In this report we investigate the solution of boundary value problems for elliptic partial differential equations on domains with corners. Previously, we observed that when, in the case of polygonal domains, the boundary value problems are formulated as boundary integral equations of classical potential theory, the solutions are representable by series of certain elementary functions. Here, we extend this observation to the general case of regions with boundaries consisting of analytic curves meeting at corners. We show that the solutions near the corners have the same leading terms as in the polygonal case, plus a series of corrections involving products of the leading terms with integer powers and powers of logarithms. Furthermore, we show that if the curve in the vicinity of a corner approximates a polygon to order k, then the correction added to the leading terms will vanish like $O(t^k)$, where t is the distance from the corner.

On the Solution of Elliptic Partial Differential Equations on Regions with Corners III: Curved Boundaries

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Contents

1 Introduction

In classical potential theory, elliptic partial differential equations are reduced to second kind boundary integral equations by representing the solutions as single-layer or double-layer potentials on the boundaries of the regions.

In general, when the region on which an elliptic partial differential equation is solved has corners, the solutions to both the differential equation and corresponding boundary integral equations will be singular at the corners. The existence and uniqueness of the solutions to both equations in the L^2-sense have long been known (see [2], [21]), and the properties of the solutions to both the differential and integral equations on such regions have been the subject of extensive research (see, for example, [9], [23], [22]; reviews of the literature can be found in, for example, [4], [10]).

Recently, the author observed that when the boundary integral equations of classical potential theory are solved on polygonal domains, the solutions in the vicinity of corners admit explicit, rapidly-convergent series representations, which appear to have been overlooked in the classical literature (see [17], [20]). These series also turn out to lend themselves to the construction of highly efficient and accurate numerical algorithms for the solution of the integral equations. However, this detailed analysis has so far been restricted to the relatively narrow case of polygons.

The subject of this report is the extension of the apparatus presented in [20] to the general case of regions with boundaries consisting of analytic curves meeting at sharp corners. It turns out that, when the smooth parts of the boundaries are allowed to be general analytic curves, the singularities near the corners have the same leading terms as in the polygonal case, plus a series of corrections involving products of the leading terms with integer powers and powers of logarithms. Furthermore, if the curve in the vicinity of a corner approximates a polygon to order k, then the correction added to the leading terms will vanish like $O(t^k)$, where t is the distance from the corner.

The structure of this report is as follows. Section 2 provides an overview of the principal results. In Section 3, we introduce the necessary mathematical preliminaries. Section 4 contains the general analytical apparatus. In Section 5, we analyze the boundary integral equations and prove the principal theorems.

2 Overview

The following two subsections 2.1 and 2.2 summarize the Dirichlet and Neumann cases respectively. The principal results of this report are theorems 2.1 and 2.2.

Suppose that $\gamma\colon [-1,1] \to \mathbb{R}^2$ is a curve in $\mathbb{R}^2$. Let γ_1 denote the function $\gamma(-t)$, for $0 \leq t \leq 1$, and γ_2 denote the function $\gamma(t)$, for $0 \leq t \leq 1$, and suppose that γ_1 and γ_2 are two analytic curves (see Definition 3.1) meeting at a corner at $\gamma(0)$ with interior angle $\pi\alpha$ (see Figure 1). Suppose that, without loss of generality, the curves are parameterized by arc length (see Theorem 3.18). Suppose further that M is a nonnegative integer and, letting κ_1 and κ_2 denote the signed curvatures of γ_1 and γ_2 respectively (see Definition 3.2), suppose that κ_1 and κ_2 are both polynomials of degree M. Finally, let $\nu(t)$ denote the inward-facing unit normal to the curve γ at t, and let Γ denote the set $\gamma([-1,1])$. By extending the ends of the curves to infinity, we divide $\mathbb{R}^2$ into two open sets Ω_1 and Ω_2 (see Figure 1).

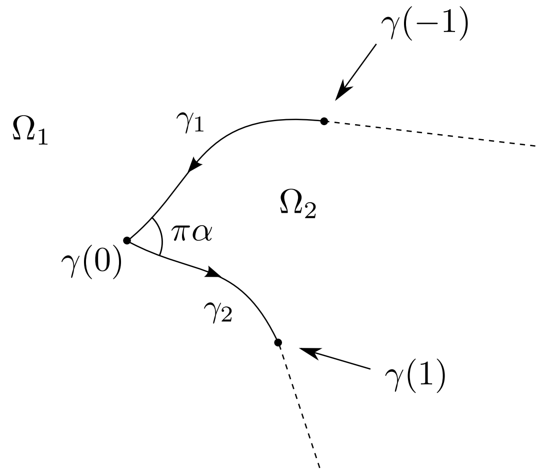

Figure 1: Two curves meeting at a corner in $\mathbb{R}^2$

2.1 The Dirichlet Case

Let $\phi\colon \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$ denote the potential induced by a dipole distribution on γ with density $\rho\colon [-1,1] \to \mathbb{R}$. In other words, let ϕ be defined by the formula

$$\phi(x) = \frac{1}{2\pi} \int_{-1}^{1} \frac{\langle x - \gamma(t), \nu(t) \rangle}{\|x - \gamma(t)\|^2} \rho(t)\, dt, \tag{1}$$

for all $x \in \mathbb{R}^2 \setminus \Gamma$, where $\|\cdot\|$ denotes the Euclidean norm and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$. Suppose that $g\colon [-1,1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \lim_{\substack{x \to \gamma(t) \\ x \in \Omega_2}} \phi(x), \tag{2}$$

for all $-1 \le t \le 1$, i.e. g is the limit of integral (1) when x approaches the point $\gamma(t)$ from inside. It is well known that

$$g(s) = \frac{1}{2}\rho(s) + \frac{1}{2\pi} \int_{-1}^{1} K(t,s)\rho(t)\, dt, \tag{3}$$

for all $-1 \le s \le 1$, where

$$K(t,s) = \frac{\langle \gamma(s) - \gamma(t), \nu(t) \rangle}{\|\gamma(s) - \gamma(t)\|^2}, \tag{4}$$

for all $-1 \le s, t \le 1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$.

The following defines the three classes of functions $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$—in Theorem 2.1, we show that if the boundary data g defined by (2) is smooth on each side of the corner, then the solution ρ to equation (3) is a linear combination of functions belonging to $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$.

Remark 2.1. Suppose that k is a positive integer and $(a_1, a_2, \ldots, a_k) \in \mathbb{R}^k$. We use the notation $P_{(a_1,a_2,\ldots,a_k)}\colon \mathbb{R}^k \to \mathbb{R}$ to denote a polynomial with the following properties:

- The polynomial $P_{(a_1,a_2,\ldots,a_k)}(x_1,x_2,\ldots,x_k)$ is a linear combination of monomials of the form

$$x_1^{n_1}x_2^{n_2}\cdots x_k^{n_k}, \tag{5}$$

where $n_i \in \mathbb{Z}$ and $0 \le n_i \le a_i$, for $i = 1,2,\ldots,k$.

- If any of $a_1,a_2,\ldots,a_k$ are less than zero, then $P_{(a_1,a_2,\ldots,a_k)} \equiv 0$.

Definition 2.1 (The class $\mathcal{F}$)**.** We define the class of functions $\mathcal{F}_{k,N,\alpha}(\mu) \subset L^2([-1,1])$ for all integers $k \ge 0$, $N \ge 0$ and real numbers $0 < \alpha < 2$, $\mu \in (-\frac{1}{2}, N]$, as follows.

Case 1: Suppose that $\alpha \notin \mathbb{Q}$. Then $\mathcal{F}_{k,N,\alpha}(\mu)$ is the set of all functions of the form

$$f(t) = |t|^{\mu} + |t|^{\mu+k+1}P_{(N-\mu-k-1,1)}(|t|,\operatorname{sgn}(t)), \tag{6}$$

for all $-1 \le t \le 1$, where $P\colon \mathbb{R}^2 \to \mathbb{R}$ is a polynomial.

Case 2: Suppose now that $\alpha \in \mathbb{Q}$. Suppose further that p and q are integers such that $\alpha = p/q$, where p/q is a reduced fraction and $q > 0$. Then $\mathcal{F}_{k,N,\alpha}(\mu)$ is the set of all functions of the form

$$f(t) = |t|^{\mu} + |t|^{\mu+k+1}P_{(N-\mu-k-1,N/q,1)}(|t|,\log|t|,\operatorname{sgn}(t)), \tag{7}$$

for all $-1 \le t \le 1$, where $P\colon \mathbb{R}^3 \to \mathbb{R}$ is a polynomial, and, for each $n \ge 0$, every monomial in f containing $(\log|t|)^n$ is multiplied by $|t|^{\mu+\ell}$, with $\ell \ge n\cdot q$.

Definition 2.2 (The class $\mathcal{G}$)**.** We define the class of functions $\mathcal{G}_{k,N,\alpha}(\mu) \subset L^2([-1,1])$ for all integers $k \ge 0$, $N \ge 0$ and real numbers $0 < \alpha < 2$, $\mu \in (-\frac{1}{2}, N]$ as the set of all functions of the form

$$f(t) = |t|^{\mu} + |t|^{\mu+k+1}P_{(N-\mu-k-1,2\cdot\lfloor N/q\rfloor,1)}(|t|,\log|t|,\operatorname{sgn}(t)), \tag{8}$$

for all $-1 \le t \le 1$, where $P\colon \mathbb{R}^3 \to \mathbb{R}$ is a polynomial and, for each $n \ge 0$, every monomial in f containing $(\log|t|)^n$ is multiplied by $|t|^{\mu+\ell}$, with $\ell \ge \lceil n/2\rceil\cdot q$.

Definition 2.3 (The class $\mathcal{H}$)**.** We define the class of functions $\mathcal{H}_{k,N,\alpha}(\mu) \subset L^2([-1,1])$ for all integers $k \ge 0$, $N \ge 0$ and real numbers $0 < \alpha < 2$, $\mu \in (-\frac{1}{2}, N]$ as the set of all functions of the form

$$f(t) = |t|^{\mu}\log(|t|) + |t|^{\mu+k+1}P_{(N-\mu-k-1,2\cdot\lfloor N/q\rfloor+1,1)}(|t|,\log|t|,\operatorname{sgn}(t)), \tag{9}$$

for all $-1 \le t \le 1$, where $P\colon \mathbb{R}^3 \to \mathbb{R}$ is a polynomial and, for each $n \ge 0$, every monomial in f containing $(\log|t|)^{n+1}$ is multiplied by $|t|^{\mu+\ell}$, with $\ell \ge \lceil n/2\rceil\cdot q$.

In this report, we prove the following theorem, which holds for any angle $0 < \pi\alpha < 2\pi$ and almost every curve γ_1 and γ_2, and which is the first of the two principal results of this report.

Theorem 2.1. *Suppose that $0 < \alpha < 2$ and that N is a nonnegative integer. Suppose further that k is a nonnegative integer and that $\kappa_1^{(n)}(0) = \kappa_2^{(n)}(0) = 0$ for $n = 0, \ldots, k-1$, or, in other words, that the curvatures of γ_1 and γ_2, and their first $k-1$ derivatives, are zero at the corner (see Theorem 3.21). Let*

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{10}$$

$$\underline{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{11}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha) N}{2} \right\rfloor, \tag{12}$$

$$\overline{M} = \left\lceil \frac{(2-\alpha) N}{2} \right\rceil, \tag{13}$$

and observe that $\overline{L}+\underline{M} = N$ and $\overline{M}+\underline{L} = N$. Then there exist functions $\chi_{1,1}, \chi_{1,2}, \ldots, \chi_{1,\overline{L}}$ and $\chi_{2,0}, \chi_{2,1}, \ldots, \chi_{2,\underline{M}}$, where

$$\chi_{1,i} \in \begin{cases} \mathcal{F}_{k,N,\alpha}\left(\frac{2i-1}{\alpha}\right) & \text{if } \frac{2i-1}{\alpha} \neq \frac{2j}{2-\alpha} \text{ for each integer } 1 \le j \le \underline{M}, \\ \mathcal{G}_{k,N,\alpha}\left(\frac{2i-1}{\alpha}\right) & \text{if } \frac{2i-1}{\alpha} = \frac{2j}{2-\alpha} \text{ for some integer } 1 \le j \le \underline{M}, \end{cases} \tag{14}$$

$$\chi_{2,j} \in \begin{cases} \mathcal{F}_{k,N,\alpha}\left(\frac{2j}{2-\alpha}\right) & \text{if } \frac{2j}{2-\alpha} \neq \frac{2i-1}{\alpha} \text{ for each integer } 1 \le i \le \overline{L}, \\ \mathcal{H}_{k,N,\alpha}\left(\frac{2j}{2-\alpha}\right) & \text{if } \frac{2j}{2-\alpha} = \frac{2i-1}{\alpha} \text{ for some integer } 1 \le i \le \overline{L}, \end{cases} \tag{15}$$

for $1 \le i \le \overline{L}$ and $0 \le j \le \underline{M}$, and functions $\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{1,\overline{M}}$ and $\eta_{2,0}, \eta_{2,1}, \ldots, \eta_{2,\underline{L}}$, where

$$\eta_{1,i} \in \begin{cases} \mathcal{F}_{k,N,\alpha}\left(\frac{2i-1}{2-\alpha}\right) & \text{if } \frac{2i-1}{2-\alpha} \neq \frac{2j}{\alpha} \text{ for each integer } 1 \le j \le \underline{L}, \\ \mathcal{G}_{k,N,\alpha}\left(\frac{2i-1}{2-\alpha}\right) & \text{if } \frac{2i-1}{2-\alpha} = \frac{2j}{\alpha} \text{ for some integer } 1 \le j \le \underline{L}, \end{cases} \tag{16}$$

$$\eta_{2,j} \in \begin{cases} \mathcal{F}_{k,N,\alpha}\left(\frac{2j}{\alpha}\right) & \text{if } \frac{2j}{\alpha} \neq \frac{2i-1}{2-\alpha} \text{ for each integer } 1 \le i \le \overline{M}, \\ \mathcal{H}_{k,N,\alpha}\left(\frac{2j}{\alpha}\right) & \text{if } \frac{2j}{\alpha} = \frac{2i-1}{2-\alpha} \text{ for some integer } 1 \le i \le \overline{M}, \end{cases} \tag{17}$$

for $1 \le i \le \overline{M}$ and $0 \le j \le \underline{L}$, such that the following holds.

Forward direction: *Suppose that ρ has the form*

$$\rho(t) = \sum_{i=1}^{\overline{L}} b_{\underline{M}+i}\chi_{1,i}(t) + \sum_{i=0}^{\underline{M}} b_i\chi_{2,i}(t) + \sum_{i=1}^{\overline{M}} c_{\underline{L}+i}\operatorname{sgn}(t)\eta_{1,i}(t) + \sum_{i=0}^{\underline{L}} c_i \operatorname{sgn}(t)\eta_{2,i}(t), \tag{18}$$

for all $-1 \le t \le 1$, where $b_0, b_1, \ldots, b_N$ and $c_0, c_1, \ldots, c_N$ are arbitrary real numbers, and let g be defined by (3). Then there exist real numbers $\beta_0, \beta_1, \ldots, \beta_N$ and $\xi_0, \xi_1, \ldots, \xi_N$ such that

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{19}$$

for all $-1 \le t \le 1$.

Converse direction: *The converse is also true, in the following sense. Suppose that N is an arbitrary nonnegative integer. Then, for each $0 < \alpha < 2$, there exists an open ball $B_{\delta(\alpha)} \subset \mathbb{R}^{M+1}$ of radius $\delta(\alpha)$, centered at zero, and a set $K(\alpha) \subset B_{\delta(\alpha)} \times B_{\delta(\alpha)}$ of measure zero, where $\mathbf{0} \notin K(\alpha)$, such that the following holds. If the curvatures κ_1 and κ_2 of the curves γ_1 and γ_2 are defined by*

$$\kappa_1(t) = a_{1,0} + a_{1,1}t + \cdots + a_{1,M}t^M, \tag{20}$$

$$\kappa_2(t) = a_{2,0} + a_{2,1}t + \cdots + a_{2,M}t^M, \tag{21}$$

for all $0 \le t \le 1$, and $(a_{1,0}, a_{1,1}, \ldots, a_{1,M}, a_{2,0}, a_{2,1}, \ldots, a_{2,M}) \in B_{\delta(\alpha)} \times B_{\delta(\alpha)} \setminus K(\alpha)$, then for any g of the form (19), there exist unique real numbers $b_0, b_1, \ldots, b_N$ and $c_0, c_1, \ldots, c_N$ such that ρ, defined by (18), solves equation (3) to within an error $o(|t|^N)$.

In other words, if ρ has the form (18), then g is smooth on each of the intervals $[-1, 0]$ and $[0, 1]$. Conversely, if g is smooth on each of the intervals $[-1, 0]$ and $[0, 1]$, then for each nonnegative integer N, and for each angle $\pi\alpha$ and almost every curve γ_1 and γ_2, there exists a unique solution ρ of the form (18) that solves equation (3) to within an error $o(|t|^N)$.

Observation 2.2. When $\alpha = 1$ we observe that, if the boundary data g is smooth on each of the intervals $[-1, 0]$ and $[0, 1]$, then the solution ρ to equation (3) is representable by a series involving products of powers of $|t|$ and powers of $|t| \log |t|$.

2.2 The Neumann Case

Let $\phi\colon \mathbb{R}^2 \setminus \Gamma \to \mathbb{R}$ denote the potential induced by a charge distribution on γ with density $\rho\colon [-1, 1] \to \mathbb{R}$. In other words, let ϕ be defined by the formula

$$\phi(x) = -\frac{1}{2\pi} \int_{-1}^{1} \log(\|\gamma(t) - x\|)\rho(t)\, dt, \tag{22}$$

for all $x \in \mathbb{R}^2 \setminus \Gamma$, where $\|\cdot\|$ denotes the Euclidean norm. Suppose that $g\colon [-1, 1] \to \mathbb{R}$ is defined by the formula

$$g(t) = \lim_{\substack{x \to \gamma(t) \\ x \in \Omega_1}} \frac{\partial \phi(x)}{\partial \nu(t)}, \tag{23}$$

for all $-1 \le t \le 1$, i.e. g is the limit of the normal derivative of integral (22) when x approaches the point $\gamma(t)$ from outside. It is well known that

$$g(s) = \frac{1}{2}\rho(s) + \frac{1}{2\pi} \int_{-1}^{1} K(s, t)\rho(t)\, dt, \tag{24}$$

for all $-1 \le s \le 1$, where

$$K(s, t) = \frac{\langle \gamma(t) - \gamma(s), \nu(s) \rangle}{\|\gamma(t) - \gamma(s)\|^2}, \tag{25}$$

for all $-1 \le s, t \le 1$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^2$.

The following theorem, which holds for any angle $0 < \pi\alpha < 2\pi$ and almost every curve γ_1 and γ_2, is the second of the two principal results of this report.

Theorem 2.2. *Suppose that $0 < \alpha < 2$ and that N is a positive integer. Suppose further that k is a nonnegative integer and that $\kappa_1^{(n)}(0) = \kappa_2^{(n)}(0) = 0$ for $n = 0, \dots, k-1$, or, in other words, that the curvatures of γ_1 and γ_2, and their first $k-1$ derivatives, are zero at the corner (see Theorem 3.21). Let*

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{26}$$

$$\underline{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{27}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{28}$$

$$\overline{M} = \left\lceil \frac{(2-\alpha)N}{2} \right\rceil, \tag{29}$$

and observe that $\overline{L} + \underline{M} = N$ and $\overline{M} + \underline{L} = N$. Then there exist functions $\chi_{1,1}, \chi_{1,2}, \dots, \chi_{1,\overline{L}}$ and $\chi_{2,1}, \chi_{2,2}, \dots, \chi_{2,\underline{M}}$, where

$$\chi_{1,i} \in \begin{cases} \mathcal{F}_{k,N-1,\alpha}\left(\frac{2i-1}{\alpha} - 1\right) & \text{if } \frac{2i-1}{\alpha} \neq \frac{2j}{2-\alpha} \text{ for each integer } 1 \le j \le \underline{M}, \\ \mathcal{G}_{k,N-1,\alpha}\left(\frac{2i-1}{\alpha} - 1\right) & \text{if } \frac{2i-1}{\alpha} = \frac{2j}{2-\alpha} \text{ for some integer } 1 \le j \le \underline{M}, \end{cases} \tag{30}$$

$$\chi_{2,j} \in \begin{cases} \mathcal{F}_{k,N-1,\alpha}\left(\frac{2j}{2-\alpha} - 1\right) & \text{if } \frac{2j}{2-\alpha} \neq \frac{2i-1}{\alpha} \text{ for each integer } 1 \le i \le \overline{L}, \\ \mathcal{H}_{k,N-1,\alpha}\left(\frac{2j}{2-\alpha} - 1\right) & \text{if } \frac{2j}{2-\alpha} = \frac{2i-1}{\alpha} \text{ for some integer } 1 \le i \le \overline{L}, \end{cases} \tag{31}$$

for $1 \le i \le \overline{L}$ and $1 \le j \le \underline{M}$, and functions $\eta_{1,1}, \eta_{1,2}, \dots, \eta_{1,\overline{M}}$ and $\eta_{2,1}, \eta_{2,2}, \dots, \eta_{2,\underline{L}}$, where

$$\eta_{1,i} \in \begin{cases} \mathcal{F}_{k,N-1,\alpha}\left(\frac{2i-1}{2-\alpha} - 1\right) & \text{if } \frac{2i-1}{2-\alpha} \neq \frac{2j}{\alpha} \text{ for each integer } 1 \le j \le \underline{L}, \\ \mathcal{G}_{k,N-1,\alpha}\left(\frac{2i-1}{2-\alpha} - 1\right) & \text{if } \frac{2i-1}{2-\alpha} = \frac{2j}{\alpha} \text{ for some integer } 1 \le j \le \underline{L}, \end{cases} \tag{32}$$

$$\eta_{2,j} \in \begin{cases} \mathcal{F}_{k,N-1,\alpha}\left(\frac{2j}{\alpha} - 1\right) & \text{if } \frac{2j}{\alpha} \neq \frac{2i-1}{2-\alpha} \text{ for each integer } 1 \le i \le \overline{M}, \\ \mathcal{H}_{k,N-1,\alpha}\left(\frac{2j}{\alpha} - 1\right) & \text{if } \frac{2j}{\alpha} = \frac{2i-1}{2-\alpha} \text{ for some integer } 1 \le i \le \overline{M}, \end{cases} \tag{33}$$

for $1 \le i \le \overline{M}$ and $1 \le j \le \underline{L}$ (see definitions 2.1, 2.2, and 2.3), such that the following holds.

Forward direction: *Suppose that ρ has the form*

$$\rho(t) = \sum_{i=1}^{\overline{L}} b_{\underline{M}+i} \chi_{1,i}(t) + \sum_{i=1}^{\underline{M}} b_i \chi_{2,i}(t) + \sum_{i=1}^{\overline{M}} c_{\underline{L}+i} \operatorname{sgn}(t) \eta_{1,i}(t) + \sum_{i=1}^{\underline{L}} c_i \operatorname{sgn}(t) \eta_{2,i}(t), \tag{34}$$

for all $-1 \le t \le 1$, where $b_1, b_2, \dots, b_N$ and $c_1, c_2, \dots, c_N$ are arbitrary real numbers, and let g be defined by (24). Then there exist real numbers $\beta_0, \beta_1, \dots, \beta_{N-1}$ and $\xi_0, \xi_1, \dots, \xi_{N-1}$ such that

$$g(t) = \sum_{n=0}^{N-1} \beta_n |t|^n + \sum_{n=0}^{N-1} \xi_n \operatorname{sgn}(t) |t|^n + o(|t|^{N-1}), \tag{35}$$

for all $-1 \le t \le 1$.

Converse direction: *The converse is also true, in the following sense. Suppose that* N *is an arbitrary positive integer. Then, for each* $0 < \alpha < 2$*, there exists an open ball* $B_{\delta(\alpha)} \subset \mathbb{R}^{M+1}$ *of radius* $\delta(\alpha)$*, centered at zero, and a set* $K(\alpha) \subset B_{\delta(\alpha)} \times B_{\delta(\alpha)}$ *of measure zero, where* $\mathbf{0} \notin K(\alpha)$*, such that the following holds. If the curvatures* κ_1 *and* κ_2 *of the curves* γ_1 *and* γ_2 *are defined by*

$$\kappa_1(t) = a_{1,0} + a_{1,1}t + \cdots + a_{1,M}t^M, \tag{36}$$

$$\kappa_2(t) = a_{2,0} + a_{2,1}t + \cdots + a_{2,M}t^M, \tag{37}$$

for all $0 \le t \le 1$*, and* $(a_{1,0}, a_{1,1}, \ldots, a_{1,M}, a_{2,0}, a_{2,1}, \ldots, a_{2,M}) \in B_{\delta(\alpha)} \times B_{\delta(\alpha)} \setminus K(\alpha)$*, then for any* g *of the form (35), there exist unique real numbers* $b_1, b_2, \ldots, b_N$ *and* $c_1, c_2, \ldots, c_N$ *such that* ρ*, defined by (34), solves equation (24) to within an error* $o(|t|^{N-1})$*.*

In other words, if ρ has the form (34), then g is smooth on each of the intervals $[-1, 0]$ and $[0, 1]$. Conversely, if g is smooth on each of the intervals $[-1, 0]$ and $[0, 1]$, then for each positive integer N, and for each angle $\pi\alpha$ and almost every curve γ_1 and γ_2, there exists a unique solution ρ of the form (34) that solves equation (24) to within an error $o(|t|^{N-1})$.

3 Mathematical Preliminaries

3.1 Boundary Value Problems

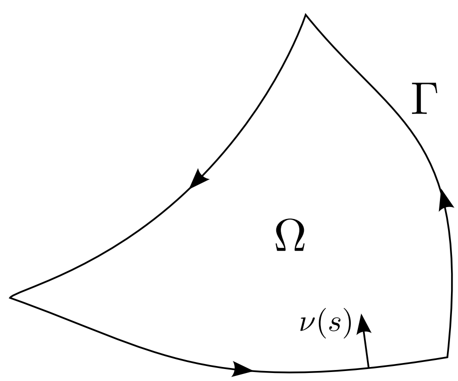

Figure 2: A simple closed curve in $\mathbb{R}^2$

Suppose that $\gamma\colon [0, L] \to \mathbb{R}^2$ is a simple closed curve of length L with a finite number of corners. Suppose further that γ is analytic except at the corners. We denote the interior of γ by Ω and the boundary of Ω by Γ, and let $\nu(t)$ denote the normalized internal normal to γ at $t \in [0, L]$. Supposing that g is some function $g\colon [0, L] \to \mathbb{R}$, we will solve the following four problems.

1) *Interior Neumann problem* (INP): Find a function $\phi\colon \Omega \to \mathbb{R}$ such that

$$\nabla^2\phi(x) = 0 \qquad \text{for } x \in \Omega, \tag{38}$$

$$\lim_{\substack{x\to\gamma(t)\\ x\in\Omega}} \frac{\partial\phi(x)}{\partial\nu(t)} = g(t) \qquad \text{for } t \in [0, L]. \tag{39}$$

2) *Exterior Neumann problem* (ENP): Find a function $\phi\colon \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{R}$ such that

$$\nabla^2\phi(x) = 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{40}$$

$$\lim_{\substack{x\to\gamma(t)\\ x\in\mathbb{R}^2\setminus\overline{\Omega}}} \frac{\partial\phi(x)}{\partial\nu(t)} = g(t) \qquad \text{for } t \in [0, L]. \tag{41}$$

3) *Interior Dirichlet problem* (IDP): Find a function $\phi\colon \Omega \to \mathbb{R}$ such that

$$\nabla^2\phi(x) = 0 \qquad \text{for } x \in \Omega, \tag{42}$$

$$\lim_{\substack{x\to\gamma(t)\\ x\in\Omega}} \phi(x) = g(t) \qquad \text{for } t \in [0, L]. \tag{43}$$

4) *Exterior Dirichlet problem* (EDP): Find a function $\phi\colon \mathbb{R}^2 \setminus \overline{\Omega} \to \mathbb{R}$ such that

$$\nabla^2\phi(x) = 0 \qquad \text{for } x \in \mathbb{R}^2 \setminus \overline{\Omega}, \tag{44}$$

$$\lim_{\substack{x\to\gamma(t)\\ x\in\mathbb{R}^2\setminus\overline{\Omega}}} \phi(x) = g(t) \qquad \text{for } t \in [0, L]. \tag{45}$$

Suppose that $g \in L^2([0, L])$. Then the interior and exterior Dirichlet problems have unique solutions. If g also satisfies the condition

$$\int_0^L g(t)\,dt = 0, \tag{46}$$

then the interior and exterior Neumann problems have unique solutions up to an additive constant (see, for example, [7], [3]).

3.2 Integral Equations of Potential Theory

In classical potential theory, boundary value problems are solved by representing the function ϕ by integrals of potentials over the boundary. The potential of a *unit charge* located at $x_0 \in \mathbb{R}^2$ is the function $\psi^0_{x_0}\colon \mathbb{R}^2 \setminus x_0 \to \mathbb{R}$, defined via the formula

$$\psi^0_{x_0}(x) = \log(\|x - x_0\|), \tag{47}$$

for all $x \in \mathbb{R}^2 \setminus x_0$, where $\|\cdot\|$ denotes the Euclidean norm. The potential of a *unit dipole* located at $x_0 \in \mathbb{R}^2$ and oriented in direction $h \in \mathbb{R}^2$, $\|h\| = 1$, is the function $\psi^1_{x_0,h}\colon \mathbb{R}^2 \setminus x_0 \to \mathbb{R}$, defined via the formula

$$\psi^1_{x_0,h}(x) = -\frac{\langle h, x - x_0\rangle}{\|x - x_0\|^2}, \tag{48}$$

for all $x \in \mathbb{R}^2 \setminus x_0$, where $\langle \cdot, \cdot \rangle$ denotes the inner product.

Charge and dipole distributions with density $\rho\colon [0, L] \to \mathbb{R}$ on Γ produce potentials given by the formulas

$$\phi(x) = \int_0^L \psi^0_{\gamma(t)}(x)\rho(t)\,dt, \tag{49}$$

and

$$\phi(x) = \int_0^L \psi^1_{\gamma(t),\nu(t)}(x)\rho(t)\,dt, \tag{50}$$

respectively, for any $x \in \mathbb{R}^2 \setminus \Gamma$.

Reduction of Boundary Value Problems to Integral Equations

The following four theorems reduce the boundary value problems of Section 3.1 to boundary integral equations. They are found in, for example, [14], [21].

Theorem 3.1 (Exterior Neumann problem)**.** *Suppose that $\rho \in L^2([0, L])$. Suppose further that $g\colon [0, L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = -\pi\rho(s) + \int_0^L \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt, \tag{51}$$

for any $s \in [0, L]$. Then g is in $L^2([0, L])$, and a solution ϕ to the exterior Neumann problem with right hand side g is obtained by substituting ρ into (49). Moreover, for any $g \in L^2([0, L])$, equation (51) has a unique solution $\rho \in L^2([0, L])$.

Theorem 3.2 (Interior Dirichlet problem)**.** *Suppose that $\rho \in L^2([0, L])$. Suppose further that $g\colon [0, L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = -\pi\rho(s) + \int_0^L \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt, \tag{52}$$

for any $s \in [0, L]$. Then g is in $L^2([0, L])$, and the solution ϕ to the interior Dirichlet problem with right hand side g is obtained by substituting ρ into (50). Moreover, for any $g \in L^2([0, L])$, equation (52) has a unique solution $\rho \in L^2([0, L])$.

The following two theorems make use of the function $\overline{\omega}\colon [0, L] \to \mathbb{R}$, defined as the solution to the equation

$$\int_0^L \overline{\omega}(t) \log(\|x - \gamma(t)\|)\,dt = 1, \tag{53}$$

for all $x \in \overline{\Omega}$. In other words, we define the function $\overline{\omega}$ as the density of the charge distribution on Γ when $\overline{\Omega}$ is a conductor.

Theorem 3.3 (Interior Neumann problem)**.** *Suppose that $\rho \in L^2([0, L])$. Suppose further that $g\colon [0, L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = \pi\rho(s) + \int_0^L \psi^1_{\gamma(s),\nu(s)}(\gamma(t))\rho(t)\,dt, \tag{54}$$

for any $s \in [0, L]$. Then g is in $L^2([0, L])$, and a solution ϕ to the exterior Neumann problem with right hand side g is obtained by substituting ρ into (49).

Now suppose that g is an arbitrary function in $L^2([0, L])$ such that

$$\int_0^L g(t)\,dt = 0. \tag{55}$$

Then equation (54) has a solution $\rho \in L^2([0, L])$. Moreover, if ρ_1 and ρ_2 are both solutions to equation (54), then there exists a real number C such that

$$\rho_1(t) - \rho_2(t) = C\overline{\omega}(t), \tag{56}$$

for $t \in [0, L]$, where $\overline{\omega}$ is the solution to (53).

Theorem 3.4 (Exterior Dirichlet problem)**.** *Suppose that $\rho \in L^2([0, L])$. Suppose further that $g \colon [0, L] \to \mathbb{R}$ is defined by the formula*

$$g(s) = \pi\rho(s) + \int_0^L \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\rho(t)\,dt, \tag{57}$$

for any $s \in [0, L]$. Then g is in $L^2([0, L])$, and the solution ϕ to the interior Dirichlet problem with right hand side g is obtained by substituting ρ into (50).

Now suppose that g is an arbitrary function in $L^2([0, L])$ such that

$$\int_0^L g(t)\overline{\omega}(t)\,dt = 0, \tag{58}$$

where $\overline{\omega}$ is the solution to (53). Then equation (57) has a solution $\rho \in L^2([0, L])$. Moreover, if ρ_1 and ρ_2 are both solutions to equation (57), then there exists a real number C such that

$$\rho_1(t) - \rho_2(t) = C, \tag{59}$$

for $t \in [0, L]$.

Observation 3.1. Equation (51) is the adjoint of equation (52), and equation (54) is the adjoint of equation (57).

Observation 3.2. Suppose that the curve $\gamma \colon [0, L] \to \mathbb{R}^2$ is *not* closed. We observe that if $\rho \in L^2([0, L])$, and g is defined by either (51), (52), (54), or (57), then $g \in L^2([0, L])$. Moreover, if $g \in L^2([0, L])$, then equations (51), (52), (54), and (57) have unique solutions $\rho \in L^2([0, L])$.

Properties of the Kernels of Equations (51), (52), (54), and (57)

The following theorem shows that if a curve γ is has k continuous derivatives, where $k \geq 2$, then the kernels of equations (51), (52), (54), and (57) have $k - 2$ continuous derivatives. It is found in, for example, [1].

Theorem 3.5. *Suppose that $\gamma\colon [0,L] \to \mathbb{R}^2$ is a curve in $\mathbb{R}^2$ that is parameterized by arc length. Suppose further that $k \ge 2$ is an integer. If γ is C^k on a neighborhood of a point s, where $0 < s < L$, then*

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)), \tag{60}$$
$$\psi^1_{\gamma(t),\nu(t)}(\gamma(s)), \tag{61}$$

are C^{k-2} functions of t on a neighborhood of s and

$$\lim_{t\to s} \psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \lim_{t\to s} \psi^1_{\gamma(t),\nu(t)}(\gamma(s)) = -\frac{1}{2}\kappa(s), \tag{62}$$

where $\kappa\colon [0,L] \to \mathbb{R}$ is the signed *curvature of γ and ψ^1 is defined in (48). Furthermore, if γ is analytic on a neighborhood of a point s, where $0 < s < L$, then (60) and (61) are analytic functions of t on a neighborhood of s.*

When the curve γ is a wedge, the kernels of equations (51), (52), (54), and (57) all have a particularly simple form, which is given by the following lemma. It is proved in [18].

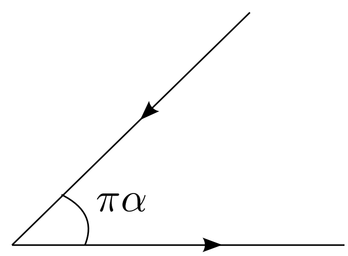

Figure 3: A wedge in $\mathbb{R}^2$

Lemma 3.6. *Suppose $\gamma\colon [-1,1] \to \mathbb{R}^2$ is defined by the formula*

$$\gamma(t) = \begin{cases} -t\cdot(\cos(\pi\alpha), \sin(\pi\alpha)) & \text{if } -1 \le t < 0, \\ (t,0) & \text{if } 0 \le t \le 1, \end{cases} \tag{63}$$

shown in Figure 3. Then, for all $0 < s \le 1$,

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \begin{cases} \dfrac{t\sin(\pi\alpha)}{s^2+t^2+2st\cos(\pi\alpha)} & \text{if } -1 \le t < 0, \\ 0 & \text{if } 0 \le t \le 1, \end{cases} \tag{64}$$

and, for all $-1 \le s < 0$,

$$\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = \begin{cases} 0 & \text{if } -1 \le t < 0, \\ \dfrac{-t\sin(\pi\alpha)}{s^2+t^2+2st\cos(\pi\alpha)} & \text{if } 0 \le t \le 1. \end{cases} \tag{65}$$

Corollary 3.7. *Identities (64) and (65) remain valid after any rotation or translation of the curve γ in $\mathbb{R}^2$.*

Corollary 3.8. *When the curve γ is a straight line, $\psi^1_{\gamma(s),\nu(s)}(\gamma(t)) = 0$ for all $-1 \le s,t \le 1$.*

3.3 Potential Theory and Complex Variables

In two dimensions, the theory of harmonic functions can be formulated in terms of complex variables. Viewing points in $\mathbb{R}^2$ as points in $\mathbb{C}$, we have

$$\psi^0_{z_0}(z) = \log(\|z - z_0\|) = \mathrm{Im}(\log(z - z_0)), \tag{66}$$

for all $z, z_0 \in \mathbb{C}$ such that $z \neq z_0$. Suppose that $h \in \mathbb{C}$ and $|h| = 1$. Then, likewise,

$$\psi^1_{z_0,ih}(z) = -\frac{\langle ih, z - z_0\rangle}{\|z - z_0\|^2} = \mathrm{Im}\Big(\frac{h}{z - z_0}\Big), \tag{67}$$

for all $z, z_0 \in \mathbb{C}$ such that $z \neq z_0$.

Viewing $\gamma\colon [0, L] \to \mathbb{R}^2$ then as a curve in $\mathbb{C}$, it is also easy to observe that

$$\psi^0_{\gamma(t)}(z) = \mathrm{Im}(\log(z - \gamma(t))), \tag{68}$$

for all $0 \le t \le L$ and $z \in \mathbb{C} \setminus \Gamma$, and

$$\psi^1_{\gamma(t),\nu(t)}(z) = \mathrm{Im}\Big(\frac{\gamma'(t)}{z - \gamma(t)}\Big), \tag{69}$$

for all $0 \le t \le L$ and $z \in \mathbb{C} \setminus \Gamma$.

3.4 The Integral Equations of Potential Theory on a Wedge

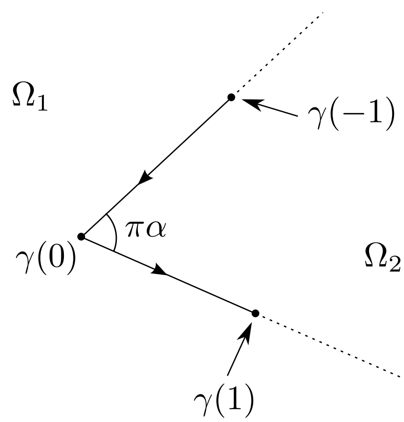

Figure 4: A wedge in $\mathbb{R}^2$

Suppose that $\gamma\colon [-1, 1] \to \mathbb{R}^2$ is a wedge with interior angle $\pi\alpha$ (i.e. both γ_1 and γ_2 are straight lines) (see Figure 4).

The following theorem is proved in [20], and is a special case of Theorem 2.1.

Theorem 3.9 (Dirichlet)**.** *Suppose that* $0 < \alpha < 2$ *and that* N *is a nonnegative integer. Letting* $\lfloor\cdot\rfloor$ *and* $\lceil\cdot\rceil$ *denote the floor and ceiling functions respectively, suppose that*

$$\bar{L} = \Big\lceil \frac{\alpha N}{2} \Big\rceil, \tag{70}$$

$$\underline{L} = \Big\lfloor \frac{\alpha N}{2} \Big\rfloor, \tag{71}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{72}$$

$$\overline{M} = \left\lceil \frac{(2-\alpha)N}{2} \right\rceil, \tag{73}$$

and observe that $\overline{L} + \underline{M} = N$ *and* $\overline{M} + \underline{L} = N$. *Suppose further that* ρ *is defined by the formula*

$$\rho(t) = \sum_{i=1}^{\overline{L}} b_{\underline{M}+i} |t|^{\frac{2i-1}{\alpha}} + \sum_{i=0}^{\underline{M}} b_i |t|^{\frac{2i}{2-\alpha}} (\log|t|)^{\sigma(i)}$$
$$+ \sum_{i=1}^{\overline{M}} c_{\underline{L}+i} \operatorname{sgn}(t) |t|^{\frac{2i-1}{2-\alpha}} + \sum_{i=0}^{\underline{L}} c_i \operatorname{sgn}(t) |t|^{\frac{2i}{\alpha}} (\log|t|)^{\nu(i)}, \tag{74}$$

for all $-1 \le t \le 1$, *where* $b_0, b_1, \ldots, b_N$ *and* $c_0, c_1, \ldots, c_N$ *are arbitrary real numbers,*

$$\sigma(i) = \begin{cases} 0 & \text{if } \frac{2i}{2-\alpha} \neq \frac{2j-1}{\alpha} \text{ for each integer } 1 \le j \le \overline{L}, \\ 1 & \text{if } \frac{2i}{2-\alpha} = \frac{2j-1}{\alpha} \text{ for some integer } 1 \le j \le \overline{L}, \end{cases} \tag{75}$$

for $0 \le i \le \underline{M}$, *and*

$$\nu(i) = \begin{cases} 0 & \text{if } \frac{2i}{\alpha} \neq \frac{2j-1}{2-\alpha} \text{ for each integer } 1 \le j \le \overline{M}, \\ 1 & \text{if } \frac{2i}{\alpha} = \frac{2j-1}{2-\alpha} \text{ for some integer } 1 \le j \le \overline{M}, \end{cases} \tag{76}$$

for $0 \le i \le \underline{L}$. *Suppose finally that* g *is defined by (3). Then there exist sequences of real numbers* $\beta_0, \beta_1, \ldots$ *and* $\gamma_0, \gamma_1, \ldots$ *such that*

$$g(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \operatorname{sgn}(t) |t|^n, \tag{77}$$

for all $-1 \le t \le 1$. *Conversely, suppose that* g *has the form (77). Suppose further that* N *is an arbitrary nonnegative integer. Then, for all angles* $\pi\alpha$, *there exist unique real numbers* $b_0, b_1, \ldots, b_N$ *and* $c_0, c_1, \ldots, c_N$ *such that* ρ, *defined by (74), solves equation (3) to within an error* $O(|t|^{N+1})$.

The following theorem is also proved in [20], and is a special case of Theorem 2.2.

Theorem 3.10 (Neumann)**.** *Suppose that* $0 < \alpha < 2$ *and that* N *is a positive integer. Letting* $\lfloor \cdot \rfloor$ *and* $\lceil \cdot \rceil$ *denote the floor and ceiling functions respectively, suppose that*

$$\overline{L} = \left\lceil \frac{\alpha N}{2} \right\rceil, \tag{78}$$

$$\underline{L} = \left\lfloor \frac{\alpha N}{2} \right\rfloor, \tag{79}$$

and

$$\underline{M} = \left\lfloor \frac{(2-\alpha)N}{2} \right\rfloor, \tag{80}$$

$$\overline{M} = \left\lceil \frac{(2-\alpha)N}{2} \right\rceil, \tag{81}$$

and observe that $\bar{L} + \underline{M} = N$ *and* $\overline{M} + \underline{L} = N$*. Suppose further that* ρ *is defined by the formula*

$$\rho(t) = \sum_{i=1}^{\bar{L}} b_{\underline{M}+i} |t|^{\frac{2i-1}{\alpha}-1} + \sum_{i=1}^{\underline{M}} b_i |t|^{\frac{2i}{2-\alpha}-1} (\log|t|)^{\sigma(i)}$$
$$+ \sum_{i=1}^{\overline{M}} c_{\underline{L}+i} \operatorname{sgn}(t) |t|^{\frac{2i-1}{2-\alpha}-1} + \sum_{i=1}^{\underline{L}} c_i \operatorname{sgn}(t) |t|^{\frac{2i}{\alpha}-1} (\log|t|)^{\nu(i)}, \tag{82}$$

for all $-1 \le t \le 1$*, where* $b_1, b_2, \ldots, b_N$ *and* $c_1, c_2, \ldots, c_N$ *are arbitrary real numbers,*

$$\sigma(i) = \begin{cases} 0 & \text{if } \frac{2i}{2-\alpha} \neq \frac{2j-1}{\alpha} \text{ for each integer } 1 \le j \le \bar{L}, \\ 1 & \text{if } \frac{2i}{2-\alpha} = \frac{2j-1}{\alpha} \text{ for some integer } 1 \le j \le \bar{L}, \end{cases} \tag{83}$$

for $1 \le i \le \underline{M}$*, and*

$$\nu(i) = \begin{cases} 0 & \text{if } \frac{2i}{\alpha} \neq \frac{2j-1}{2-\alpha} \text{ for each integer } 1 \le j \le \overline{M}, \\ 1 & \text{if } \frac{2i}{\alpha} = \frac{2j-1}{2-\alpha} \text{ for some integer } 1 \le j \le \overline{M}, \end{cases} \tag{84}$$

for $1 \le i \le \underline{L}$*. Suppose finally that* g *is defined by (24). Then there exist sequences of real numbers* $\beta_0, \beta_1, \ldots$ *and* $\gamma_0, \gamma_1, \ldots$ *such that*

$$g(t) = \sum_{n=0}^{\infty} \beta_n |t|^n + \sum_{n=0}^{\infty} \gamma_n \operatorname{sgn}(t) |t|^n, \tag{85}$$

for all $-1 \le t \le 1$*. Conversely, suppose that* g *has the form (85). Suppose further that* N *is an arbitrary positive integer. Then, for all angles* $\pi\alpha$*, there exist unique real numbers* $b_1, b_2, \ldots, b_N$ *and* $c_1, c_2, \ldots, c_N$ *such that* ρ*, defined by (82), solves equation (24) to within an error* $O(|t|^N)$*.*

3.5 Ordinary Differential Equations in the Complex Domain

The following theorem is found in, for example, [6].

Theorem 3.11. *Let* U *be an open simply connected set in* $\mathbb{C}$*, and suppose that* $A\colon U \to M_n(\mathbb{C})$ *is holomorphic, where* $M_n(\mathbb{C})$ *denotes the set of all* $n \times n$ *matrices over* $\mathbb{C}$*. Suppose further that* $z_0 \in U$*. Then for any* $y_0 \in \mathbb{C}^n$*, there exists a unique holomorphic function* $y\colon U \to \mathbb{C}^n$ *such that*

$$y'(z) = A(z) y(z), \tag{86}$$

for all $z \in U$*, and*

$$y(z_0) = y_0. \tag{87}$$

Furthermore, the solution $y(z)$ *at any point* $z \in U$ *depends holomophically on the initial value* $y_0 \in \mathbb{C}^n$*.*

The following corollary states that if the matrix A in (86) depends holomorphically on some additional parameters $w \in \mathbb{C}^m$, then the solution y also depends holomorphically on w. It follows from Theorem 3.11 by treating the additional parameters as fictitious dependent variables (see, for example, [6]).

Corollary 3.12. *Suppose that U is an open simply connected set in $\mathbb{C}$. Suppose further that m is a nonnegative integer and that $V \subset \mathbb{C}^m$ is also open and simply connected. Suppose finally that $A\colon U \times V \to M_n(\mathbb{C})$ is holomorphic, where $M_n(\mathbb{C})$ denotes the set of all $n \times n$ matrices over $\mathbb{C}$. Suppose further that $z_0 \in U$. Then for any $y_0 \in \mathbb{C}^n$, there exists a unique holomorphic function $y\colon U \to \mathbb{C}^n$ such that,*

$$y'(z) = A(z, w)y(z), \tag{88}$$

for all $z \in U$, and

$$y(z_0) = y_0. \tag{89}$$

Furthermore, the solution $y(z)$ at any point $z \in U$ depends holomophically on both the initial value $y_0 \in \mathbb{C}^n$ and the parameters $w \in \mathbb{C}^m$.

3.6 Complex Functions of Several Variables

Suppose that $U \subset \mathbb{C}^n$ is open and connected. A function $f\colon U \to \mathbb{C}$ is called partially holomorphic if it is holomorphic in each variable separately. Specifically, f is partially homomorphic if, for each fixed $(z_1^0, z_2^0, \ldots, z_n^0) \in U$ and each integer $1 \le j \le n$, the function $f(z_1^0, \ldots, z_{j-1}^0, z_j, z_{j+1}^0, \ldots, z_n^0)$ is a holomorphic function of z_j. If f is both partially holomorphic and continuous, then f is called holomorphic. It turns out that, as a result of Hartogs' extension theorem, every partially holomorphic function is necessarily continuous (see, for example, [16]).

The following theorem states that the zero set of a holomorphic function $f : \mathbb{C}^n \to \mathbb{C}$ has codimension 1 (see, for example, [8]).

Theorem 3.13. *Suppose that $U \subset \mathbb{C}^n$ is a open connected set and that $f\colon U \to \mathbb{C}$ is a holomorphic function. Suppose further that $f \not\equiv 0$, and let $N(f)$ denote the set $\{x \in U : f(x) = 0\}$. Then, for each point $a \in N(f)$, there exists a 1-dimensional affine subspace Γ of $\mathbb{C}^n$ such that a is an isolated point of $\Gamma \cap N(f)$.*

The following theorem states that a certain kernel involving an analytic function is equal to the Cauchy kernel plus an analytic function. For completeness, a proof is provided in Appendix A.

Theorem 3.14. *Suppose that $U \subset \mathbb{C}$ is a open simply connected set and that $F\colon U \to \mathbb{C}$ is a analytic function such that $F'(z) \neq 0$ for all $z \in U$. Suppose further that $K\colon \{(s,t) \in U \times U : s \neq t\} \to \mathbb{C}$ is defined by*

$$K(s,t) = \frac{F'(t)}{F(s) - F(t)}, \tag{90}$$

for all $s, t \in U$ such that $s \neq t$. Then there exists some holomorphic function $R\colon U \times U \to \mathbb{C}$ such that

$$K(s,t) = \frac{1}{s-t} - \frac{F''(t)}{2F'(t)} + (s-t)R(s,t) \tag{91}$$

for all $s, t \in U$ such that $s \neq t$.

3.7 Real Analytic Functions of Several Variables

The following theorem states that any real analytic function has a complex analytic extension.

Theorem 3.15. *Suppose that n is a positive integer and that $U \subset \mathbb{R}^n$ is an open simply-connected set. Suppose further that $f\colon U \to \mathbb{R}$ is a real analytic function. Then there exists an open simply-connected set $V \subset \mathbb{C}^n$ which contains U, and a complex analytic function $F\colon V \to \mathbb{C}$ such that $F|_U = f$.*

The following theorem states that the zero set of a real analytic function has zero measure and is proved in, for example, [12].

Theorem 3.16. *Suppose that $U \subset \mathbb{R}^n$ is a open connected set and that $f\colon U \to \mathbb{R}$ is a real analytic function. Suppose further that $f \not\equiv 0$, and let $N(f)$ denote the set $\{x \in U : f(x) = 0\}$. Then the set $N(f)$ has zero measure.*

3.8 Analytic Curves

This section contains some elementary definitions and lemmas related to analytic curves in $\mathbb{R}^2$.

Definition 3.1. Suppose that $a < b$ are real numbers. We refer to a curve $\gamma\colon [a,b] \to \mathbb{C}$ as an analytic curve if, at each point $a \le t_0 \le b$, there is a neighborhood of t_0 on which the curve γ is representable by a Taylor series centered at t_0, and $\gamma'(t_0) \neq 0$ for each $a \le t_0 \le b$.

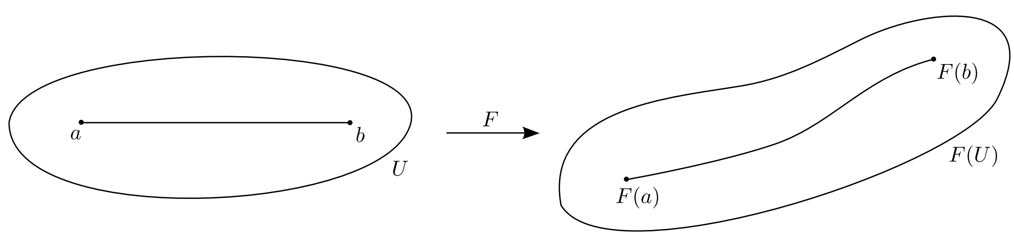

Figure 5: An analytic curve in $\mathbb{C}$

The following theorem provides an alternative definition of an analytic curve.

Theorem 3.17. *Suppose that $a < b$ are real numbers. Then a curve $\gamma\colon [a,b] \to \mathbb{C}$ is an analytic curve if and only if there exists some simply connected open set $[a,b] \subset U \subset \mathbb{C}$, and some analytic function $F\colon U \to \mathbb{C}$ such that $F|_{[a,b]} = \gamma$ and $F'(z) \neq 0$ for all $z \in U$ (see Figure 5).*

The following theorem states that if an analytic curve is reparameterized by arc length, then the result is also an analytic curve. A proof is found in, for example, [13].

Theorem 3.18. *Suppose that $a < b$ are real numbers and that $\gamma\colon [a,b] \to \mathbb{C}$ is an analytic curve. Let $s\colon [a,b] \to \mathbb{R}$ denote the arc length of the curve, defined by*

$$s(t) = \int_a^t |\gamma'(u)|\, du, \tag{92}$$

for all $a \le t \le b$. Then $\gamma \circ s^{-1}\colon [0, s(b)] \to \mathbb{C}$ is also an analytic curve.

3.9 Elementary Differential Geometry

The following are some elementary facts from differential geometry, which can be found in, for example, [11].

Definition 3.2. Suppose that $L > 0$ is a real number and $\gamma\colon [0,L] \to \mathbb{C}$ is an analytic curve parameterized by arc length. Then we define the *unit tangent vector* $T\colon [0,L] \to \mathbb{C}$ by

$$T(x) = \gamma'(x), \tag{93}$$

for all $0 \le x \le L$, the *unit normal vector* $N\colon [0,L] \to \mathbb{C}$ by

$$N(x) = iT(x), \tag{94}$$

for all $0 \le x \le L$, and the *signed curvature* $\kappa\colon [0,L] \to \mathbb{R}$ by

$$\kappa(x) = \frac{T'(x)}{N(x)}, \tag{95}$$

for all $0 \le x \le L$. It is straightforward to show that κ is always real-valued.

The following theorem states that the signed curvature of an analytic curve is also an analytic function.

Theorem 3.19. *Suppose that $L > 0$ is a real number an that $\gamma\colon [0,L] \to \mathbb{C}$ is an analytic curve. Then there exists some open simply connected set $[0,L] \subset U \subset \mathbb{C}$ and some analytic function $\kappa\colon U \to \mathbb{C}$ such that $\kappa|_{[0,L]}$ is the signed curvature of γ.*

Proof. Without loss of generality, we suppose that γ is parameterized by arc length (see Theorem 3.18). By Theorem 3.17, there exists some open simply connect set $[0,L] \subset U \subset \mathbb{C}$ and some analytic function $F\colon U \to \mathbb{C}$ such that $F'(z) \neq 0$ for any $z \in U$ and $F|_{[0,L]} = \gamma$. Let $\kappa\colon U \to \mathbb{C}$ be defined by

$$\kappa(z) = -\frac{iF''(z)}{F'(z)}, \tag{96}$$

for all $z \in U$. Clearly, κ is analytic, and

$$\kappa(x) = \frac{T'(x)}{N(x)}, \tag{97}$$

for all $0 \le x \le L$, where T is the unit tangent vector of γ and N is the unit normal vector (see Definition 3.2).

■

The following theorem states that an analytic signed curvature uniquely determines an analytic curve parameterized by arc length.

Theorem 3.20. *Suppose that $L > 0$ is an integer and that $[0,L] \subset U \subset \mathbb{C}$ is an open simply connected set. Suppose further that $\kappa\colon U \to \mathbb{C}$ is an analytic function such that $\mathrm{Im}(\kappa(x)) = 0$ when $x \in [0,L]$. Then, for each $\gamma_0 \in \mathbb{C}$ and real number $0 \le \theta < 2\pi$, there exists a unique analytic curve parameterized by arc length $\gamma\colon [0,L] \to \mathbb{C}$ with curvature $\kappa|_{[0,L]}$, such that $\gamma(0) = \gamma_0$ and $\gamma'(0) = e^{i\theta}$. Furthermore, there exists an analytic function $F\colon U \to \mathbb{C}$ such that $F|_{[0,L]} = \gamma$.*

Proof. By Theorem 3.11, there exist unique holomorphic functions $T\colon U \to \mathbb{C}$ and $N\colon U \to \mathbb{C}$ such that

$$\begin{pmatrix} T'(z) \\ N'(z) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(z) \\ -\kappa(z) & 0 \end{pmatrix} \begin{pmatrix} T(z) \\ N(z) \end{pmatrix} \tag{98}$$

for all $z \in U$, and

$$T(0) = e^{i\theta}, \tag{99}$$

$$N(0) = iT(0). \tag{100}$$

Since $\kappa(x)$ is real for all $x \in [0, L]$, it follows that equation (98), on the interval $[0, L]$, describes the Frenet-Serret apparatus for a curve, so $|T(x)| = 1$ and $N(x) = iT(x)$ for all $x \in [0, L]$ (see, for example, [11]). Let $F\colon U \to C$ be defined by

$$F(z) = \int_0^z T(w)\,dw + \gamma_0, \tag{101}$$

for all $z \in U$, and let $\gamma\colon [0, L] \to \mathbb{C}$ be defined by $\gamma = F|_{[0,L]}$. Then γ is an analytic curve parameterized by arc length with curvature $\kappa|_{[0,L]}$, such that $\gamma(0) = \gamma_0$ and $\gamma'(0) = e^{i\theta}$. ■

The following theorem states that the 2nd to the $(m+1)$th derivatives of a curve vanish at a point if and only if the curvature, as well as the curvature's first $m-1$ derivatives, vanish at that point.

Theorem 3.21. *Suppose that $L > 0$ is a real number and that $\gamma\colon [0, L] \to \mathbb{C}$ is an analytic curve parameterized by arc length with curvature $\kappa\colon [0, L] \to \mathbb{R}$. Suppose further that m is a nonnegative integer. Then, for each $0 \le x \le L$, $\gamma^{(n+2)}(x) = 0$ for $n = 0, 1, 2, \ldots, m-1$ if and only if $\kappa^{(n)}(x) = 0$ for $n = 0, 1, 2, \ldots, m-1$.*

Proof. The proof follows from repeated differentiation of the formula

$$T'(x) = \kappa(x) i T(x), \tag{102}$$

for all $0 \le x \le L$, where T is the unit tangent vector of γ (see Definition 3.2). ■

The following theorem states that if the curvature is bounded by ϵ, then the curve lies in a certain sector in $\mathbb{R}^2$ with subtended angle ϵ.

Theorem 3.22. *Suppose that $\gamma\colon [0, 1] \to \mathbb{R}^2$ is an analytic curve, and that $\gamma(0) = \gamma_0$, where $\gamma_0 \in \mathbb{R}^2$, and $\gamma'(0) = e^{i\theta_0}$, where $0 \le \theta_0 < 2\pi$. Let κ denote the signed curvature of γ, and suppose further that $\epsilon > 0$ and $|\kappa(t)| < \epsilon$ for all $0 \le t \le 1$. Then*

$$\gamma([0, 1]) \subset \left\{ \gamma_0 + re^{i(\theta+\theta_0)} : -\frac{\epsilon}{2} \le \theta \le \frac{\epsilon}{2},\ 0 \le r \le 1 \right\}. \tag{103}$$

3.10 Conformal Mapping and Analytic Curves

The principal results of this section are theorems 3.24 and 3.25.

The following lemma states that, under certain conditions, the maximum of a continuous function is also a continuous function of any additional parameters.

Lemma 3.23. *Suppose that n and m are positive integers and that $K \subset \mathbb{C}^n$ is a compact set. Suppose further that $g\colon K \times \mathbb{C}^m \to \mathbb{C}$ is a continuous function and that $h\colon \mathbb{C}^m \to [0,\infty)$ is defined by*

$$h(w) = \max_{z \in K} |g(z,w)| \tag{104}$$

for all $w \in \mathbb{C}^m$. Then h is also continuous.

The following theorem states that all curves with sufficiently small curvatures are representable as restrictions of conformal mappings of the unit disc.

Theorem 3.24. *Suppose that M is a nonnegative integer and that the function $\kappa\colon [0,1] \to \mathbb{R}$ is defined by*

$$\kappa(t) = a_0 + a_1 t + \cdots + a_M t^M, \tag{105}$$

for all $0 \le t \le 1$, where $a_0, a_1, \ldots, a_M \in \mathbb{R}$. Suppose further that $0 \le \theta < 2\pi$ and $\gamma_0 \in \mathbb{C}$, and let $\gamma\colon [0,1] \to \mathbb{C}$ be the analytic curve with curvature κ, where $\gamma'(0) = e^{i\theta}$ and $\gamma(0) = \gamma_0$ (see Theorem 3.20). Suppose now that $\epsilon > 0$ and let $D_{\gamma_0,1+\epsilon} \subset \mathbb{C}$ denote the open disc of radius $1+\epsilon$ centered at γ_0. Then there exists a $\delta > 0$ such that, for each $(a_0, a_1, \ldots, a_M) \in B_\delta$, where $B_\delta \subset \mathbb{R}^{M+1}$ is the open ball of radius δ centered at zero, there is an open simply-connected set $[0,1] \subset U \subset \mathbb{C}$ and a conformal mapping (a biholomorphic function) $F\colon U \to D_{\gamma_0,1+\epsilon}$, such that $F|_{[0,1]} = \gamma$.

Proof. Suppose that $\kappa\colon \mathbb{C} \times \mathbb{C}^{M+1} \to \mathbb{C}$ is defined by

$$\kappa(z,\mathbf{a}) = a_0 + a_1 z + \cdots + a_M z^M, \tag{106}$$

for all $z \in \mathbb{C}$, where $\mathbf{a} = (a_0, a_1, \ldots, a_M) \in \mathbb{C}^{M+1}$. By Corollary 3.12, there exist unique entire functions $T\colon \mathbb{C} \to \mathbb{C}$ and $N\colon \mathbb{C} \to \mathbb{C}$ such that

$$\begin{pmatrix} T'(z) \\ N'(z) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(z,\mathbf{a}) \\ -\kappa(z,\mathbf{a}) & 0 \end{pmatrix} \begin{pmatrix} T(z) \\ N(z) \end{pmatrix} \tag{107}$$

for all $z \in U$, and

$$T(0) = e^{i\theta}, \tag{108}$$

$$N(0) = ie^{i\theta}. \tag{109}$$

Furthermore, for any fixed $z \in \mathbb{C}$, the solutions $T(z)$ and $N(z)$ depend holomorphically on $\mathbf{a} \in \mathbb{C}^{M+1}$. Let $F\colon \mathbb{C} \times \mathbb{C}^{M+1} \to \mathbb{C}$ be defined by

$$F(z,\mathbf{a}) = \int_0^z T(w,\mathbf{a})\, dw + \gamma_0, \tag{110}$$

for all $z \in \mathbb{C}$ and $\mathbf{a} \in \mathbb{C}^{M+1}$. Note that since (107) is a Frenet-Seret apparatus, we have that, for any real $\mathbf{a} \in \mathbb{R}^{M+1}$, $\gamma = F|_{[0,1]}$ is an analytic curve parameterized by arc length with curvature $\kappa|_{[0,1]}$, where $\gamma'(0) = e^{i\theta}$ and $\gamma(0) = \gamma_0$. Clearly,

$$F(z, \mathbf{0}) = e^{i\theta} z + \gamma_0, \tag{111}$$

for all $z \in \mathbb{C}$.

Suppose that $g\colon \mathbb{C} \times \mathbb{C}^{M+1}$ is defined by

$$g(z, \mathbf{a}) = F(z, \mathbf{a}) - e^{i\theta} z - \gamma_0, \tag{112}$$

for all $z \in \mathbb{C}$ and $\mathbf{a} \in \mathbb{C}^{M+1}$. Suppose further that $\overline{D}_{2+2\epsilon} \subset \mathbb{C}$ denotes the closed disc of radius $2 + 2\epsilon$ centered at zero, and let $h\colon \mathbb{C}^{M+1} \to [0, \infty)$ be defined by

$$h(\mathbf{a}) = \max_{z \in \overline{D}_{2+2\epsilon}} |g'(z, \mathbf{a})|, \tag{113}$$

for all $\mathbf{a} \in \mathbb{C}^{M+1}$, where g' denotes the derivative of g with respect to z. Clearly, $h(\mathbf{0}) = 0$. By Lemma 3.23, h is continuous, so there exists some $\delta > 0$ such that

$$|h(\mathbf{a})| < \frac{1}{2}, \tag{114}$$

for all $\mathbf{a} \in B_\delta$, where $B_\delta \subset \mathbb{C}^{M+1}$ is the ball of radius δ centered at zero.

Suppose that now that $\mathbf{a} \in B_\delta$ is fixed; in an abuse of notation, we omit $\mathbf{a}$ where the meaning is clear. We now show that F, defined by (110), is injective on $\overline{D}_{2+2\epsilon}$. Suppose $w \in \overline{D}_{2+2\epsilon}$ is fixed. We observe that

$$F(z) - F(w) = g(z) - g(w) + e^{i\theta} z - e^{i\theta} w, \tag{115}$$

for all $z \in \overline{D}_{2+2\epsilon}$. Clearly, $e^{i\theta} z - e^{i\theta} w$ is an analytic function of z with only one zero at $z = w$. We also observe that, for any $z, w \in \overline{D}_{2+2\epsilon}$,

$$\begin{aligned}
|g(z) - g(w)| &= \Big| \int_w^z g'(u)\, du \Big| \\
&\leq \max_{u \in \overline{D}_{2+2\delta}} |g'(u)| \cdot |z - w| \\
&= h(\mathbf{a}) \cdot |z - w| \\
&< \frac{1}{2} |z - w|.
\end{aligned} \tag{116}$$

Thus,

$$|g(z) - g(w)| < |e^{i\theta} z - e^{i\theta} w|, \tag{117}$$

for all $z \in \overline{D}_{2+2\epsilon}$, so, by Rouche's theorem, $F(z) - F(w)$ also has only one zero at $z = w$. Since this holds for any $w \in \overline{D}_{2+2\epsilon}$, it follows that F is injective on $\overline{D}_{2+2\epsilon}$. We now show that $F(\overline{D}_{2+2\epsilon}) \supset \overline{D}_{\gamma_0, 1+\epsilon}$. Since F is an entire function, and is injective on $\overline{D}_{2+2\epsilon}$, we observe that $F(\partial D_{2+2\epsilon}) = \partial F(D_{2+2\epsilon})$, where $\partial D_{2+2\epsilon}$ denotes the boundary of the set $D_{2+2\epsilon}$ and $\partial F(D_{2+2\epsilon})$ denotes the boundary of the set $F(D_{2+2\epsilon})$. By (116), we have that, for all $z \in \partial D_{2+2\epsilon}$,

$$
\begin{aligned}
&|F(z) - \gamma_0| = |F(z) - F(0)| \\
&= |g(z) - g(0) - e^{i\theta} z| \\
&\geq |z| - |g(z) - g(0)| \\
&> \frac{1}{2}|z| \\
&= 1 + \epsilon.
\end{aligned} \tag{118}
$$

Therefore, if $w \in \partial F(D_{2+2\epsilon})$, then $|w - \gamma_0| > 1 + \epsilon$. Hence, $F(\overline{D}_{2+2\epsilon}) \supset \overline{D}_{\gamma_0,1+\epsilon}$. Letting the set $[0,1] \subset U \subset \mathbb{C}$ be defined by the formula $U = F^{-1}(D_{\gamma_0,1+\epsilon})$, we have that F is a conformal map from U to $D_{\gamma_0,1+\epsilon}$.

■

The following theorem states that a curve with curvature κ, defined by (106), depends analytically on the parameters $a_0, a_1, \ldots, a_M$. It follows immediately from the proof of Theorem 3.24.

Theorem 3.25. *Suppose that M is a nonnegative integer and that $\kappa\colon [0,1] \to \mathbb{R}$ is defined by (106), where $a_0, a_1, \ldots, a_M \in \mathbb{R}$. Suppose further that $0 \leq \theta < 2\pi$ and $\gamma_0 \in \mathbb{C}$, and let $\gamma\colon [0,1] \to \mathbb{C}$ be the analytic curve with curvature κ, where $\gamma'(0) = e^{i\theta}$ and $\gamma(0) = \gamma_0$ (see Theorem 3.20). Then, for any fixed $t \in [0,1]$, $\gamma(t)$ is a real-analytic function of the parameters $a_0, a_1, \ldots, a_M$.*

Remark 3.3. It is straightforward to show that, in a small enough neighborhood of zero, all analytic curves satisfy the conditions of...

3.11 Elementary Analytical Facts

This section contains a number of miscellaneous elementary lemmas.

Theorem 3.26. *Suppose that $0 < \alpha < 2$. Then*

$$
\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} + 1 = 0, \tag{119}
$$

if and only if

$$
\mu = \frac{2n-1}{\alpha}, \tag{120}
$$

or

$$
\mu = \frac{2n}{2-\alpha}, \tag{121}
$$

for some integer n.

Theorem 3.27. *Suppose that* $0 < \alpha < 2$ *and that* $\alpha \notin \mathbb{Q}$*. Suppose further that* $J \neq 0$ *is an integer and that either*

$$\mu = \frac{2n-1}{\alpha} + J, \tag{122}$$

or

$$\mu = \frac{2n}{2-\alpha} + J, \tag{123}$$

for some integer n*. Then*

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} + 1 \neq 0. \tag{124}$$

Theorem 3.28. *Suppose that* $0 < \alpha < 2$ *and that* $\alpha \in \mathbb{Q}$*, and let* p *and* q *be integers such that* $\alpha = p/q$*, where* p/q *is a reduced fraction and* $q > 0$*. Suppose further that* J *is an integer and that either*

$$\mu = \frac{2n-1}{\alpha} + J, \tag{125}$$

or

$$\mu = \frac{2n}{2-\alpha} + J, \tag{126}$$

for some integer n*. If*

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} + 1 = 0, \tag{127}$$

then it must be the case that

$$J = q \cdot m, \tag{128}$$

for some integer m*.*

Theorem 3.29. *Suppose that* $0 < \alpha < 2$ *and that* $\alpha \in \mathbb{Q}$*, and let* p *and* q *be integers such that* $\alpha = p/q$*, where* p/q *is a reduced fraction and* $q > 0$*. Suppose further that* n *and* J *are integers and that either*

$$\mu = \frac{2n-1}{\alpha} + J, \text{ where } \frac{2n-1}{\alpha} \neq \frac{2j}{2-\alpha} \text{ for each integer } j, \tag{129}$$

or

$$\mu = \frac{2n}{2-\alpha} + J, \text{ where } \frac{2n}{2-\alpha} \neq \frac{2j-1}{\alpha} \text{ for each integer } j. \tag{130}$$

If

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} + 1 = 0, \tag{131}$$

then

$$(1-\alpha)\cos(\pi(1-\alpha)\mu) - \cot(\pi\mu)\sin(\pi(1-\alpha)\mu) \neq 0. \tag{132}$$

Theorem 3.30. *Suppose that* $0 < \alpha < 2$. *Suppose further that*

$$\mu = \frac{2n}{2-\alpha}, \tag{133}$$

where n *is an integer. Then*

$$(1-\alpha)\cos(\pi(1-\alpha)\mu) - \cot(\pi\mu)\sin(\pi(1-\alpha)\mu) = 0, \tag{134}$$

if and only if

$$\frac{2n}{2-\alpha} = \frac{2j-1}{\alpha}, \tag{135}$$

for some integer j.

Theorem 3.31. *Suppose that* $0 < \alpha < 2$ *and that* $\alpha \in \mathbb{Q}$, *and let* p *and* q *be integers such that* $\alpha = p/q$, *where* p/q *is a reduced fraction and* $q > 0$. *Suppose further that* n *and* J *are integers and that*

$$\mu = \frac{2n-1}{\alpha} + J, \text{ where } \frac{2n-1}{\alpha} = \frac{2j}{2-\alpha} \text{ for some integer } j. \tag{136}$$

If

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} + 1 = 0, \tag{137}$$

then

$$(1-\alpha)\cos(\pi(1-\alpha)\mu) - \cot(\pi\mu)\sin(\pi(1-\alpha)\mu) = 0, \tag{138}$$

and

$$(1-\alpha)\cot(\pi\mu)\cos(\pi(1-\alpha)\mu) - \big((\cot(\pi\mu))^2 - \alpha + \frac{\alpha^2}{2}\big)\sin(\pi(1-\alpha)\mu) \neq 0. \tag{139}$$

Theorem 3.32. *Suppose that* $0 < \alpha < 2$. *Then*

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} - 1 = 0, \tag{140}$$

if and only if

$$\mu = \frac{2n}{\alpha}, \tag{141}$$

or

$$\mu = \frac{2n-1}{2-\alpha}, \tag{142}$$

for some integer n.

Theorem 3.33. *Suppose that* $0 < \alpha < 2$ *and that* $\alpha \notin \mathbb{Q}$*. Suppose further that* $J \neq 0$ *is an integer and that either*

$$\mu = \frac{2n}{\alpha} + J, \tag{143}$$

or

$$\mu = \frac{2n-1}{2-\alpha} + J, \tag{144}$$

for some integer n*. Then*

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} - 1 \neq 0. \tag{145}$$

Theorem 3.34. *Suppose that* $0 < \alpha < 2$ *and that* $\alpha \in \mathbb{Q}$*, and let* p *and* q *be integers such that* $\alpha = p/q$*, where* p/q *is a reduced fraction and* $q > 0$*. Suppose further that* J *is an integer and that either*

$$\mu = \frac{2n}{\alpha} + J, \tag{146}$$

or

$$\mu = \frac{2n-1}{2-\alpha} + J, \tag{147}$$

for some integer n*. If*

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} - 1 = 0, \tag{148}$$

then it must be the case that

$$J = q \cdot m, \tag{149}$$

for some integer m*.*

Theorem 3.35. *Suppose that* $0 < \alpha < 2$ *and that* $\alpha \in \mathbb{Q}$*, and let* p *and* q *be integers such that* $\alpha = p/q$*, where* p/q *is a reduced fraction and* $q > 0$*. Suppose further that* n *and* J *are integers and that either*

$$\mu = \frac{2n-1}{2-\alpha} + J, \ \textit{where} \ \frac{2n-1}{2-\alpha} \neq \frac{2j}{\alpha} \ \textit{for each integer} \ j, \tag{150}$$

or

$$\mu = \frac{2n}{\alpha} + J, \ \textit{where} \ \frac{2n}{\alpha} \neq \frac{2j-1}{2-\alpha} \ \textit{for each integer} \ j. \tag{151}$$

If

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} - 1 = 0, \tag{152}$$

then

$$(1-\alpha)\cos(\pi(1-\alpha)\mu) - \cot(\pi\mu)\sin(\pi(1-\alpha)\mu) \neq 0. \tag{153}$$

Theorem 3.36. *Suppose that* $0 < \alpha < 2$. *Suppose further that*

$$\mu = \frac{2n}{\alpha}, \tag{154}$$

where n *is an integer. Then*

$$(1-\alpha)\cos(\pi(1-\alpha)\mu) - \cot(\pi\mu)\sin(\pi(1-\alpha)\mu) = 0, \tag{155}$$

if and only if

$$\frac{2n}{\alpha} = \frac{2j-1}{2-\alpha}, \tag{156}$$

for some integer j.

Theorem 3.37. *Suppose that* $0 < \alpha < 2$ *and that* $\alpha \in \mathbb{Q}$, *and let* p *and* q *be integers such that* $\alpha = p/q$, *where* p/q *is a reduced fraction and* $q > 0$. *Suppose further that* n *and* J *are integers and that*

$$\mu = \frac{2n-1}{2-\alpha} + J, \text{ where } \frac{2n-1}{2-\alpha} = \frac{2j}{\alpha} \text{ for some integer } j. \tag{157}$$

If

$$\frac{\sin(\pi\mu(1-\alpha))}{\sin(\pi\mu)} - 1 = 0, \tag{158}$$

then

$$(1-\alpha)\cos(\pi(1-\alpha)\mu) - \cot(\pi\mu)\sin(\pi(1-\alpha)\mu) = 0, \tag{159}$$

and

$$(1-\alpha)\cot(\pi\mu)\cos(\pi(1-\alpha)\mu) - \Big((\cot(\pi\mu))^2 - \alpha + \frac{\alpha^2}{2}\Big)\sin(\pi(1-\alpha)\mu) \neq 0. \tag{160}$$

4 Analytical Apparatus

4.1 Functions with Branch Points

Suppose that $F(z)$ is an analytic function with a branch point at $z = 0$ and the branch cut $[0, \infty)$. In this section, we characterize the functions F for which

$$F(t) - F(e^{2\pi i}t) = t^{\mu}(\log(t))^n, \tag{161}$$

where $\mu \in \mathbb{R}$ and n is a nonnegative integer. The principal results of this section are theorems 4.1 and 4.2.

Remark 4.1. When a function $F(z)$ has a branch point singularity at $z = 0$, we use the notation $F(e^{i\theta}t)$, where $t > 0$ is real, to denote the value of F along the ray $\arg(z) = \theta$.

Theorem 4.1. *Suppose that $\mu \notin \mathbb{Z}$ is a real number and that $n \geq 0$ is an integer. Then there exists some polynomial P_{n-3} such that if $F\colon \{z \in \mathbb{C} : 0 \leq \arg(z) \leq 2\pi\} \to \mathbb{C}$ is defined by the formula*

$$\begin{aligned} F(z) &= \frac{1}{1-e^{2\pi i\mu}} \cdot z^\mu (\log(z))^n + \frac{2\pi i n\, e^{2\pi i\mu}}{(1-e^{2\pi i\mu})^2} \cdot z^\mu (\log(z))^{n-1} \mathbb{1}_{\{n\geq 1\}} \\ &- \frac{(1+e^{2\pi i\mu})e^{2\pi i\mu} 2\pi^2 n(n-1)}{(1-e^{2\pi i\mu})^3} \cdot z^\mu (\log(z))^{n-2} \mathbb{1}_{\{n\geq 2\}} + z^\mu P_{n-3}(\log(z)), \end{aligned} \tag{162}$$

for all $0 \leq \arg(z) \leq 2\pi$, where z^μ and $\log(z)$ have the branch cut $[0,\infty)$, then

$$F(t) - F(e^{2\pi i} t) = t^\mu (\log(t))^n, \tag{163}$$

for all real $t > 0$.

Proof. Let

$$F(z) = \frac{1}{1-e^{2\pi i\mu}} \cdot z^\mu, \tag{164}$$

for all $z \in \mathbb{C} \setminus [0,\infty)$. Then

$$F(t) - F(e^{2\pi i} t) = t^\mu, \tag{165}$$

for all real $t > 0$. Thus, Theorem 4.1 is true for $n = 0$.

Now suppose that $k \geq 1$ is an integer and that Theorem 4.1 is true for $n = 0, 1, \ldots, k-1$. We will show that Theorem 4.1 is true for $n = k$. Let $G\colon \mathbb{C} \setminus [0,\infty) \to \mathbb{C}$ be defined by

$$G(z) = \frac{1}{1-e^{2\pi i\mu}} \cdot z^\mu (\log(z))^k, \tag{166}$$

for all $0 \leq \arg(z) \leq 2\pi$. Then

$$\begin{aligned} G(t) - G(e^{2\pi i} t) &= \frac{1}{1-e^{2\pi i\mu}} \big(t^\mu (\log(t))^k - e^{2\pi i\mu} \cdot t^\mu (\log(t) + 2\pi i)^k \big) \\ &= \frac{1}{1-e^{2\pi i\mu}} \left(t^\mu (\log(t))^k - e^{2\pi i\mu} \cdot t^\mu \sum_{j=0}^{k} \binom{k}{j} (\log(t))^j (2\pi i)^{k-j} \right) \\ &= t^\mu (\log(t))^k - \frac{e^{2\pi i\mu}}{1-e^{2\pi i\mu}} \cdot t^\mu \sum_{j=0}^{k-1} \binom{k}{j} (\log(t))^j (2\pi i)^{k-j} \\ &= t^\mu (\log(t))^k - \frac{e^{2\pi i\mu}}{1-e^{2\pi i\mu}} 2\pi i k \cdot t^\mu (\log(t))^{k-1} \mathbb{1}_{\{k\geq 1\}} \\ &+ \frac{e^{2\pi i\mu}}{1-e^{2\pi i\mu}} 2\pi^2 k(k-1) \cdot t^\mu (\log(t))^{k-2} \mathbb{1}_{\{k\geq 2\}} \\ &- \frac{e^{2\pi i\mu}}{1-e^{2\pi i\mu}} \cdot t^\mu \sum_{j=0}^{k-3} \binom{k}{j} (\log(t))^j (2\pi i)^{k-j}, \end{aligned} \tag{167}$$

for all real $t > 0$. By the induction hypothesis, there exist functions $F_0, F_1, \ldots, F_{k-1} \colon \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ defined by

$$F_j(z) = \frac{1}{1 - e^{2\pi i\mu}} \cdot z^\mu (\log(z))^j + \frac{2\pi i j\ e^{2\pi i\mu}}{(1 - e^{2\pi i\mu})^2} \cdot z^\mu (\log(z))^{j-1} \mathbb{1}_{\{j \geq 1\}} + z^\mu P_{j-2}(\log(z)), \tag{168}$$

for all $0 \leq \arg(z) \leq 2\pi$, where P_{j-2} is a polynomial of order $j - 2$, such that

$$F_j(t) - F_j(e^{2\pi i} t) = t^\mu (\log(t))^j, \tag{169}$$

for all real $t > 0$, for all $j = 0, 1, \ldots, k - 1$. Suppose now that F is defined by

$$\begin{aligned} F(z) = G(z) &+ \frac{e^{2\pi i\mu}}{1 - e^{2\pi i\mu}} 2\pi i k \cdot F_{k-1}(z) \mathbb{1}_{\{k \geq 1\}} \\ &- \frac{e^{2\pi i\mu}}{1 - e^{2\pi i\mu}} 2\pi^2 k(k-1) \cdot F_{k-2}(z) \mathbb{1}_{\{k \geq 2\}} \\ &+ \frac{e^{2\pi i\mu}}{1 - e^{2\pi i\mu}} \cdot \sum_{j=0}^{k-3} \binom{k}{j} (2\pi i)^{k-j} F_j(z), \end{aligned} \tag{170}$$

for all $0 \leq \arg(z) \leq 2\pi$. Combining (167) and (169), we observe that

$$F(t) - F(e^{2\pi i} t) = t^\mu (\log(t))^k, \tag{171}$$

for all real $t > 0$. Moreover, combining (170), (166), and (168), we see that

$$\begin{aligned} F(z) = &\frac{1}{1 - e^{2\pi i\mu}} \cdot z^\mu (\log(z))^k + \frac{2\pi i k\ e^{2\pi i\mu}}{(1 - e^{2\pi i\mu})^2} \cdot z^\mu (\log(z))^{k-1} \mathbb{1}_{\{k \geq 1\}} \\ &- \frac{(1 + e^{2\pi i\mu}) e^{2\pi i\mu} 2\pi^2 k(k-1)}{(1 - e^{2\pi i\mu})^3} \cdot z^\mu (\log(z))^{k-2} \mathbb{1}_{\{k \geq 2\}} + z^\mu P_{k-3}(\log(z)), \end{aligned} \tag{172}$$

where P_{k-3} is a polynomial of order $k - 3$. Thus, Theorem 4.1 is true for $n = k$.

■

Remark 4.2. The polynomial P_{n-3} in formula (162) of Theorem 4.1 can be computed explicitly. Specifically, the induction proof of Theorem 4.1 defines the polynomials $P_0, P_1, \ldots, P_{n-3}$ recursively—a straightforward calculation yields a recurrence relation for their coefficients.

Theorem 4.2. *Suppose that m and n are nonnegative integers. Then there exists some polynomial P_{n-1} such that if F is defined by the formula*

$$F(z) = -\frac{1}{2\pi i(n+1)} \cdot z^m (\log(z))^{n+1} + \frac{1}{2} \cdot z^m (\log(z))^n + z^m P_{n-1}(\log(z)), \tag{173}$$

for all $0 \leq \arg(z) \leq 2\pi$, where $\log(z)$ has the cut $[0, \infty)$, then

$$F(t) - F(e^{2\pi i} t) = t^m (\log(t))^n, \tag{174}$$

for all real $t > 0$.

Proof. Let

$$F(z) = -\frac{1}{2\pi i} \cdot z^m \log(z), \tag{175}$$

for all $0 \le \arg(z) \le 2\pi$. Then

$$F(t) - F(e^{2\pi i} t) = t^m, \tag{176}$$

for all real $t > 0$. Thus, Theorem 4.2 is true for $n = 0$.

Now suppose that $k \ge 1$ is an integer and that Theorem 4.2 is true for $n = 0, 1, \dots, k-1$. We will show that Theorem 4.2 is true for $n = k$. Let $G\colon \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ be defined by

$$G(z) = -\frac{1}{2\pi i(k+1)} \cdot z^m (\log(z))^{k+1}, \tag{177}$$

for all $0 \le \arg(z) \le 2\pi$. Then

$$\begin{aligned}
G(t) - G(e^{2\pi i} t) &= -\frac{1}{2\pi i(k+1)} \left(t^m (\log(t))^{k+1} - t^m (\log(t) + 2\pi i)^{k+1} \right) \\
&= -\frac{1}{2\pi i(k+1)} \left(-2\pi i(k+1) \cdot t^m (\log(t))^k - t^m \sum_{j=0}^{k-1} \binom{k+1}{j} (\log(t))^j (2\pi i)^{k+1-j} \right) \\
&= t^m (\log(t))^k + \frac{1}{2\pi i(k+1)} \cdot t^m \sum_{j=0}^{k-1} \binom{k+1}{j} (\log(t))^j (2\pi i)^{k+1-j} \\
&= t^m (\log(t))^k + \pi i k \cdot t^m (\log(t))^{k-1} + \frac{1}{2\pi i(k+1)} \cdot t^m \sum_{j=0}^{k-2} \binom{k+1}{j} (\log(t))^j (2\pi i)^{k+1-j},
\end{aligned} \tag{178}$$

for all real $t > 0$. By the induction hypothesis, there exist functions $F_0, F_1, \dots, F_{k-1}\colon \mathbb{C} \setminus [0, \infty) \to \mathbb{C}$ defined by

$$F_j(z) = -\frac{1}{2\pi i(j+1)} \cdot z^m (\log(z))^{j+1} + z^m P_j(\log(z)), \tag{179}$$

for all $0 \le \arg(z) \le 2\pi$, where P_j is a polynomial of order j, such that

$$F_j(t) - F_j(e^{2\pi i} t) = t^m (\log(t))^j, \tag{180}$$

for all real $t > 0$, for all $j = 0, 1, \dots, k-1$. Suppose now that F is defined by

$$F(z) = G(z) - \pi i k \cdot F_{k-1}(z) - \frac{1}{2\pi i(k+1)} \cdot \sum_{j=0}^{k-2} \binom{k+1}{j} (2\pi i)^{k+1-j} F_j(z), \tag{181}$$

for all $0 \le \arg(z) \le 2\pi$. Combining (178) and (180), we observe that

$$F(t) - F(e^{2\pi i} t) = t^m (\log(t))^k, \tag{182}$$

for all real $t > 0$. Moreover, combining (181), (177), and (179), we see that

$$F(z) = -\frac{1}{2\pi i(k+1)} \cdot z^m (\log(z))^{k+1} + \frac{1}{2} \cdot z^m (\log(z))^k + z^m P_{k-1}(\log(z)), \tag{183}$$

where P_{k-1} is a polynomial of order $k-1$. Thus, Theorem 4.2 is true for $n = k$.

■

Remark 4.3. Like in Theorem 4.1, the polynomial P_{n-1} in formula (173) of Theorem 4.2 can be computed explicitly. Specifically, the induction proof of Theorem 4.2 defines the polynomials $P_0, P_1, \ldots, P_{n-1}$ recursively—a straightforward calculation yields a recurrence relation for their coefficients.

4.2 Integrals Involving the Cauchy Kernel

In this section, we show that the integral of the Cauchy kernel multiplied by $t^\mu (\log(t))^n$ is given by an explicit analytic formula. The principal results of this section are theorems 4.3 and 4.4.

The following theorem evaluates the integral of the Cauchy kernel multiplied by $t^\mu (\log(t))^n$, when $\mu \notin \mathbb{Z}$.

Theorem 4.3. *Suppose that n is a positive integer and that $\mu > -1$ is a real number such that $\mu \notin \mathbb{Z}$. Suppose further that $C \subset \mathbb{C}$ is a simple closed contour, and let U denote the interior of C; suppose also that $[0, 1) \subset U$ and that $1 \in C$ (see Figure 6). Then*

$$\begin{aligned}\int_0^1 \frac{1}{t-z} t^\mu (\log(t))^n \, dt &= \frac{2\pi i}{1 - e^{2\pi i\mu}} \cdot z^\mu (\log(z))^n - \frac{4\pi^2 n \, e^{2\pi i \mu}}{(1 - e^{2\pi i\mu})^2} \cdot z^\mu (\log(z))^{n-1} \mathbb{1}_{\{n \geq 1\}} \\ &- \frac{(1 + e^{2\pi i\mu}) e^{2\pi i\mu} \cdot 4\pi^3 i n(n-1)}{(1 - e^{2\pi i\mu})^3} \cdot z^\mu (\log(z))^{n-2} \mathbb{1}_{\{n \geq 2\}} + z^\mu P_{n-3}(\log(z)) + \varphi(z),\end{aligned} \tag{184}$$

for all $z \in U \setminus [0, 1]$, where z^μ and $\log(z)$ have the cut $[0, \infty)$, P_{n-3} is a polynomial of degree $n-3$, and $\varphi \colon U \to \mathbb{C}$ is an analytic function.

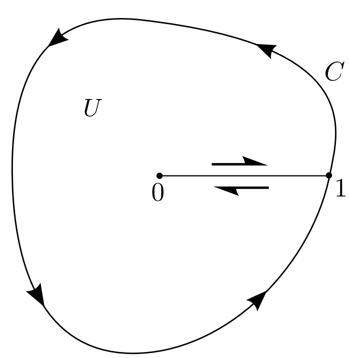

Figure 6: A contour in $\mathbb{C}$

Proof. By Theorem 4.1, there exists some polynomial P_{n-3} such that if $F\colon \mathbb{C}\setminus[0,\infty)\to\mathbb{C}$ is defined by the formula

$$\begin{aligned} F(z) &= \frac{1}{1-e^{2\pi i\mu}}\cdot z^{\mu}(\log(z))^{n} + \frac{2\pi in\,e^{2\pi i\mu}}{(1-e^{2\pi i\mu})^2}\cdot z^{\mu}(\log(z))^{n-1}\mathbb{1}_{\{n\geq 1\}} \\ &- \frac{(1+e^{2\pi i\mu})e^{2\pi i\mu}2\pi^2 n(n-1)}{(1-e^{2\pi i\mu})^3}\cdot z^{\mu}(\log(z))^{n-2}\mathbb{1}_{\{n\geq 2\}} + z^{\mu}P_{n-3}(\log(z)), \end{aligned} \tag{185}$$

for all $0\leq \arg(z)\leq 2\pi$, where z^{μ} and $\log(z)$ have the branch cut $[0,\infty)$, then

$$F(t)-F(e^{2\pi i}t) = t^{\mu}(\log(t))^{n}, \tag{186}$$

for all real $t>0$. Suppose that F is defined by (185). Suppose further that $\widetilde{C}$ is a closed contour defined by the contour C together with the intervals $[0,1]$ and $e^{2\pi i}[0,1]$, oriented in the counter-clockwise direction (see Figure 6). Since F is analytic on $\mathbb{C}\setminus[0,\infty)$, by Cauchy's theorem

$$\frac{1}{2\pi i}\oint_{\widetilde{C}}\frac{1}{w-z}F(w)\,dw = F(z), \tag{187}$$

for all $z\in U\setminus[0,1]$. We observe that

$$\oint_{\widetilde{C}}\frac{1}{w-z}F(w)\,dw = \int_C\frac{1}{w-z}F(w)\,dw + \int_0^1\frac{1}{t-z}(F(t)-F(e^{2\pi i}t))\,dt, \tag{188}$$

for all $z\in U\setminus[0,1]$. Combining (188) and (186),

$$\oint_{\widetilde{C}}\frac{1}{w-z}F(w)\,dw = \int_C\frac{1}{w-z}F(w)\,dw + \int_0^1\frac{1}{t-z}t^{\mu}(\log(t))^{n}\,dt, \tag{189}$$

for all $z\in U\setminus[0,1]$. Also,

$$\int_C\frac{1}{w-z}F(w)\,dw \tag{190}$$

is clearly analytic for all $z\in U$. Therefore, combining (189), (187), and (185), we have

$$\begin{aligned} \int_0^1\frac{1}{t-z}t^{\mu}(\log(t))^{n}\,dt &= \frac{2\pi i}{1-e^{2\pi i\mu}}\cdot z^{\mu}(\log(z))^{n} - \frac{4\pi^2 n\,e^{2\pi i\mu}}{(1-e^{2\pi i\mu})^2}\cdot z^{\mu}(\log(z))^{n-1}\mathbb{1}_{\{n\geq 1\}} \\ &- \frac{(1+e^{2\pi i\mu})e^{2\pi i\mu}\cdot 4\pi^3 in(n-1)}{(1-e^{2\pi i\mu})^3}\cdot z^{\mu}(\log(z))^{n-2}\mathbb{1}_{\{n\geq 2\}} + z^{\mu}P_{n-3}(\log(z)) + \varphi(z), \end{aligned} \tag{191}$$

for all $z\in D\setminus[0,1]$, where $\varphi\colon U\to\mathbb{C}$ is an analytic function.

■

The following theorem evaluates the integral of the Cauchy kernel multiplied by $t^m(\log(t))^n$, where $m\in\mathbb{Z}$. The proof is essentially identical to the proof of Theorem 4.3, except that it uses Theorem 4.2 instead of Theorem 4.1.

Theorem 4.4. *Suppose that both n and m are nonnegative integers. Suppose further that $C \subset \mathbb{C}$ is a simple closed contour, and let U denote the interior of C; suppose also that $[0,1) \subset U$ and that $1 \in C$ (see Figure 6). Then*

$$\int_0^1 \frac{1}{t-z} t^m (\log(t))^n \, dt = -\frac{1}{n+1} \cdot z^m (\log(z))^{n+1} + \pi i \cdot z^m (\log(z))^n \\ + z^m P_{n-1}(\log(z)) + \varphi(z), \tag{192}$$

for all $z \in U \setminus [0,1]$, where $\log(z)$ has the cut $[0,\infty)$, P_{n-1} is a polynomial of degree $n-1$, and $\varphi \colon U \to \mathbb{C}$ is an analytic function.

4.3 Two Technical Lemmas

In this section, we prove two elementary technical lemmas involving functions of complex variables.

Lemma 4.5. *Suppose that $0 \in U_1, U_2 \subset \mathbb{C}$ are both open simply connected sets, and that $F_1 \colon U_1 \to V$ and $F_2 \colon U_2 \to V$ are conformal mappings, where $V \subset \mathbb{C}$. Suppose further that $F_1'(0) \neq 0$, $F_2'(0) \neq 0$, and that $F_1'(0)/F_2'(0) = e^{i\pi\alpha}$, where $0 < \alpha < 2$ is a real number. Finally, suppose that $F_1^{(n)}(0) = 0$, $F_2^{(n)}(0) = 0$ for $n = 2, 3, \dots, k+1$, where k is a nonnegative integer (if $k = 0$ assume that possibly $F_1''(0) \neq 0$ or $F_2''(0) \neq 0$). Then*

$$F_2^{-1} \circ F_1(z) = e^{i\pi\alpha} z + z^{k+2} \varphi(z), \tag{193}$$

for all $z \in U_1$, where $\varphi \colon U_1 \to \mathbb{C}$ is an analytic function.

Proof. This lemma follows immediately from the Lagrange inversion theorem.

■

Lemma 4.6. *Suppose that $0 \in U \subset \mathbb{C}$ is an open simply connected set and that $\varphi \colon U \to \mathbb{C}$ is an analytic function. Let $F : U \to \mathbb{C}$ be defined by*

$$F(z) = e^{i\pi\alpha} z + z^{k+2} \varphi(z), \tag{194}$$

for all $z \in U$, where k is a nonnegative integer and $0 < \alpha < 2$ is a real number. Suppose further that $G \colon \mathbb{C} \setminus [0,\infty) \to \mathbb{C}$ is defined by

$$G(z) = z^\mu (\log(z))^n, \tag{195}$$

for all $z \in \mathbb{C} \setminus [0,\infty)$, where n is a nonnegative integer, $\mu > -1$ is a real number, and both z^μ and $\log(z)$ have the branch cut $[0,\infty)$. Then, for each positive integer N,

$$G \circ F(z) = e^{i\pi\alpha\mu} z^\mu (\log(z))^n + e^{i\pi\alpha\mu} \cdot i\pi\alpha n \cdot z^\mu (\log(z))^{n-1} \mathbb{1}_{\{n \geq 1\}} \\ - e^{i\pi\alpha\mu} \frac{\pi^2 \alpha^2 n(n-1)}{2} \cdot z^\mu (\log(z))^{n-2} \mathbb{1}_{\{n \geq 2\}} + z^\mu P_{n-3}(\log(z)) \\ + z^{\mu+k+1} Q_{(N-\mu-k-1,n)}(z, \log(z)) + o(z^N), \tag{196}$$

for all $z \in U \setminus e^{-i\pi\alpha}[0,\infty)$, where z^μ and $\log(z)$ have the branch cut $e^{-i\pi\alpha}[0,\infty)$, and both $P \colon \mathbb{C} \to \mathbb{C}$ and $Q \colon \mathbb{C}^2 \to \mathbb{C}$ are polynomials. Furthermore, P depends only on n, α, and μ.

Proof. We observe that

$$\begin{aligned} G\circ F(z) &= (e^{i\pi\alpha}z + z^{k+2}\varphi(z))^{\mu}\cdot\big(\log(e^{i\pi\alpha}z + z^{k+2}\varphi(z))\big)^n \\ &= (e^{i\pi\alpha}z)^{\mu}\cdot(1+e^{-i\pi\alpha}z^{k+1}\varphi(z))^{\mu}\cdot\big(\log(e^{i\pi\alpha}z)+\log(1+e^{-i\pi\alpha}z^{k+1}\varphi(z))\big)^n, \end{aligned} \tag{197}$$

for all $z\in U\setminus e^{-i\pi\alpha}[0,\infty)$. Clearly (for example, by the chain rule), there exists an open simply connected set $0\in U'\subset U$ and analytic functions $\varphi_2,\varphi_3\colon U'\to\mathbb{C}$ such that

$$1+z^{k+1}\varphi_2(z) = (1+e^{-i\pi\alpha}z^{k+1}\varphi(z))^{\mu}, \tag{198}$$

$$z^{k+1}\varphi_3(z) = \log(1+e^{-i\pi\alpha}z^{k+1}\varphi(z)), \tag{199}$$

for all $z\in U'$. Thus,

$$\begin{aligned} G\circ F(z) &= (e^{i\pi\alpha}z)^{\mu}\cdot(1+z^{k+1}\varphi_2(z))\cdot\big(\log(e^{i\pi\alpha}z)+z^{k+1}\varphi_3(z)\big)^n \\ &= (e^{i\pi\alpha}z)^{\mu}\cdot(1+z^{k+1}\varphi_2(z))\cdot\Big((\log(e^{i\pi\alpha}z))^n+\sum_{j=0}^{n-1}\binom{n}{j}(\log(e^{i\pi\alpha}z))^j(z^{k+1}\varphi_3(z))^{n-j}\Big) \\ &= (e^{i\pi\alpha}z)^{\mu}\cdot(1+z^{k+1}\varphi_2(z))\cdot\Big((\log(e^{i\pi\alpha}z))^n \\ &\quad + z^{k+1}\varphi_3(z)\sum_{j=0}^{n-1}\binom{n}{j}(\log(e^{i\pi\alpha}z))^j(z^{k+1}\varphi_3(z))^{n-j-1}\Big) \\ &= (e^{i\pi\alpha}z)^{\mu}(\log(e^{i\pi\alpha}z))^n + z^{\mu+k+1}Q_{(N-\mu-k-1,n)}(z,\log(z))+o(z^N), \end{aligned} \tag{200}$$

for all $z\in U'\setminus e^{-i\pi\alpha}[0,\infty)$, where $Q\colon\mathbb{C}^2\to\mathbb{C}$ is a polynomial. Clearly, identity (200) extends to $z\in U\setminus e^{-i\pi\alpha}[0,\infty)$, so

$$G\circ F(z) = (e^{i\pi\alpha}z)^{\mu}(\log(e^{i\pi\alpha}z))^n + z^{\mu+k+1}Q_{(N-\mu-k-1,n)}(z,\log(z))+o(z^N), \tag{201}$$

for all $z\in U\setminus e^{-i\pi\alpha}[0,\infty)$. Thus,

$$\begin{aligned} G\circ F(z) &= e^{i\pi\alpha\mu}z^{\mu}(\log(z))^n + e^{i\pi\alpha\mu}\cdot i\pi\alpha n\cdot z^{\mu}(\log(z))^{n-1}\mathbb{1}_{\{n\ge1\}} \\ &- e^{i\pi\alpha\mu}\frac{\pi^2\alpha^2 n(n-1)}{2}\cdot z^{\mu}(\log(z))^{n-2}\mathbb{1}_{\{n\ge2\}} + z^{\mu}P_{n-3}(\log(z)) \\ &+ z^{\mu+k+1}Q_{(N-\mu-k-1,n)}(z,\log(z))+o(z^N), \end{aligned} \tag{202}$$

for all $z\in U\setminus e^{-i\pi\alpha}[0,\infty)$, where z^{μ} and $\log(z)$ now have the cut $e^{-i\pi\alpha}[0,\infty)$, and P_{n-3} is a polynomial of degree $n-3$. Furthermore, P clearly depends only on n, α, and μ. ■

5 Analysis of the Integral Equation

In this section, we assume the following. Suppose that $\gamma\colon[-1,1]\to\mathbb{C}$ is a curve in $\mathbb{C}$, and let $\gamma_1\colon[0,1]\to\mathbb{C}$ denote $\gamma(-t)$ for $0\le t\le1$ and $\gamma_2\colon[0,1]\to\mathbb{C}$ denote $\gamma(t)$ for $0\le t\le1$. Suppose that γ_1 and γ_2 are analytic curves parameterized by arc length and meeting at a corner at $\gamma(0)$ with interior angle $\pi\alpha$ (see Figure 7). Suppose that M is

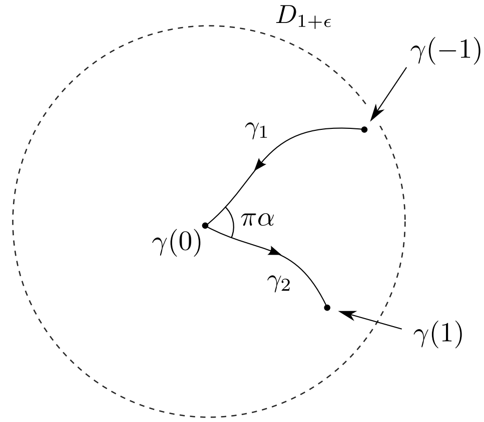

Figure 7: Two curves meeting at a corner in $\mathbb{R}^2$

a nonnegative integer and that the signed curvatures $\kappa_1, \kappa_2 \colon [0,1] \to \mathbb{R}$ of γ_1 and γ_2, respectively, are given by the formulas

$$\kappa_1(t) = a_{1,0} + a_{1,1}t + \cdots + a_{1,M}t^M, \tag{203}$$

$$\kappa_2(t) = a_{2,0} + a_{2,1}t + \cdots + a_{2,M}t^M, \tag{204}$$

for $0 \le t \le 1$, where $a_{1,i}$ and $a_{2,i}$ are real numbers. Suppose further that $\epsilon > 0$, and that $\delta(\alpha) > 0$ is sufficiently small so that, if $(a_{1,0}, a_{1,1}, \ldots, a_{1,M})$ and $(a_{2,0}, a_{2,1}, \ldots, a_{2,M})$ are in $B_{\delta(\alpha)} \subset \mathbb{R}^{M+1}$, then the conditions of Theorem 3.24 for ϵ are satisfied, and the curves γ_1 and γ_2 do not intersect (see Theorem 3.22). Then there exist open, simply connected sets $[0,1] \subset U_1, U_2 \subset \mathbb{C}$ and conformal mappings $F_1 \colon U_1 \to D_{1+\epsilon}$, $F_2 \colon U_2 \to D_{1+\epsilon}$, where $D_{1+\epsilon}$ is the unit disc of radius $1+\epsilon$, such that $F_1|_{[0,1]} = \gamma_1$ and $F_2|_{[0,1]} = \gamma_2$ (see Figure 7). Finally, suppose that k is a nonnegative integer such that $\kappa_1^{(n)}(0) = 0$ and $\kappa_2^{(n)}(0) = 0$ for $n = 0, 1, \ldots, k-1$ (see Theorem 3.21).

In Section 5.1, we describe the principal apparatus which we will use to analyze the integral equation (52). In Section 5.2, we prove the forward direction of Theorem 2.1. In Section 5.3, we prove the converse direction of Theorem 2.1. The proof of Theorem 2.2 is essentially identical and is omitted.

5.1 Principal Analytical Apparatus

In this section, we derive explicit formulas for integrals of the form

$$\int_{-1}^{1} \psi^1_{\gamma(t),\nu(t)}(\gamma(s)) \cdot |t|^{\mu} (\log|t|)^n \, dt, \tag{205}$$

$$\int_{-1}^{1} \psi^1_{\gamma(t),\nu(t)}(\gamma(s)) \cdot \operatorname{sgn}(t)|t|^{\mu} (\log|t|)^n \, dt, \tag{206}$$

where $\mu > -1$ is real and n is a nonnegative integer. The principal results of this section are theorems 5.7 and 5.8.

The following lemma and corollary show that the self interaction on each side of the corner is smooth.

Lemma 5.1. *Suppose that $n \geq 0$ is an integer and that $\mu > -1$ is a real number. Then*

$$\text{p.v.} \int_0^1 \frac{F_1'(t)}{F_1(s) - F_1(t)} t^\mu (\log(t))^n \, dt = \text{p.v.} \int_0^1 \frac{1}{t-s} t^\mu (\log(t))^n \, dt + \varphi(s), \tag{207}$$

for all $0 < s < 1$, where $\varphi \colon U_1 \to \mathbb{C}$ is analytic.

Proof. Identity (207) follows immediately from Lemma 3.14.

■

Corollary 5.2. *Suppose that $n \geq 0$ is an integer and that $\mu > -1$ is a real number. Then*

$$\int_0^1 \psi^1_{\gamma_1(t),\nu_1(t)}(\gamma_1(s)) t^\mu (\log(t))^n \, dt = \varphi(s), \tag{208}$$

for all $0 < s < 1$, where $\varphi \colon U_1 \to \mathbb{R}$ is real analytic.

Proof. Identity (208) follows by taking the imaginary part of (207) and applying (69). The principal value then disappears because the integrand is smooth (see Theorem 3.5).

■

The following lemma and corollary show that when the density of the double layer potential on one side of the corner is equal to $t^\mu(\log(t))^n$, where $\mu \notin \mathbb{Z}$ and n is a nonnegative integer, the potential induced on the other side of the corner takes a certain explicit form.

Lemma 5.3. *Suppose that $n \geq 0$ is an integer and that $\mu > -1$ is a real number such that $\mu \notin \mathbb{Z}$. Then, for each positive integer N,*

$$\begin{aligned}
\int_0^1 \frac{F_2'(t)}{F_1(s) - F_2(t)} t^\mu (\log(t))^n \, dt &= \frac{\pi e^{i\pi(\alpha-1)\mu}}{\sin(\pi\mu)} s^\mu (\log(s))^n \\
&- \Big(-\frac{i\alpha\pi^2 n \cdot e^{i\pi(\alpha-1)\mu}}{\sin(\pi\mu)} + \frac{\pi^2 n \cdot e^{i\pi\alpha\mu}}{(\sin(\pi\mu))^2} \Big) s^\mu (\log(s))^{n-1} \mathbb{1}_{\{n\geq 1\}} \\
&- \Big(\frac{\alpha^2\pi^3 n(n-1) \cdot e^{i\pi(\alpha-1)\mu}}{2\sin(\pi\mu)} + \frac{i\alpha\pi^3 n(n-1) \cdot e^{i\pi\alpha\mu}}{(\sin(\pi\mu))^2} \\
&+ \frac{\pi^3 n(n-1)\cos(\pi\mu) \cdot e^{i\pi\alpha\mu}}{(\sin(\pi\mu))^3} \Big) s^\mu (\log(s))^{n-2} \mathbb{1}_{\{n\geq 2\}} + s^\mu P_{n-3}(\log(s)) \\
&+ s^{\mu+k+1} Q_{(N-\mu-k-1,n)}(s, \log(s)) + R_N(s) + o(s^N)
\end{aligned} \tag{209}$$

for all $0 < s < 1$, where $P \colon \mathbb{C} \to \mathbb{C}$, $Q \colon \mathbb{C}^2 \to \mathbb{C}$, and $R \colon \mathbb{C} \to \mathbb{C}$ are polynomials. Furthermore, the polynomial P depends only on n, α, and μ.

Proof. We observe that

$$\int_0^1 \frac{F_2'(t)}{F_1(s) - F_2(t)} t^\mu (\log(t))^n \, dt = \int_0^1 \frac{F_2'(t)}{F_2(F_2^{-1} \circ F_1(s)) - F_2(t)} t^\mu (\log(t))^n \, dt, \tag{210}$$

for all $0 < s < 1$. Applying Theorem 3.14 to (210),

$$\int_0^1 \frac{F_2'(t)}{F_1(s) - F_2(t)} t^{\mu} (\log(t))^n \, dt = \int_0^1 \frac{1}{F_2^{-1} \circ F_1(s) - t} t^{\mu} (\log(t))^n \, dt + \varphi(s), \tag{211}$$

where $\varphi \colon U_1 \to \mathbb{C}$ is an analytic function. Combining (211) with Theorem 4.3 and lemmas 4.5 and 4.6, we obtain (209).

■

Corollary 5.4. *Suppose that $n \geq 0$ is an integer and that $\mu > -1$ is a real number such that $\mu \notin \mathbb{Z}$. Then, for each positive integer N,*

$$\begin{aligned}
&\int_0^1 \psi^1_{\gamma_2(t),\nu_2(t)}(\gamma_1(s)) t^{\mu} (\log(t))^n \, dt = -\pi \frac{\sin(\pi(1-\alpha)\mu)}{\sin(\pi\mu)} s^{\mu} (\log(s))^n \\
&- \big((1-\alpha)\cos(\pi(1-\alpha)\mu) - \cot(\pi\mu)\sin(\pi(1-\alpha)\mu)\big) \frac{\pi^2 n}{\sin(\pi\mu)} \cdot s^{\mu} (\log(s))^{n-1} \mathbb{1}_{\{n\geq 1\}} \\
&- \Big((1-\alpha)\cot(\pi\mu)\cos(\pi(1-\alpha)\mu) - \big((\cot(\pi\mu))^2 - \alpha + \frac{\alpha^2}{2}\big)\sin(\pi(1-\alpha)\mu)\Big) \\
&\cdot \frac{\pi^3 n(n-1)}{\sin(\mu\pi)} \cdot s^{\mu} (\log(s))^{n-2} \mathbb{1}_{\{n\geq 2\}} + s^{\mu} P_{n-3}(\log(s)) \\
&+ s^{\mu+k+1} Q_{(N-\mu-k-1,n)}(s, \log(s)) + R_N(s) + o(s^N),
\end{aligned} \tag{212}$$

for all $0 < s < 1$, where $P \colon \mathbb{R} \to \mathbb{R}$, $Q \colon \mathbb{R}^2 \to \mathbb{R}$, and $R \colon \mathbb{R} \to \mathbb{R}$ are polynomials. Furthermore, the polynomial P depends only on n, α, and μ.

Proof. Identity (212) follows by taking the imaginary part of (209) and applying (69).

■

Similarly, the following lemma and corollary show that when the density of the double layer potential on one side of the corner is equal to $t^m(\log(t))^n$, where m and n are nonnegative integers, the potential induced on the other side of the corner takes a certain explicit form. The proofs are essentially identical to the ones above.

Lemma 5.5. *Suppose that m and n are nonnegative integers. Then, for each positive integer N,*

$$\begin{aligned}
&\int_0^1 \frac{F_2'(t)}{F_1(s) - F_2(t)} t^m (\log(t))^n \, dt = \frac{e^{i\pi\alpha m}}{n+1} s^m (\log(s))^{n+1} \\
&+ \Big(\frac{i\pi\alpha(n+1)\cdot e^{i\pi\alpha m}}{n+1} - \pi i \cdot e^{i\pi\alpha m}\Big) s^m (\log(s))^n + s^m P_{n-1}(\log(s)) \\
&+ s^{m+k+1} Q_{(N-m-k-1,n+1)}(s, \log(s)) + R_N(s) + o(s^N)
\end{aligned} \tag{213}$$

for all $0 < s < 1$, where $P \colon \mathbb{C} \to \mathbb{C}$, $Q \colon \mathbb{C}^2 \to \mathbb{C}$, and $R \colon \mathbb{C} \to \mathbb{C}$ are polynomials. Furthermore, the polynomial P depends only on n, α, and m.

Corollary 5.6. *Suppose that m and n are nonnegative integers. Then, for each positive integer N,*

$$\begin{aligned}
&\int_0^1 \psi^1_{\gamma_2(t),\nu_2(t)}(\gamma_1(s))t^m(\log(t))^n\,dt = \frac{\sin(\pi\alpha m)}{n+1}s^m(\log(s))^{n+1}\\
&-\pi(1-\alpha)\cos(\pi\alpha m)s^m(\log(s))^n + s^m P_{n-1}(\log(s))\\
&+s^{m+k+1}Q_{(N-m-k-1,n+1)}(s,\log(s)) + R_N(s) + o(s^N),
\end{aligned} \tag{214}$$

for all $0 < s < 1$, where $P\colon \mathbb{R}\to\mathbb{R}$, $Q\colon \mathbb{R}^2\to\mathbb{R}$, and $R\colon \mathbb{R}\to\mathbb{R}$ are polynomials. Furthermore, the polynomial P depends only on n, α, and m.

The following two theorems are the principal results of this section. They follow immediately from corollaries 5.2, 5.4, and 5.6.

Theorem 5.7. *Suppose that $n \ge 0$ is an integer, $\mu > -1$ is a real number such that $\mu \notin \mathbb{Z}$, and $\sigma \in \{0,1\}$. Then, for each positive integer N,*

$$\begin{aligned}
&\int_{-1}^1 \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\cdot(\operatorname{sgn}(t))^\sigma|t|^\mu(\log|t|)^n\,dt\\
&= -\pi\frac{\sin(\pi(1-\alpha)\mu)}{\sin(\pi\mu)}(-\operatorname{sgn}(s))^\sigma|s|^\mu(\log|s|)^n + \big((1-\alpha)\cos(\pi(1-\alpha)\mu)\\
&+\cot(\pi\mu)\sin(\pi(1-\alpha)\mu)\big)\frac{\pi^2 n}{\sin(\pi\mu)}\cdot(-\operatorname{sgn}(s))^\sigma|s|^\mu(\log|s|)^{n-1}\mathbb{1}_{\{n\ge1\}}\\
&-\Big((1-\alpha)\cot(\pi\mu)\cos(\pi(1-\alpha)\mu) - \big((\cot(\pi\mu))^2-\alpha+\frac{\alpha^2}{2}\big)\sin(\pi(1-\alpha)\mu)\Big)\\
&\cdot\frac{\pi^3 n(n-1)}{\sin(\mu\pi)}\cdot(-\operatorname{sgn}(s))^\sigma|s|^\mu(\log|s|)^{n-2}\mathbb{1}_{\{n\ge2\}} + (-\operatorname{sgn}(s))^\sigma|s|^\mu P_{n-3}(\log|s|)\\
&+|s|^{\mu+k+1}Q_{(N-\mu-k-1,n,1)}(|s|,\log|s|,\operatorname{sgn}(s)) + R_{(N,1)}(|s|,\operatorname{sgn}(s)) + o(|s|^N),
\end{aligned} \tag{215}$$

for all $0 < s < 1$, where $P\colon \mathbb{R}\to\mathbb{R}$, $Q\colon \mathbb{R}^3\to\mathbb{R}$, and $R\colon \mathbb{R}^2\to\mathbb{R}$ are polynomials. Furthermore, the polynomial P depends only on n, α, and μ.

Theorem 5.8. *Suppose that m and n are nonnegative integers, and that $\sigma \in \{0,1\}$. Then, for each positive integer N,*

$$\begin{aligned}
&\int_{-1}^1 \psi^1_{\gamma(t),\nu(t)}(\gamma(s))\cdot(\operatorname{sgn}(t))^\sigma|t|^m(\log|t|)^n\,dt\\
&= \frac{\sin(\pi\alpha m)}{n+1}(-\operatorname{sgn}(s))^\sigma|s|^m(\log|s|)^{n+1}\\
&-\pi(1-\alpha)\cos(\pi\alpha m)(-\operatorname{sgn}(s))^\sigma|s|^m(\log|s|)^n + (-\operatorname{sgn}(s))^\sigma|s|^m P_{n-1}(\log|s|)\\
&+|s|^{m+k+1}Q_{(N-m-k-1,n+1,1)}(|s|,\log|s|,\operatorname{sgn}(s)) + R_{(N,1)}(|s|,\operatorname{sgn}(s)) + o(|s|^N),
\end{aligned} \tag{216}$$

for all $0 < s < 1$, where $P\colon \mathbb{R}\to\mathbb{R}$, $Q\colon \mathbb{R}^3\to\mathbb{R}$, and $R\colon \mathbb{R}^2\to\mathbb{R}$ are polynomials. Furthermore, the polynomial P depends only on n, α, and m.

5.2 Proof of the Forward Direction of Theorem 2.1

Suppose that N is a positive integer. In this section, we prove that for certain values of μ, there exist functions $\rho\colon [-1,1] \to \mathbb{R}$ of the forms

$$\rho(t) \in \mathcal{F}_{N,k,\alpha}(\mu), \tag{217}$$
$$\rho(t) \in \mathcal{G}_{N,k,\alpha}(\mu), \tag{218}$$
$$\rho(t) \in \mathcal{H}_{N,k,\alpha}(\mu), \tag{219}$$
$$\operatorname{sgn}(t)\rho(t) \in \mathcal{F}_{N,k,\alpha}(\mu), \tag{220}$$
$$\operatorname{sgn}(t)\rho(t) \in \mathcal{G}_{N,k,\alpha}(\mu), \tag{221}$$
$$\operatorname{sgn}(t)\rho(t) \in \mathcal{H}_{N,k,\alpha}(\mu), \tag{222}$$

such that the resulting boundary data g, defined by (52), is smooth on each side of the corner, to order N (see definitions 2.1, 2.2, and 2.3). The principal results of this section are theorems 5.9, 5.10, 5.11 and 5.12 in the even case (for functions of the forms (217)–(219)), and theorems 5.13, 5.14, 5.15, and 5.16 in the odd case (for functions of the forms (220)–(222)).

Even Case

The following theorem shows that when $\alpha \notin \mathbb{Q}$, there exist certain values of μ and functions $\rho(t)$, where $\rho(t) \in \mathcal{F}_{N,k,\alpha}(\mu)$, such that the resulting boundary data g is smooth on each side of the corner, to order N.

Theorem 5.9. *Suppose that $\alpha \notin \mathbb{Q}$ and that N is a positive integer, and let $\overline{L} = \lceil \alpha N/2 \rceil$. Then, for each integer $1 \le n \le \overline{L}$, there exists a function*

$$\rho(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \tag{223}$$

where $-1 \le t \le 1$, such that g, defined by (52), has the form

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{224}$$

where $-1 \le t \le 1$ and $\beta_0, \beta_1, \ldots, \beta_N$ and $\xi_0, \xi_1, \ldots, \xi_N$ are real numbers.

Now let $\underline{M} = \lfloor (2-\alpha)N/2 \rfloor$. Then, likewise, for each integer $1 \le n \le \underline{M}$, there exists a function

$$\rho(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n}{2-\alpha}\Big), \tag{225}$$

where $-1 \le t \le 1$, such that g, defined by (52), also has the form (224).

Proof. Suppose that J is a nonnegative integer, and let P_J be the proposition that there exists some function

$$\rho_J(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big) \tag{226}$$

such that, if g_J is defined by (52) (where g replaced by g_J), then g_J has the form

$$g_J(t) = R_{(N,1)}(|t|, \mathrm{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+1} Q_{(N-\frac{2n-1}{\alpha}-J-1,1)}(|t|, \mathrm{sgn}(t)) + o(|t|^N), \quad (227)$$

for all $-1 \le t \le 1$, where $R\colon \mathbb{R}^2 \to \mathbb{R}$ and $Q\colon \mathbb{R}^2 \to \mathbb{R}$ are polynomials. First, combining Theorem 5.7 with Theorem 3.26, we observe that if

$$\rho_k(t) = |t|^{\frac{2n-1}{\alpha}}, \quad (228)$$

for $-1 \le t \le 1$, then

$$g_k(t) = R_{(N,1)}(|t|, \mathrm{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+k+1} Q_{(N-\frac{2n-1}{\alpha}-k-1,1)}(|t|, \mathrm{sgn}(t)) + o(|t|^N), \quad (229)$$

for $-1 \le t \le 1$, so clearly all of $P_0, P_1, \ldots, P_k$ are true.

Now suppose that P_J is true. We will show that this implies that P_{J+1} is true. If P_J is true, then there exists some function

$$\rho_J(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \quad (230)$$

for $-1 \le t \le 1$, such that

$$g_J(t) = R_{(N,1)}(|t|, \mathrm{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+1} Q_{(N-\frac{2n-1}{\alpha}-J-1,1)}(|t|, \mathrm{sgn}(t)) + o(|t|^N), \quad (231)$$

for $-1 \le t \le 1$. By combining Theorem 5.7 with Theorem 3.27, we observe that there exist numbers $b, c \in \mathbb{R}$ such that if

$$\rho_{J+1}(t) = \rho_J(t) + b|t|^{\frac{2n-1}{\alpha}+J+1} + c \cdot \mathrm{sgn}(t)|t|^{\frac{2n-1}{\alpha}+J+1}, \quad (232)$$

for $-1 \le t \le 1$, then

$$g_{J+1}(t) = R_{(N,1)}(|t|, \mathrm{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+2} Q_{(N-\frac{2n-1}{\alpha}-J-2,1)}(|t|, \mathrm{sgn}(t)) + o(|t|^N), \quad (233)$$

for $-1 \le t \le 1$. Clearly, since $\rho_{J+1} \in \mathcal{F}_{N,k,\alpha}(\frac{2n-1}{\alpha})$, it follows that P_J implies P_{J+1}. The proof of the remaining part of this theorem (in which $\frac{2n-1}{\alpha}$ is replaced by $\frac{2n}{2-\alpha}$) is essentially identical.

■

The following theorem shows that when $\alpha \in \mathbb{Q}$, there exist certain values of μ and functions $\rho(t)$, where $\rho(t) \in \mathcal{F}_{N,k,\alpha}(\mu)$, such that the resulting boundary data g is smooth on each side of the corner, to order N.

Theorem 5.10. *Suppose that $\alpha \in \mathbb{Q}$ and that N is a positive integer, and let $\overline{L} = \lceil \alpha N/2 \rceil$. Then, for each integer $1 \le n \le \overline{L}$ for which*

$$\frac{2n-1}{\alpha} \neq \frac{2j}{2-\alpha} \text{ for each integer } j, \quad (234)$$

there exists a function

$$\rho(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \quad (235)$$

where $-1 \le t \le 1$, such that g, defined by (52), has the form

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{236}$$

where $-1 \le t \le 1$ and $\beta_0, \beta_1, \dots, \beta_N$ and $\xi_0, \xi_1, \dots, \xi_N$ are real numbers.

Now let $\underline{M} = \lfloor (2-\alpha)N/2 \rfloor$. Then, likewise, for each integer $1 \le n \le \underline{M}$ for which

$$\frac{2n}{2-\alpha} \neq \frac{2j-1}{\alpha} \text{ for each integer } j, \tag{237}$$

there exists a function

$$\rho(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n}{2-\alpha}\Big), \tag{238}$$

where $-1 \le t \le 1$, such that g, defined by (52), also has the form (236).

Proof. Let p and q be integers such that $\alpha = p/q$ is a reduced fraction, with $q > 0$. Suppose for now that

$$\frac{2n-1}{\alpha} \notin \mathbb{Z}. \tag{239}$$

Suppose further that J is a nonnegative integer, and let P_J be the proposition that there exists some function

$$\rho_J(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \tag{240}$$

such that, if g_J is defined by (52) (where g replaced by g_J), then g_J has the form

$$g_J(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+1} Q_{(N-\frac{2n-1}{\alpha}-J-1, J/q, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \tag{241}$$

for all $-1 \le t \le 1$, where $R\colon \mathbb{R}^2 \to \mathbb{R}$ and $Q\colon \mathbb{R}^3 \to \mathbb{R}$ are polynomials. First, by combining Theorem 5.7 with Theorem 3.26, we observe that if

$$\rho_k(t) = |t|^{\frac{2n-1}{\alpha}}, \tag{242}$$

for $-1 \le t \le 1$, then

$$g_k(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+k+1} Q_{(N-\frac{2n-1}{\alpha}-k-1, 0, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \tag{243}$$

for $-1 \le t \le 1$, so clearly all of $P_0, P_1, \dots, P_k$ are true.

Now suppose that P_J is true. We will show that this implies that P_{J+1} is true. If P_J is true, then there exists some function

$$\rho_J(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \tag{244}$$

for $-1 \le t \le 1$, such that

$$g_J(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+1} Q_{(N-\frac{2n-1}{\alpha}-J-1, J/q, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \quad (245)$$

for all $-1 \le t \le 1$, where $R\colon \mathbb{R}^2 \to \mathbb{R}$ and $Q\colon \mathbb{R}^3 \to \mathbb{R}$ are polynomials.

Case 1: Suppose first that $J+1 \neq m \cdot q$, for each integer m. Then, combining Theorem 5.7 with Theorem 3.28, there exist real numbers $b_0, b_1, \ldots, b_{\lfloor J/q \rfloor}$ and $c_0, c_1, \ldots, c_{\lfloor J/q \rfloor}$ such that if

$$\rho_{J+1}(t) = \rho_J(t) + \sum_{i=0}^{\lfloor J/q \rfloor} b_i |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i + \sum_{i=0}^{\lfloor J/q \rfloor} c_i \cdot \operatorname{sgn}(t) |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i, \quad (246)$$

for $-1 \le t \le 1$, then

$$g_{J+1}(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+2} Q_{(N-\frac{2n-1}{\alpha}-J-2, J/q, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \quad (247)$$

for $-1 \le t \le 1$. Clearly, since $\rho_{J+1} \in \mathcal{F}_{N,k,\alpha}(\frac{2n-1}{\alpha})$, it follows that P_J implies P_{J+1}.

Case 2: Suppose now that $J+1 = m \cdot q$, for some integer m. Then, combining Theorem 5.7 with Theorem 3.29, there exist real numbers $b_0, b_1, \ldots, b_{\lfloor J/q \rfloor+1}$ and $c_0, c_1, \ldots, c_{\lfloor J/q \rfloor+1}$ such that if

$$\rho_{J+1}(t) = \rho_J(t) + \sum_{i=0}^{\lfloor J/q \rfloor+1} b_i |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i + \sum_{i=0}^{\lfloor J/q \rfloor+1} c_i \cdot \operatorname{sgn}(t) |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i, \quad (248)$$

for $-1 \le t \le 1$, then

$$g_{J+1}(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+2} Q_{(N-\frac{2n-1}{\alpha}-J-2, J/q+1, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \quad (249)$$

for $-1 \le t \le 1$. Since, in this case, $\lfloor J/q \rfloor + 1 = \lfloor (J+1)/q \rfloor$, we observe that $\rho_{J+1} \in \mathcal{F}_{N,k,\alpha}(\frac{2n-1}{\alpha})$ and that P_{J+1} is true. Thus, P_J implies P_{J+1}.

The proof is similar in the case when $\frac{2n-1}{\alpha} \in \mathbb{Z}$, except that it uses Theorem 5.8 instead of Theorem 5.7. The proof of the rest of this theorem (in which $\frac{2n}{2-\alpha}$ replaces $\frac{2n-1}{\alpha}$) is essentially identical.

∎

The following theorem shows that when $\alpha \in \mathbb{Q}$, there exist certain values of μ and functions $\rho(t)$, where $\rho(t) \in \mathcal{G}_{N,k,\alpha}(\mu)$, such that the resulting boundary data g is smooth on each side of the corner, to order N.

Theorem 5.11. *Suppose that $\alpha \in \mathbb{Q}$ and that N is a positive integer, and let $\overline{L} = \lceil \alpha N/2 \rceil$. Then, for each integer $1 \le n \le \overline{L}$ for which*

$$\frac{2n-1}{\alpha} = \frac{2j}{2-\alpha} \text{ for some integer } j, \tag{250}$$

there exists a function

$$\rho(t) \in \mathcal{G}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \tag{251}$$

where $-1 \le t \le 1$, such that g, defined by (52), has the form

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{252}$$

where $-1 \le t \le 1$ and $\beta_0, \beta_1, \dots, \beta_N$ and $\xi_0, \xi_1, \dots, \xi_N$ are real numbers.

Now let $\underline{M} = \lfloor (2-\alpha)N/2 \rfloor$. Then, likewise, for each integer $1 \le n \le \underline{M}$ for which

$$\frac{2n}{2-\alpha} = \frac{2j-1}{\alpha} \text{ for some integer } j, \tag{253}$$

there exists a function

$$\rho(t) \in \mathcal{G}_{N,k,\alpha}\Big(\frac{2n}{2-\alpha}\Big), \tag{254}$$

where $-1 \le t \le 1$, such that g, defined by (52), also has the form (252).

Proof. Let p and q be integers such that $\alpha = p/q$ is a reduced fraction, with $q > 0$. Suppose that J is a nonnegative integer, and let P_J be the proposition that there exists some function

$$\rho_J(t) \in \mathcal{G}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \tag{255}$$

such that, if g_J is defined by (52) (where g replaced by g_J), then g_J has the form

$$g_J(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+1} Q_{(N-\frac{2n-1}{\alpha}-J-1, 2\cdot\lfloor J/q \rfloor, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \tag{256}$$

for all $-1 \le t \le 1$, where $R\colon \mathbb{R}^2 \to \mathbb{R}$ and $Q\colon \mathbb{R}^3 \to \mathbb{R}$ are polynomials. First, by combining Theorem 5.7 with Theorem 3.26, we observe that if

$$\rho_k(t) = |t|^{\frac{2n-1}{\alpha}}, \tag{257}$$

for $-1 \le t \le 1$, then

$$g_k(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+k+1} Q_{(N-\frac{2n-1}{\alpha}-k-1, 0, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \tag{258}$$

for $-1 \le t \le 1$, so clearly all of $P_0, P_1, \dots, P_k$ are true.

Now suppose that P_J is true. We will show that this implies that P_{J+1} is true. If P_J is true, then there exists some function

$$\rho_J(t) \in \mathcal{G}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \tag{259}$$

for $-1 \le t \le 1$, such that

$$g_J(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+1} Q_{(N-\frac{2n-1}{\alpha}-J-1, 2\cdot\lfloor J/q\rfloor, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \tag{260}$$

for all $-1 \le t \le 1$, where $R\colon \mathbb{R}^2 \to \mathbb{R}$ and $Q\colon \mathbb{R}^3 \to \mathbb{R}$ are polynomials.

Case 1: Suppose first that $J+1 \neq m\cdot q$, for each integer m. Then, combining Theorem 5.7 with Theorem 3.28, there exist real numbers $b_0, b_1, \dots, b_{2\cdot\lfloor J/q\rfloor}$ and $c_0, c_1, \dots, c_{2\cdot\lfloor J/q\rfloor}$ such that if

$$\rho_{J+1}(t) = \rho_J(t) + \sum_{i=0}^{2\cdot\lfloor J/q\rfloor} b_i |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i + \sum_{i=0}^{2\cdot\lfloor J/q\rfloor} c_i \cdot \operatorname{sgn}(t) |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i, \tag{261}$$

for $-1 \le t \le 1$, then

$$g_{J+1}(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+2} Q_{(N-\frac{2n-1}{\alpha}-J-2, 2\cdot\lfloor J/q\rfloor, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \tag{262}$$

for $-1 \le t \le 1$. Clearly, since $\rho_{J+1} \in \mathcal{G}_{N,k,\alpha}(\frac{2n-1}{\alpha})$, it follows that P_J implies P_{J+1}.

Case 2: Suppose now that $J+1 = m\cdot q$, for some integer m. Then, combining Theorem 5.7 with Theorem 3.31, there exist real numbers $b_0, b_1, \dots, b_{2\cdot\lfloor J/q\rfloor+2}$ and $c_0, c_1, \dots, c_{2\cdot\lfloor J/q\rfloor+2}$ such that if

$$\rho_{J+1}(t) = \rho_J(t) + \sum_{i=0}^{2\cdot\lfloor J/q\rfloor+2} b_i |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i + \sum_{i=0}^{2\cdot\lfloor J/q\rfloor+2} c_i \cdot \operatorname{sgn}(t) |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i, \tag{263}$$

for $-1 \le t \le 1$, then

$$g_{J+1}(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+2} Q_{(N-\frac{2n-1}{\alpha}-J-2, 2\cdot\lfloor J/q\rfloor+2, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \tag{264}$$

for $-1 \le t \le 1$. Since, in this case, $\lfloor J/q\rfloor + 1 = \lfloor (J+1)/q\rfloor$, we observe that $\rho_{J+1} \in \mathcal{G}_{N,k,\alpha}(\frac{2n-1}{\alpha})$ and that P_{J+1} is true. Thus, P_J implies P_{J+1}.

The proof of the rest of this theorem (in which $\frac{2n}{2-\alpha}$ replaces $\frac{2n-1}{\alpha}$) is essentially identical.

■

The following theorem shows that when $\alpha \in \mathbb{Q}$, there exist certain values of μ and functions $\rho(t)$, where $\rho(t) \in \mathcal{H}_{N,k,\alpha}(\mu)$, such that the resulting boundary data g is smooth on each side of the corner, to order N.

Theorem 5.12. *Suppose that $\alpha \in \mathbb{Q}$ and that N is a positive integer, and let $\overline{L} = \lceil \alpha N/2 \rceil$. Then, for each integer $1 \le n \le \overline{L}$ for which*

$$\frac{2n-1}{\alpha} = \frac{2j}{2-\alpha} \text{ for some integer } j, \tag{265}$$

there exists a function

$$\rho(t) \in \mathcal{H}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \tag{266}$$

where $-1 \le t \le 1$, such that g, defined by (52), has the form

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{267}$$

where $-1 \le t \le 1$ and $\beta_0, \beta_1, \ldots, \beta_N$ and $\xi_0, \xi_1, \ldots, \xi_N$ are real numbers.

Now let $\underline{M} = \lfloor (2-\alpha)N/2 \rfloor$. Then, likewise, for each integer $1 \le n \le \underline{M}$ for which

$$\frac{2n}{2-\alpha} = \frac{2j-1}{\alpha} \text{ for some integer } j, \tag{268}$$

there exists a function

$$\rho(t) \in \mathcal{H}_{N,k,\alpha}\Big(\frac{2n}{2-\alpha}\Big), \tag{269}$$

where $-1 \le t \le 1$, such that g, defined by (52), also has the form (267).

Proof. Let p and q be integers such that $\alpha = p/q$ is a reduced fraction, with $q > 0$. Suppose that J is a nonnegative integer, and let P_J be the proposition that there exists some function

$$\rho_J(t) \in \mathcal{H}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \tag{270}$$

such that, if g_J is defined by (52) (where g replaced by g_J), then g_J has the form

$$g_J(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+1} Q_{(N-\frac{2n-1}{\alpha}-J-1, 2\cdot\lfloor J/q \rfloor+1, 1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \tag{271}$$

for all $-1 \le t \le 1$, where $R\colon \mathbb{R}^2 \to \mathbb{R}$ and $Q\colon \mathbb{R}^3 \to \mathbb{R}$ are polynomials. First, by combining Theorem 5.7 with theorems 3.26 and 3.30, we observe that if

$$\rho_k(t) = |t|^{\frac{2n-1}{\alpha}} \log|t|, \tag{272}$$

for $-1 \le t \le 1$, then

$$g_k(t) = R_{(N,1)}(|t|, \operatorname{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+k+1} Q_{(N-\frac{2n-1}{\alpha}-k-1,1,1)}(|t|, \log|t|, \operatorname{sgn}(t)) + o(|t|^N), \tag{273}$$

for $-1 \le t \le 1$, so clearly all of $P_0, P_1, \ldots, P_k$ are true.

Now suppose that P_J is true. We will show that this implies that P_{J+1} is true. If P_J is true, then there exists some function

$$\rho_J(t) \in \mathcal{H}_{N,k,\alpha}\Big(\frac{2n-1}{\alpha}\Big), \tag{274}$$

for $-1 \le t \le 1$, such that

$$g_J(t) = R_{(N,1)}(|t|, \mathrm{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+1} Q_{(N-\frac{2n-1}{\alpha}-J-1, 2\cdot\lfloor J/q\rfloor+1,1)}(|t|, \log|t|, \mathrm{sgn}(t)) + o(|t|^N), \tag{275}$$

for all $-1 \le t \le 1$, where $R\colon \mathbb{R}^2 \to \mathbb{R}$ and $Q\colon \mathbb{R}^3 \to \mathbb{R}$ are polynomials.

Case 1: Suppose first that $J+1 \neq m\cdot q$, for each integer m. Then, combining Theorem 5.7 with Theorem 3.28, there exist real numbers $b_0, b_1, \ldots, b_{2\cdot\lfloor J/q\rfloor+1}$ and $c_0, c_1, \ldots, c_{2\cdot\lfloor J/q\rfloor+1}$ such that if

$$\rho_{J+1}(t) = \rho_J(t) + \sum_{i=0}^{2\cdot\lfloor J/q\rfloor+1} b_i |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i + \sum_{i=0}^{2\cdot\lfloor J/q\rfloor+1} c_i \cdot \mathrm{sgn}(t) |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i, \tag{276}$$

for $-1 \le t \le 1$, then

$$g_{J+1}(t) = R_{(N,1)}(|t|, \mathrm{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+2} Q_{(N-\frac{2n-1}{\alpha}-J-2, 2\cdot\lfloor J/q\rfloor+1,1)}(|t|, \log|t|, \mathrm{sgn}(t)) + o(|t|^N), \tag{277}$$

for $-1 \le t \le 1$. Clearly, since $\rho_{J+1} \in \mathcal{H}_{N,k,\alpha}(\frac{2n-1}{\alpha})$, it follows that P_J implies P_{J+1}.

Case 2: Suppose now that $J+1 = m\cdot q$, for some integer m. Then, combining Theorem 5.7 with Theorem 3.31, there exist real numbers $b_0, b_1, \ldots, b_{2\cdot\lfloor J/q\rfloor+3}$ and $c_0, c_1, \ldots, c_{2\cdot\lfloor J/q\rfloor+3}$ such that if

$$\rho_{J+1}(t) = \rho_J(t) + \sum_{i=0}^{2\cdot\lfloor J/q\rfloor+3} b_i |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i + \sum_{i=0}^{2\cdot\lfloor J/q\rfloor+3} c_i \cdot \mathrm{sgn}(t) |t|^{\frac{2n-1}{\alpha}+J+1} (\log|t|)^i, \tag{278}$$

for $-1 \le t \le 1$, then

$$g_{J+1}(t) = R_{(N,1)}(|t|, \mathrm{sgn}(t)) + |t|^{\frac{2n-1}{\alpha}+J+2} Q_{(N-\frac{2n-1}{\alpha}-J-2, 2\cdot\lfloor J/q\rfloor+3,1)}(|t|, \log|t|, \mathrm{sgn}(t)) + o(|t|^N), \tag{279}$$

for $-1 \le t \le 1$. Since, in this case, $\lfloor J/q\rfloor + 1 = \lfloor (J+1)/q\rfloor$, we observe that $\rho_{J+1} \in \mathcal{H}_{N,k,\alpha}(\frac{2n-1}{\alpha})$ and that P_{J+1} is true. Thus, P_J implies P_{J+1}.

The proof of the rest of this theorem (in which $\frac{2n}{2-\alpha}$ replaces $\frac{2n-1}{\alpha}$) is essentially identical.

$\blacksquare$

Odd Case

The proofs in this section are essentially identical to those in the even case, and are omitted.

The following theorem shows that when $\alpha \notin \mathbb{Q}$, there exist certain values of μ and functions $\rho(t)$, where $\operatorname{sgn}(t)\rho(t) \in \mathcal{F}_{N,k,\alpha}(\mu)$, such that the resulting boundary data g is smooth on each side of the corner, to order N.

Theorem 5.13. *Suppose that $\alpha \notin \mathbb{Q}$ and that N is a positive integer, and let $\overline{M} = \lceil (2-\alpha)N/2 \rceil$. Then, for each integer $1 \le n \le \overline{M}$, there exists a function $\rho(t)$, where*

$$\operatorname{sgn}(t)\rho(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n-1}{2-\alpha}\Big) \tag{280}$$

and $-1 \le t \le 1$, such that g, defined by (52), has the form

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{281}$$

where $-1 \le t \le 1$ and $\beta_0, \beta_1, \dots, \beta_N$ and $\xi_0, \xi_1, \dots, \xi_N$ are real numbers.

Now let $\underline{L} = \lfloor \alpha N/2 \rfloor$. Then, likewise, for each integer $1 \le n \le \underline{L}$, there exists a function $\rho(t)$, where

$$\operatorname{sgn}(t)\rho(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n}{\alpha}\Big) \tag{282}$$

and $-1 \le t \le 1$, such that g, defined by (52), also has the form (281).

The following theorem shows that when $\alpha \in \mathbb{Q}$, there exist certain values of μ and functions $\rho(t)$, where $\operatorname{sgn}(t)\rho(t) \in \mathcal{F}_{N,k,\alpha}(\mu)$, such that the resulting boundary data g is smooth on each side of the corner, to order N.

Theorem 5.14. *Suppose that $\alpha \in \mathbb{Q}$ and that N is a positive integer, and let $\overline{M} = \lceil (2-\alpha)N/2 \rceil$. Then, for each integer $1 \le n \le \overline{M}$ for which*

$$\frac{2n-1}{2-\alpha} \neq \frac{2j}{\alpha} \text{ for each integer } j, \tag{283}$$

there exists a function $\rho(t)$, where

$$\operatorname{sgn}(t)\rho(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n-1}{2-\alpha}\Big) \tag{284}$$

and $-1 \le t \le 1$, such that g, defined by (52), has the form

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{285}$$

where $-1 \le t \le 1$ and $\beta_0, \beta_1, \dots, \beta_N$ and $\xi_0, \xi_1, \dots, \xi_N$ are real numbers.

Now let $\underline{L} = \lfloor \alpha N/2 \rfloor$. Then, likewise, for each integer $1 \le n \le \underline{L}$ for which

$$\frac{2n}{\alpha} \neq \frac{2j-1}{2-\alpha} \text{ for each integer } j, \tag{286}$$

there exists a function $\rho(t)$, where

$$\operatorname{sgn}(t)\rho(t) \in \mathcal{F}_{N,k,\alpha}\Big(\frac{2n}{\alpha}\Big) \tag{287}$$

and $-1 \le t \le 1$, such that g, defined by (52), also has the form (285).

The following theorem shows that when $\alpha \in \mathbb{Q}$, there exist certain values of μ and functions $\rho(t)$, where $\operatorname{sgn}(t)\rho(t) \in \mathcal{G}_{N,k,\alpha}(\mu)$, such that the resulting boundary data g is smooth on each side of the corner, to order N.

Theorem 5.15. *Suppose that $\alpha \in \mathbb{Q}$ and that N is a positive integer, and let $\overline{M} = \lceil (2-\alpha)N/2 \rceil$. Then, for each integer $1 \le n \le \overline{M}$ for which*

$$\frac{2n-1}{2-\alpha} = \frac{2j}{\alpha} \text{ for some integer } j, \tag{288}$$

there exists a function $\rho(t)$, where

$$\operatorname{sgn}(t)\rho(t) \in \mathcal{G}_{N,k,\alpha}\Big(\frac{2n-1}{2-\alpha}\Big) \tag{289}$$

and $-1 \le t \le 1$, such that g, defined by (52), has the form

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{290}$$

where $-1 \le t \le 1$ and $\beta_0, \beta_1, \dots, \beta_N$ and $\xi_0, \xi_1, \dots, \xi_N$ are real numbers.

Now let $\underline{L} = \lfloor \alpha N/2 \rfloor$. Then, likewise, for each integer $1 \le n \le \underline{L}$ for which

$$\frac{2n}{\alpha} = \frac{2j-1}{2-\alpha} \text{ for some integer } j, \tag{291}$$

there exists a function $\rho(t)$, where

$$\operatorname{sgn}(t)\rho(t) \in \mathcal{G}_{N,k,\alpha}\Big(\frac{2n}{\alpha}\Big) \tag{292}$$

and $-1 \le t \le 1$, such that g, defined by (52), also has the form (290).

The following theorem shows that when $\alpha \in \mathbb{Q}$, there exist certain values of μ and functions $\rho(t)$, where $\operatorname{sgn}(t)\rho(t) \in \mathcal{H}_{N,k,\alpha}(\mu)$, such that the resulting boundary data g is smooth on each side of the corner, to order N.

Theorem 5.16. *Suppose that $\alpha \in \mathbb{Q}$ and that N is a positive integer, and let $\overline{M} = \lceil (2-\alpha)N/2 \rceil$. Then, for each integer $1 \le n \le \overline{M}$ for which*

$$\frac{2n-1}{2-\alpha} = \frac{2j}{\alpha} \text{ for some integer } j, \tag{293}$$

there exists a function $\rho(t)$, where

$$\operatorname{sgn}(t)\rho(t) \in \mathcal{H}_{N,k,\alpha}\Big(\frac{2n-1}{2-\alpha}\Big) \tag{294}$$

and $-1 \le t \le 1$*, such that* g*, defined by (52), has the form*

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{295}$$

where $-1 \le t \le 1$ *and* $\beta_0, \beta_1, \ldots, \beta_N$ *and* $\xi_0, \xi_1, \ldots, \xi_N$ *are real numbers.*

Now let $\underline{L} = \lfloor \alpha N/2 \rfloor$*. Then, likewise, for each integer* $1 \le n \le \underline{L}$ *for which*

$$\frac{2n}{\alpha} = \frac{2j-1}{2-\alpha} \text{ for some integer } j, \tag{296}$$

there exists a function $\rho(t)$*, where*

$$\operatorname{sgn}(t)\rho(t) \in \mathcal{H}_{N,k,\alpha}\Big(\frac{2n}{\alpha}\Big) \tag{297}$$

and $-1 \le t \le 1$*, such that* g*, defined by (52), also has the form (295).*

5.3 Proof of the Converse Direction of Theorem 2.1

The following theorem is the principal result of this section.

Theorem 5.17. *Suppose that* $0 < \alpha < 2$ *and that* N *is a nonnegative integer. Let*

$$\overline{L} = \Big\lceil \frac{\alpha N}{2} \Big\rceil, \tag{298}$$

$$\underline{L} = \Big\lfloor \frac{\alpha N}{2} \Big\rfloor, \tag{299}$$

and

$$\underline{M} = \Big\lfloor \frac{(2-\alpha)N}{2} \Big\rfloor, \tag{300}$$

$$\overline{M} = \Big\lceil \frac{(2-\alpha)N}{2} \Big\rceil, \tag{301}$$

and observe that $\overline{L} + \underline{M} = N$ *and* $\overline{M} + \underline{L} = N$*. Suppose further that the functions* $\chi_{1,1}, \chi_{1,2}, \ldots, \chi_{1,\overline{L}}$ *and* $\chi_{2,0}, \chi_{2,1}, \ldots, \chi_{2,\underline{M}}$ *satisfy*

$$\chi_{1,i} \in \begin{cases} \mathcal{F}_{k,N,\alpha}\big(\frac{2i-1}{\alpha}\big) & \text{if } \frac{2i-1}{\alpha} \ne \frac{2j}{2-\alpha} \text{ for each integer } 1 \le j \le \underline{M}, \\ \mathcal{G}_{k,N,\alpha}\big(\frac{2i-1}{\alpha}\big) & \text{if } \frac{2i-1}{\alpha} = \frac{2j}{2-\alpha} \text{ for some integer } 1 \le j \le \underline{M}, \end{cases} \tag{302}$$

$$\chi_{2,j} \in \begin{cases} \mathcal{F}_{k,N,\alpha}\big(\frac{2j}{2-\alpha}\big) & \text{if } \frac{2j}{2-\alpha} \ne \frac{2i-1}{\alpha} \text{ for each integer } 1 \le i \le \overline{L}, \\ \mathcal{H}_{k,N,\alpha}\big(\frac{2j}{2-\alpha}\big) & \text{if } \frac{2j}{2-\alpha} = \frac{2i-1}{\alpha} \text{ for some integer } 1 \le i \le \overline{L}, \end{cases} \tag{303}$$

for $1 \le i \le \overline{L}$ *and* $0 \le j \le \underline{M}$*, and the functions* $\eta_{1,1}, \eta_{1,2}, \ldots, \eta_{1,\overline{M}}$ *and* $\eta_{2,0}, \eta_{2,1}, \ldots, \eta_{2,\underline{L}}$ *satisfy*

$$\eta_{1,i} \in \begin{cases} \mathcal{F}_{k,N,\alpha}\big(\frac{2i-1}{2-\alpha}\big) & \text{if } \frac{2i-1}{2-\alpha} \ne \frac{2j}{\alpha} \text{ for each integer } 1 \le j \le \underline{L}, \\ \mathcal{G}_{k,N,\alpha}\big(\frac{2i-1}{2-\alpha}\big) & \text{if } \frac{2i-1}{2-\alpha} = \frac{2j}{\alpha} \text{ for some integer } 1 \le j \le \underline{L}, \end{cases} \tag{304}$$

$$\eta_{2,j} \in \begin{cases} \mathcal{F}_{k,N,\alpha}\big(\frac{2j}{\alpha}\big) & \text{if } \frac{2j}{\alpha} \ne \frac{2i-1}{2-\alpha} \text{ for each integer } 1 \le i \le \overline{M}, \\ \mathcal{H}_{k,N,\alpha}\big(\frac{2j}{\alpha}\big) & \text{if } \frac{2j}{\alpha} = \frac{2i-1}{2-\alpha} \text{ for some integer } 1 \le i \le \overline{M}, \end{cases} \tag{305}$$

for $1 \le i \le \overline{M}$ and $0 \le j \le \underline{L}$, and that, if ρ has the form

$$\rho(t) = \sum_{i=1}^{\bar{L}} b_{\underline{M}+i}\chi_{1,i}(t) + \sum_{i=0}^{\underline{M}} b_i\chi_{2,i}(t) + \sum_{i=1}^{\overline{M}} c_{\underline{L}+i}\operatorname{sgn}(t)\eta_{1,i}(t) + \sum_{i=0}^{\underline{L}} c_i \operatorname{sgn}(t)\eta_{2,i}(t), \tag{306}$$

for all $-1 \le t \le 1$, then, for any arbitrary real numbers $b_0, b_1, \dots, b_N$ and $c_0, c_1, \dots, c_N$, the function g defined by (52) has the form

$$g(t) = \sum_{n=0}^{N} \beta_n |t|^n + \sum_{n=0}^{N} \xi_n \operatorname{sgn}(t)|t|^n + o(|t|^N), \tag{307}$$

for all $-1 \le t \le 1$, where $\beta_0, \beta_1, \dots, \beta_N$ and $\xi_0, \xi_1, \dots, \xi_N$ are real numbers (see theorems 5.9–5.16). Then, for each $0 < \alpha < 2$, there exists an open ball $B_{\delta(\alpha)}$ of radius $\delta(\alpha)$, centered at zero, and a set $K(\alpha) \subset B_{\delta(\alpha)} \times B_{\delta(\alpha)}$ of measure zero, where $\mathbf{0} \notin K(\alpha)$, such that the following holds. If the curvatures κ_1 and κ_2 of the curves γ_1 and γ_2 are defined by

$$\kappa_1(t) = a_{1,0} + a_{1,1}t + \dots + a_{1,M}t^M, \tag{308}$$
$$\kappa_2(t) = a_{2,0} + a_{2,1}t + \dots + a_{2,M}t^M, \tag{309}$$

for all $0 \le t \le 1$, and $(a_{1,0}, a_{1,1}, \dots, a_{1,M}, a_{2,0}, a_{2,1}, \dots, a_{2,M}) \in B_{\delta(\alpha)} \times B_{\delta(\alpha)} \setminus K(\alpha)$, then for any g of the form (307), there exist unique real numbers $b_0, b_1, \dots, b_N$ and $c_0, c_1, \dots, c_N$ such that ρ, defined by (306), solves equation (52) to within an error $o(|t|^N)$.

Proof. Suppose that $0 < \alpha < 2$ is fixed, and let $A(\boldsymbol{a_1}, \boldsymbol{a_2})$ denote the $(2N+2) \times (2N+2)$ real matrix satisfying

$$A(\boldsymbol{a_1}, \boldsymbol{a_2}) \begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_N \\ c_0 \\ c_1 \\ \vdots \\ c_N \end{pmatrix} = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \beta_N \\ \xi_0 \\ \xi_1 \\ \vdots \\ \xi_N \end{pmatrix}, \tag{310}$$

where

$$\boldsymbol{a_1} = (a_{1,0}, a_{1,1}, \dots, a_{1,M}), \tag{311}$$
$$\boldsymbol{a_2} = (a_{2,0}, a_{2,1}, \dots, a_{2,M}). \tag{312}$$

By combining Theorem 3.25 with the proofs of theorems 5.7 and 5.8, it is straightforward to show that the entries of $A(\boldsymbol{a_1}, \boldsymbol{a_2})$ are real analytic functions from $B_{\delta(\alpha)} \times B_{\delta(\alpha)} \to \mathbb{R}$. Thus,

$$\det(A(\boldsymbol{a_1}, \boldsymbol{a_2})) \tag{313}$$

is a real analytic function from $B_{\delta(\alpha)} \times B_{\delta(\alpha)} \to \mathbb{R}$. By Theorem 3.9,

$$\det(A(\mathbf{0},\mathbf{0})) \neq 0, \tag{314}$$

so (313) is not identically equal to zero. Hence, by Theorem 3.16, there exists some set $K(\alpha) \subset B_{\delta(\alpha)} \times B_{\delta(\alpha)}$ of measure zero, where $\mathbf{0} \notin K(\alpha)$, such that

$$\det(A(\boldsymbol{a_1},\boldsymbol{a_2})) \neq 0 \tag{315}$$

for all $(\boldsymbol{a_1},\boldsymbol{a_2}) \in B_{\delta(\alpha)} \times B_{\delta(\alpha)} \setminus K(\alpha)$.

■

6 Extensions and Generalizations

6.1 Numerical Algorithms

In [5], Jeremy Hoskins, Vladimir Rokhlin, and the author use the explicit and rapidly convergent series representations provided in [17] to construct accurate and efficient numerical algorithms for the solution of the integral equations of potential theory on polygonal domains. The numerical apparatus presented in [5] extends in a straightforward manner to the general case of analytic curves meeting at corners. In particular, the generalized quadratures and discretizations described there can be augmented by the additional terms arising from the presence of curved boundaries (i.e. powers of logarithms).

6.2 Helmholtz Equation

In [19], the author and Vladimir Rokhlin show that, on polygonal domains, the solutions to the boundary integral equations associated with the Helmholtz equation are representable by explicit series of certain Bessel functions of noninteger orders. It turns out that, when the boundaries of the regions consist of analytic curves meeting at corners, the solutions are representable by series identical to those in the Laplace case (presented in this report). The proof proceeds in a similar fashion, with lemmas involving Cauchy's formula replaced by lemmas involving Green's theorem. A detail analysis is in preparation.

6.3 Stokes Equation

In [15], Manas Rachh (together with the author) demonstrates that when, on polygonal domains, Stokes equation is formulated as a boundary integral equation, the solution is representable by the real part of a series of powers of the form t^z, where t is the distance from the corner and z is complex. Preliminary work suggests that, in the more general case of regions with boundaries consisting of analytic curves meeting at corners, the solution is, like in the Laplace case, augmented by correction terms involving the leading terms t^z multiplied by integer powers of t and powers of $\log(t)$. The detailed analysis is currently underway.

7 Acknowledgements

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8 Appendix A

In this section we prove Theorem 8.3, which is a restatement of Theorem 3.14 in the main text.

The following two lemmas state that, locally, the kernel $K(s,t)$ is partially holomorphic in both s and t.

Lemma 8.1. *Suppose that $U \subset C$ is a open simply connected set and that $F\colon U \to \mathbb{C}$ is a analytic function such that $F'(z) \neq 0$ for all $z \in U$. Suppose further that $K\colon \{(s,t) \in U \times U : s \neq t\} \to \mathbb{C}$ is defined by*

$$K(s,t) = \frac{F'(t)}{F(s) - F(t)}, \tag{316}$$

for all $s,t \in U$ such that $s \neq t$. Then, for each $t \in U$, there exists an open simply connected set $t \in U_t \subset \mathbb{C}$ and an analytic function $\phi_t\colon U_t \to \mathbb{C}$ such that

$$K(s,t) = \frac{1}{s-t} - \frac{F''(t)}{2F'(t)} + (s-t)\phi_t(s), \tag{317}$$

for all $s \in U_t \setminus \{t\}$.

Proof. Suppose that $t \in U$. Since F is analytic on U, there exists an open simply connected set $t \in U_t \subset U$ and an analytic function $\phi_{0,t}\colon U_t \to \mathbb{C}$, such that

$$F(s) = F(t) + F'(t)(s-t) + \frac{F''(t)}{2}(s-t)^2 + (s-t)^3\phi_{0,t}(s) \tag{318}$$

for all $s \in U_t$. Thus,

$$\begin{aligned} K(s,t) &= \frac{F'(t)}{F(s) - F(t)} \\ &= \frac{F'(t)}{F'(t)(s-t) + \frac{F''(t)}{2}(s-t)^2 + (s-t)^3\phi_{0,t}(s)} \\ &= \frac{1}{s-t} \cdot \frac{1}{1 + \frac{F''(t)}{2F'(t)}(s-t) + \frac{1}{F'(t)}(s-t)^2\phi_{0,t}(s)}, \end{aligned} \tag{319}$$

for all $s \in U_t \setminus \{t\}$. By examining the Taylor series of the quotient in (319), we observe that there exists an analytic function $\phi_t\colon U_t \to \mathbb{C}$ such that

$$\begin{aligned} K(s,t) &= \frac{1}{s-t} \cdot \Big(1 - \frac{F''(t)}{2F'(t)}(s-t) + (s-t)^2\phi_t(s)\Big) \\ &= \frac{1}{s-t} - \frac{F''(t)}{2F'(t)} + (s-t)\phi_t(s), \end{aligned} \tag{320}$$

for all $s \in U_t \setminus \{t\}$.

$\blacksquare$

Lemma 8.2. *Suppose that $U \subset C$ is a open simply connected set and that $F\colon U \to \mathbb{C}$ is a analytic function such that $F'(z) \neq 0$ for all $z \in U$. Suppose further that $K\colon \{(s,t) \in U \times U : s \neq t\} \to \mathbb{C}$ is defined by*

$$K(s,t) = \frac{F'(t)}{F(s) - F(t)}, \tag{321}$$

for all $s, t \in U$ such that $s \neq t$. Then, for each $s \in U$, there exists an open simply connected set $s \in U_s \subset \mathbb{C}$ and an analytic function $\phi_s\colon U_s \to \mathbb{C}$ such that

$$K(s,t) = \frac{1}{s-t} - \frac{F''(t)}{2F'(t)} + (s-t)\phi_s(t), \tag{322}$$

for all $t \in U_s \setminus \{s\}$.

Proof. Suppose that $s \in U$. Since F is analytic on U, there exists an open simply connected set $s \in U_s \subset U$ and an analytic function $\phi_{0,s}\colon U_s \to \mathbb{C}$, such that

$$F(t) = F(s) + F'(s)(t-s) + \frac{F''(s)}{2}(t-s)^2 + (t-s)^3\phi_{0,s}(t), \tag{323}$$

for all $t \in U_s$. Thus,

$$\begin{aligned}
K(s,t) &= \frac{F'(t)}{F(s) - F(t)} \\
&= \frac{F'(s) + F''(s)(t-s) + 3(t-s)^2\phi_{0,s}(t) + (t-s)^3\phi_{0,s}'(t)}{-F'(s)(t-s) - \frac{F''(s)}{2}(t-s)^2 - (t-s)^3\phi_{0,s}(t)} \\
&= \frac{F'(s)}{-F'(s)(t-s) - \frac{F''(s)}{2}(t-s)^2 - (t-s)^3\phi_{0,s}(t)} \\
&+ \frac{F''(s)(t-s) + 3(t-s)^2\phi_{0,s}(t) + (t-s)^3\phi_{0,s}'(t)}{-F'(s)(t-s) - \frac{F''(s)}{2}(t-s)^2 - (t-s)^3\phi_{0,s}(t)} \\
&= \frac{1}{s-t} \cdot \frac{1}{1 + \frac{F''(s)}{2F'(s)}(t-s) + \frac{1}{F'(s)}(t-s)^2\phi_{0,s}(t)} \\
&+ \frac{F''(s) + 3(t-s)\phi_s(t) + (t-s)^2\phi_{0,s}'(t)}{-F'(s) - \frac{F''(s)}{2}(t-s) - (t-s)^2\phi_{0,s}(t)},
\end{aligned} \tag{324}$$

for all $t \in U_s \setminus \{s\}$. By examining the Taylor series of the two quotients in (324), we observe that there exist analytic functions $\phi_{1,s}\colon U_s \to \mathbb{C}$ and $\phi_{2,s}\colon U_s \to \mathbb{C}$ such that

$$\begin{aligned}
K(s,t) &= \frac{1}{s-t} \cdot \Big(1 - \frac{F''(s)}{2F'(s)}(t-s) + (t-s)^2\phi_{1,s}(t)\Big) \\
&- \frac{F''(s)}{F'(s)} + (t-s)\phi_{2,s}(t) \\
&= \frac{1}{s-t} - \frac{F''(s)}{2F'(s)} + (t-s)\big(\phi_{2,s}(t) - \phi_{1,s}(t)\big) \\
&= \frac{1}{s-t} - \frac{F''(t)}{2F'(t)} + \Big(\frac{F''(t)}{2F'(t)} - \frac{F''(s)}{2F'(s)}\Big) + (t-s)\big(\phi_{2,s}(t) - \phi_{1,s}(t)\big),
\end{aligned} \tag{325}$$

for all $t \in U_s \setminus \{s\}$. Clearly then, there exists another analytic function $\phi_s\colon U_s \to \mathbb{C}$ such that

$$K(s,t) = \frac{1}{s-t} - \frac{F''(t)}{2F'(t)} + (s-t)\phi_s(t), \tag{326}$$

for all $t \in U_s \setminus \{s\}$.

■

The following theorem states that the kernel $K(s,t)$ is the sum of the Cauchy kernel and a holomorphic function.

Theorem 8.3. *Suppose that $U \subset C$ is a open simply connected set and that $F\colon U \to \mathbb{C}$ is a analytic function such that $F'(z) \neq 0$ for all $z \in U$. Suppose further that $K\colon \{(s,t) \in U \times U : s \neq t\} \to \mathbb{C}$ is defined by*

$$K(s,t) = \frac{F'(t)}{F(s) - F(t)}, \tag{327}$$

for all $s,t \in U$ such that $s \neq t$. Then there exists some holomorphic function $R\colon U \times U \to \mathbb{C}$ such that

$$K(s,t) = \frac{1}{s-t} - \frac{F''(t)}{2F'(t)} + (s-t)R(s,t) \tag{328}$$

for all $s,t \in U$ such that $s \neq t$.

Proof. Suppose that $R\colon \{(s,t) \in U \times U : s \neq t\} \to \mathbb{C}$ is defined by

$$R(s,t) = \frac{K(s,t) - \frac{1}{s-t} + \frac{F''(t)}{2F'(t)}}{s-t}, \tag{329}$$

for all $(s,t) \in U \times U$ such that $s \neq t$. Clearly, R is holomorphic on the set $\{(s,t) \in U \times U : s \neq t\}$. By Lemma 8.1, for each $t \in U$, the function $R(\cdot,t)\colon U \setminus \{t\} \to \mathbb{C}$ is bounded on $U \setminus \{t\}$ and is thus analytic on U. Likewise, by Lemma 8.2, for each $s \in U$, the function $R(s,\cdot)\colon U \setminus \{s\} \to \mathbb{C}$ is bounded on $U \setminus \{s\}$ and is thus analytic on U. Therefore, R is partially holomorphic on $U \times U$, from which it follows that R is holomorphic on $U \times U$.

■

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