}(0)}{i!} \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} s^i ds \tag{107}$$

where $f_m^{(k)}$ is the k^{th} derivative of f_m (see (5) and $P(m,x)$ is defined in (4). Furthermore, for all $k \in \mathbb{N}$,

$$\begin{aligned}\left|P(m+1,x) - \frac{m^m e^{-m}}{\Gamma(m+1)} \sum_{i=0}^{k-1} \left(\frac{f_m^{(i)}(0)}{i!} \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} s^i ds\right)\right| &< \\ 15\frac{\Gamma(\frac{k+1}{2})2^{(k+1)/2}}{m^{k/6}} + m^{-1/6} e^{\frac{-m^{1/3}}{2}+1}&,\end{aligned} \tag{108}$$

where $\Gamma(k)$ is defined in (7).

Proof. Clearly, using (2) and Observation 4.4,

$$\begin{aligned}\overline{\gamma}(m+1,x) &= \int_{-m}^{-m^{2/3}} e^{\phi(s)}ds + \int_{-m^{2/3}}^{x-m} e^{\phi(s)}ds \\ &= \int_{-m}^{-m^{2/3}} e^{\phi(s)}ds + \int_{-m^{2/3}}^{x-m} e^{-s^2/2m}(1 + f_m'(0)s + \ldots \\ &\quad + \frac{f_m^{(k-1)}(0)}{(k-1)!}s^{k-1} + R_k(s))ds,\end{aligned} \tag{109}$$

where R_k is the Taylor remainder term (101). Now, rearranging the terms of (109),

$$\begin{aligned}\overline{\gamma}(m+1,x) - \sum_{i=0}^{k-1}\left(\frac{f_m^{(i)}(0)}{i!}\int_{-m^{2/3}}^{x-m} e^{-s^2/2m} s^i ds\right) \\ = \int_{-m}^{-m^{2/3}} e^{\phi(s)} ds + \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} R_k(s) ds.\end{aligned} \tag{110}$$

Using (4) and scaling both sides of (110) by

$$\frac{m^m e^{-m}}{\Gamma(m+1)} \tag{111}$$

we obtain,

$$\begin{aligned}P(m+1,x) - \frac{m^m e^{-m}}{\Gamma(m+1)} \sum_{i=0}^{k-1}\left(\frac{f_m^{(i)}(0)}{i!}\int_{-m^{2/3}}^{x-m} e^{-s^2/2m} s^i ds\right) \\ = \frac{m^m e^{-m}}{\Gamma(m+1)} \int_{-m}^{-m^{2/3}} e^{\phi(s)} ds + \frac{m^m e^{-m}}{\Gamma(m+1)} \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} R_k(s) ds.\end{aligned} \tag{112}$$

Combining Remark 2.1, (2), and (1), we obtain

$$\frac{m^m e^{-m}}{\Gamma(m+1)} \int_{-m}^{-m^{2/3}} e^{\phi(s)} ds = P(m+1, m-m^{2/3}) < m^{-1/6} e^{\frac{-m^{1/3}}{2}+1}. \tag{113}$$

Furthermore, according to Lemma 4.3 and Observation 2.1,

$$\frac{m^m e^{-m}}{\Gamma(m+1)} \left|\int_{-m^{2/3}}^{x-m} e^{-s^2/2m} R_k(s) ds\right| < \frac{15}{m^{2k/3+1/2}} \Gamma\left(\frac{k+1}{2}\right) (2m)^{(k+1)/2}. \tag{114}$$

It follows immediately from applying the triangle inequality and combining (112), (113), and (114) that

$$\begin{aligned}\left|P(m+1,x) - \frac{m^m e^{-m}}{\Gamma(m+1)} \sum_{i=0}^{k-1}\left(\frac{f_m^{(i)}(0)}{i!}\int_{-m^{2/3}}^{x-m} e^{-s^2/2m} s^i ds\right)\right| \leq \\ 15\frac{\Gamma(\frac{k+1}{2}) 2^{(k+1)/2}}{m^{k/6}} + m^{-1/6} e^{\frac{-m^{1/3}}{2}+1}.\end{aligned} \tag{115}$$

□

5 Description of Algorithm

Suppose we wish to evaluate $P(m,x)$ for some $m, x > 0$. We consider two regimes.

5.1 $m \leq 10{,}000$

Numerical experiments show that in this regime, for all $x > 0$, evaluation of $P(m,x)$ using formula (83) is faster than evaluation by asymptotic expansion (107). Hence, in this regime we evaluate $P(m,x)$ directly using formula (83). A bound on the error of approximation (83) is provided in Lemma 3.2.

For $x > m/2$, we compute recursively the factors ω_n of (83) defined by the formula,

$$\omega_{m+i} = \frac{x^{m+i}e^{-x}}{\Gamma(m+i+1)} \tag{116}$$

by observing that

$$\omega_{k+1} = \frac{x^k e^{-x}}{\Gamma(k+1)} = \frac{x}{k}\omega_k. \tag{117}$$

and evaluating the initial recursive step ω_m by observing that

$$\begin{aligned}\omega_m &= \frac{x^m e^{-x}}{\Gamma(m+1)} \\ &= \exp\left(m\log(x) - x - \log(\Gamma(m+1))\right).\end{aligned} \tag{118}$$

We then use (12) to evaluate $\log(\Gamma(m+1))$.

Remark 5.1. *For nearly full extended precision accuracy, for $m \in (2000, 10^6)$ and for $x > m$, the function $P(m,x)$ should be evaluated via formula (89). Numerical experiments show that in this regime, evaluation via sum (89) results in nearly full extended precision accuracy, in some cases three digits more accuracy than evalution via (89).*

5.2 $m > 10,000$

We first check if $x < m$. If so, we determine whether $P(m,x)$ is well-approximated by 0 to some user-specified accuracy via Lemma 3.1. If $x > m$, we check if $P(m,x)$ is well-approximated by 1 via Corollary 2.2.

If $P(m,x)$ is neither well-approximated by 0 nor by 1, further analysis remains to show under what conditions algorithm (107) is computationally less expensive than (83). However, numerical experiments show that for "most" x, evaluation of $P(m,x)$ by asymptotic expansion (107) is significantly faster than evaluation by (83). Hence, we evaluate $P(m,x)$ by asymptotic expansion (107). In the remainder of this section, we provide a detailed explanation of asymptotic algorithm (107).

5.2.1 Precomputing

Asymptotic expansion (107) includes the factors,

$$\frac{f_m^{(k)}(0)}{k!} \tag{119}$$

where $k \in \mathbb{N}$ and $f_m^{(k)}$ is the k^{th} derivative of f_m (see (5)). Straightforward differentiation shows that for all k, the values $f_m^{(k)}(0)$ are defined by the formula,

$$\frac{f_m^{(k)}(0)}{k!} = \sum_{i=1}^{n_k} \frac{a_{k,i}}{k!\, m^{j_k+i}}, \tag{120}$$

for some $j_k, n_k \in \mathbb{N}$ and some $a_{k,i} \in \mathbb{R}$ where $i \in \{1, ..., n_k\}$. The values,

$$\left\{\frac{a_{k,i}}{k!}\right\}. \tag{121}$$

are computed in Mathematica and stored in Fortran DATA statements.

5.2.2 Evaluation

The inputs to this stage of the algorithm are $m > 10,000$ and $x > 0$.

Step 0. Given some requirement on the precision of the approximation, we use (108) to determine the number of terms in the expansion. For the remainder of this section, we assume that we require an expansion of K terms.

Step 1. For all $k \in \{1, ..., K\}$, compute the powers from (120),

$$\frac{1}{m^{j_k+i}}, \tag{122}$$

and store them for all $i \in \{1, ..., n_k\}$, where n_k is defined in (120).

Step 2. Evaluate the factors in (107) defined by (120). Specifically, for each $k \leq K$, evaluate

$$\frac{f_m^{(k)}(0)}{k!} = \sum_{i=1}^{n_k} \frac{a_{k,i}}{k!\, m^{j_k+i}}, \tag{123}$$

where $a_{k,i}$ and m^{j_k+i} are defined in (120). Observe that we have already computed the quotients

$$\left\{\frac{a_{k,i}}{k!}\right\} \tag{124}$$

in the precomputation stage while the necessary powers of $1/m$ were computed in Step 1.

Step 3. Evaluate the integrals of (107) of the form,

$$\int_a^b t^{2n+1} e^{-t^2/2m} dt, \tag{125}$$

where $n \in \mathbb{N}$ and $m > 10,000$ via Lemma 2.5.

Step 4. Evaluate the integrals of (107) of the form,

$$\int_a^b t^{2n} e^{-t^2/2m} dt, \tag{126}$$

where $n \in \mathbb{N}$ and $m > 10,000$ via Lemma 2.6.

6 Numerical Experiments

The algorithm of this paper was implemented in Fortran 77. We used the Lahey/Fujitsu compiler on a 2.9 GHz Intel i7-3520M Lenovo laptop; all examples in this section were run in double precision arithmetic.

Throughout this section, we report numerical results relating to the evaluation of $P(m, x)$ via asymptotic expansion (107) and via summation (83) for various values of m and x. In each table in this section, the column labeled "m" denotes the value of m in $P(m, x)$. The column labeled "x" denotes the value of x in $P(m, x)$. The column labeled "k" denotes the number of terms of expansion (107) used to approximate $P(m, x)$. The column labeled "time (μs)" denotes the time, in microseconds, required to run each evaluation. The column labeled "relative error" denotes the relative error of the approximation. The column labeled "absolute error" denotes the absolute error of the approximation. The column labeled $P(m, x)$ denotes the true value (obtained via a calculation in extended precision) that is being approximated.

In Table 3, the column labeled $\alpha_k(m, x)$ denotes $\log_{10}$ of the magnitude of the k^{th} term of asymptotic expansion (107). Specifically, $\alpha_k(m, x)$ is defined via the formula

$$\alpha_k(m,x) = \log_{10} \left| \frac{m^m e^{-m}}{\Gamma(m+1)} \frac{f_m^{(k)}(0)}{k!} \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} s^k ds \right|. \tag{127}$$

In Table 3, the column labeled $\sigma_k(m, x)$ denotes the relative error of the k-term approximation (107). Specifically, $\sigma_k(m, x)$ is defined via the formula

$$\sigma_k(m,x) = \\ P(m+1,x)^{-1} \left| P(m+1,x) - \frac{m^m e^{-m}}{\Gamma(m+1)} \sum_{i=0}^{k} \left(\frac{f_m^{(i)}(0)}{i!} \int_{-m^{2/3}}^{x-m} e^{-s^2/2m} s^i ds \right) \right|. \tag{128}$$

In Table 6, the column labeled "evaluator" indicates whether $P(m, x)$ was evaluated via sum (83) or asymptotic expansion (107).

The primary purpose of Table 1 and Figure 1 is to demonstrate that for fixed m and fixed k, evaluation of $P(m, x)$ via k-term asymptotic expansion (107) results in a smaller error for larger x.

Table 2 and Figure 2 report the numerical costs of evaluation of $P(m, x)$ via k-term asymptotic expansion (107) for different k. We report runtimes for different k with $m = x = 10^7$.

The primary purpose of Table 3 and Figure 3 is to report the decrease in the magnitude of the k^{th} term of asymptotic expansion (107) along with the corresponding error of k-term expansion (107). In Table 3, we report these numerical results for the case $m = x = 10^4$. In Figure 3, we plot $\log_{10}$ of the magnitude of the k^{th} term of expansion (107) for $k \le 28$. We do this for the cases $m = x = 10^4$, $m = x = 10^7$, and $m = x = 10^{10}$.

In Table 4 and Figure 4 we report the numerical costs of evaluation of $P(m, x)$ via summation (83) for different m. We report runtimes for different m with $x = m$. In Figure 4, the horizontal line corresponds to the runtime required for evaluation of $P(m, x)$ via asymptotic expansion (107) with $k = 28$.

Table 5 and Figure 5 report the numerical costs of evaluation of summation (83) for fixed m and different x. We report runtimes for $m = 1000$ with various x.

Table 6 demonstrates that both sum (83) and asymptotic expansion (107) achieve nearly full extended precision accuracy when evaluating $P(m, x)$. We demonstrate this for various values of m and x. We truncate numbers in the $P(m, x)$ column after 23 digits due to space constraints.

Observation 6.1. *Figure 4 demonstrates that for $m < 10^3$, evaluation of $P(m, m)$ via asymptotic expansion (107) is computationally more expensive than evaluation of $P(m, m)$ via sum (83).*

m	$x\ (\cdot 10^3)$	k	relative error	$P(m,x)$
10^6	996	10	0.21472×10^{-7}	0.0000310071182110
10^6	997	10	0.15395×10^{-8}	0.0013381041673135
10^6	998	10	0.11619×10^{-9}	0.0226961140067368
10^6	999	10	0.16768×10^{-10}	0.1586552135743036
10^6	1000	10	0.53194×10^{-11}	0.5001329807608725
10^6	1001	10	0.31623×10^{-11}	0.8413447863683402
10^6	1002	10	0.27477×10^{-11}	0.9771959041012301
10^6	1003	10	0.32794×10^{-11}	0.9986382593537824
10^6	1004	10	0.46816×10^{-11}	0.9999676545526865

Table 1: Relative errors for the evaluation of $P(m,x)$ via 10-term asymptotic asymptotic expansion (107) for different x

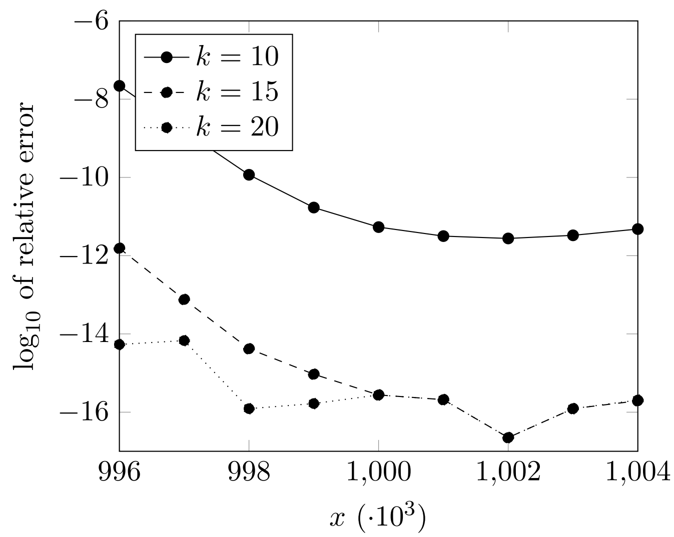

Figure 1: $\log_{10}$ of relative errors for evaluation of $P(m,x)$ via k-term asymptotic expansion (107) for different x and for $k = 10$, $k = 15$, and $k = 20$

m	x	k	time (μs)	relative error	$P(m,x)$
10^7	10^7	4	1.68	0.83327×10^{-7}	0.500042052208723698
10^7	10^7	8	2.08	0.59881×10^{-10}	0.500042052208723698
10^7	10^7	12	2.50	0.40330×10^{-15}	0.500042052208723698
10^7	10^7	16	2.92	0.60328×10^{-15}	0.500042052208723698
10^7	10^7	20	3.44	0.60328×10^{-15}	0.500042052208723698
10^7	10^7	24	3.75	0.60328×10^{-15}	0.500042052208723698
10^7	10^7	28	4.26	0.60328×10^{-15}	0.500042052208723698

Table 2: Times for the evaluation of $P(m,x)$ via k-term asymptotic asymptotic expansion (107) for different k

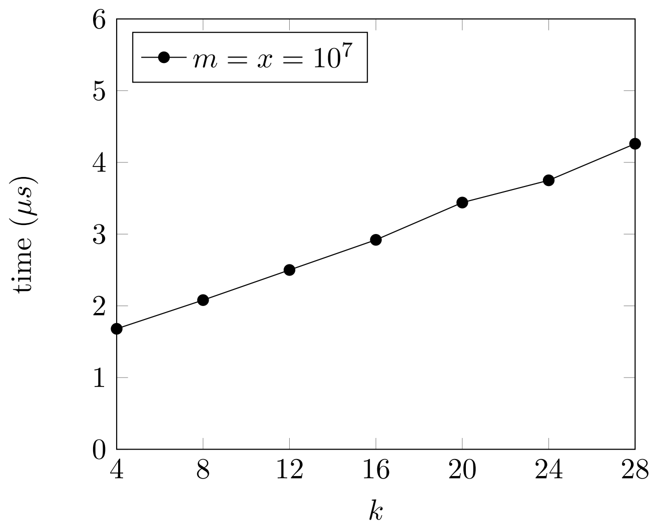

Figure 2: Times for evaluation of $P(m,x)$ via k-term asymptotic expansion (107) for different k and $m = x = 10^7$

m	x	k	$\alpha_k(m,x)$	$\sigma_k(m,x)$	$P(m,x)$
10^4	10^4	0	-0.2976	0.52970×10^{-2}	0.501329808339955200
10^4	10^4	4	-4.4259	0.83144×10^{-4}	0.501329808339955200
10^4	10^4	6	-4.3803	0.13220×10^{-5}	0.501329808339955200
10^4	10^4	8	-7.2890	0.19636×10^{-5}	0.501329808339955200
10^4	10^4	10	-7.1832	0.50570×10^{-7}	0.501329808339955200
10^4	10^4	12	-7.5752	0.41020×10^{-8}	0.501329808339955200
10^4	10^4	14	-9.6979	0.19454×10^{-8}	0.501329808339955200
10^4	10^4	16	-9.8870	0.50321×10^{-10}	0.501329808339955200
10^4	10^4	18	-10.496	0.10907×10^{-10}	0.501329808339955200
10^4	10^4	20	-12.127	0.31368×10^{-11}	0.501329808339955200
10^4	10^4	22	-12.492	0.48601×10^{-13}	0.501329808339955200
10^4	10^4	24	-13.264	0.33580×10^{-13}	0.501329808339955200
10^4	10^4	26	-14.519	0.68569×10^{-14}	0.501329808339955200
10^4	10^4	28	-15.016	0.52361×10^{-16}	0.501329808339955200

Table 3: Numerical results for $\log_{10}$ of the magnitude of the k^{th} term of asymptotic expansion (107) and errors of the k-term expansion

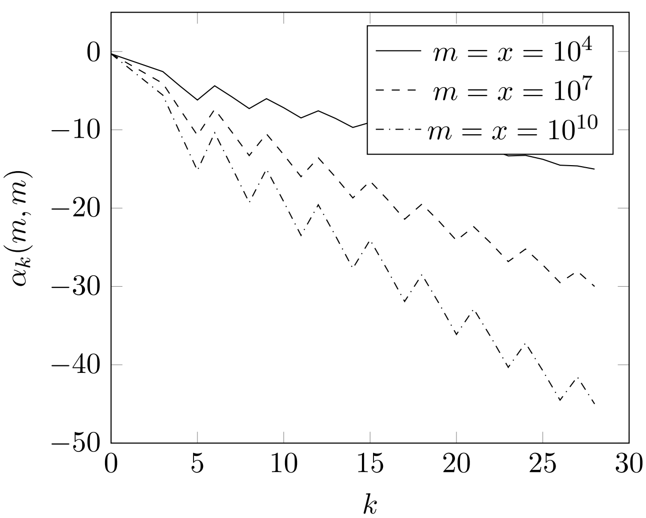

Figure 3: $\log_{10}$ of the magnitude of the k^{th} term of asymptotic expansion (107) for different m

m	x	time (μs)	relative error	$P(m,x)$
10^0	10^0	0.62	0.34164×10^{-16}	0.632120558828557678
10^1	10^1	0.96	0.16914×10^{-15}	0.542070285528147791
10^2	10^2	1.34	0.71081×10^{-15}	0.513298798279148664
10^3	10^3	3.00	0.41353×10^{-15}	0.504205244180215508
10^4	10^4	7.80	0.39818×10^{-15}	0.501329808339955200
10^5	10^5	20.88	0.45496×10^{-14}	0.500420522110365176
10^6	10^6	62.32	0.95799×10^{-14}	0.500132980760872591

Table 4: Times and errors for the evaluation of $P(m,m)$ by direct summation (83) for different m

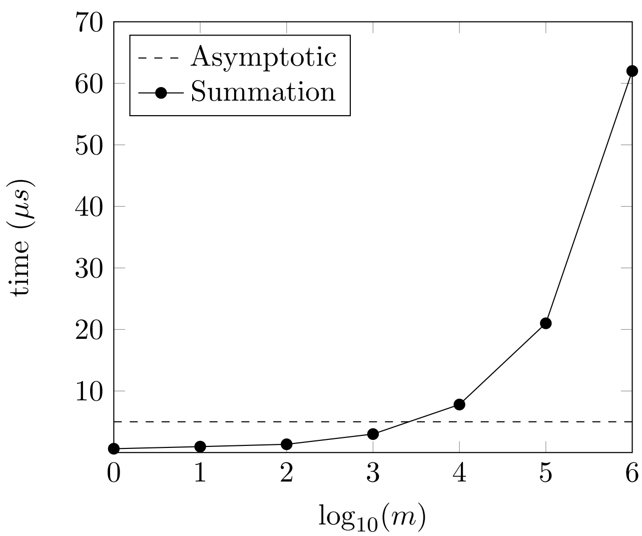

Figure 4: Times for evaluation of $P(m,x)$ by direct summation (83) and by asymptotic expansion (107)

m	x	time (μs)	relative error	$P(m,x)$
1000	900	1.93	0.50370×10^{-15}	0.000549902265711782
1000	925	2.16	0.93887×10^{-15}	0.007693713246846007
1000	950	2.28	0.99996×10^{-16}	0.055054686230738034
1000	975	2.49	0.27405×10^{-14}	0.215731105240819891
1000	1000	2.65	0.41353×10^{-15}	0.504205244180215508
1000	1025	2.90	0.62391×10^{-15}	0.786575483861807090
1000	1050	3.06	0.34303×10^{-15}	0.941328888622681922
1000	1075	3.35	0.13468×10^{-14}	0.989973597928674133
1000	1100	3.55	0.15243×10^{-14}	0.998940676746070022

Table 5: Times for the evaluation of $P(m,x)$ by direct summation (83) for different x

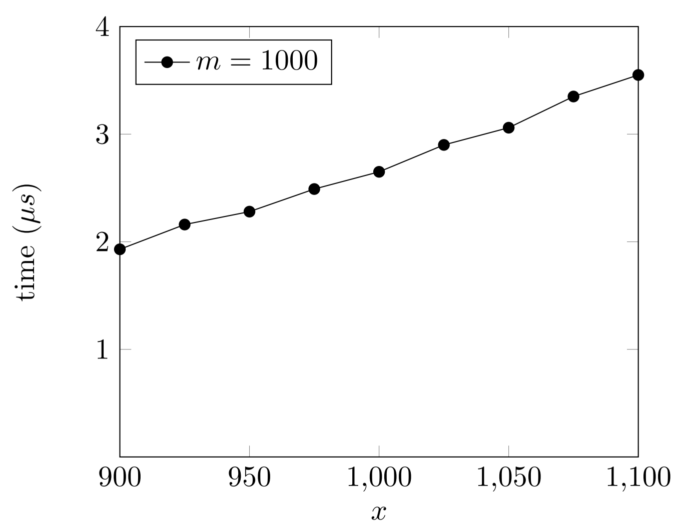

Figure 5: Times of evaluation of $P(m,x)$ via direct summation (83) for different x

m	x	evaluator	absolute error	$P(m, x)$
1	0.5	Sum (83)	0.48148×10^{-34}	0.39346934028736657639620046500 9
1	1	Sum (83)	0.10000×10^{-34}	0.63212055882855767840447622983 9
1	10	Sum (83)	0.13482×10^{-32}	0.99995460007023751514846440848 4
100	80	Sum (83)	0.10803×10^{-32}	0.01710831303513311416587730763 6
100	100	Sum (83)	0.33415×10^{-31}	0.51329879827914866485731425656 4
100	120	Sum (83)	0.80504×10^{-31}	0.97213626010947933851581483214 4
$10,000$	$9,000$	Sum (83)	0.13501×10^{-34}	0.00000000000000000000000020732 9
$10,000$	$10,000$	Sum (83)	0.49111×10^{-32}	0.50132980833995520038274225130 0
$10,000$	$11,000$	Sum (83)	0.19356×10^{-31}	0.99999999999999999999983071468 5
10^5	$10^5 - 10^3$	Sum (83)	0.31597×10^{-34}	0.00075741992117476797411846530 4
10^5	10^5	Sum (83)	0.80889×10^{-32}	0.50042052211036517669331257904 4
10^5	$10^5 + 10^3$	Sum (83)	0.45221×10^{-30}	0.99919157848707440926753122654 4
10^6	$10^6 - 10^3$	Sum (83)	0.43733×10^{-30}	0.15865521357430365246303274349 5
10^6	10^6	Sum (83)	0.51519×10^{-31}	0.50013298076087259124432281750 3
10^6	$10^6 + 10^3$	Sum (83)	0.87534×10^{-31}	0.84134478636834029162756385146 6
10^7	$10^7 - 10^3$	Exp. (107)	0.11700×10^{-30}	0.37595081883144316041616276154 6
10^7	10^7	Exp. (107)	0.24941×10^{-31}	0.50004205220872369833375616478 3
10^7	$10^7 + 10^3$	Exp. (107)	0.63748×10^{-31}	0.62412118350555233953180996493 9

Table 6: Absolute errors for the evaluation of $P(m, x)$ by direct summation (83) and asymptotic expansion (107) in extended precision

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