Recently, B. Vioreanu and V. Rokhlin described a numerical procedure for the construction of stable interpolation and quadrature formulae for functions on bounded convex regions in the plane, based on treating the spectra of certain operators as interpolation nodes. The construction depends on the analytical observation that the spectrum of the complex multiplication operator on a bounded convex domain, restricted to any linear subspace of functions, is contained entirely in the domain. In this note, we observe that this theorem admits an extremely brief and elementary proof.

A Note on the Use of the Spectra of Multiplication Operators as a Numerical Tool

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Suppose that X is a bounded convex region in $\mathbb{R}^2$ (which we treat as a region in $\mathbb{C}$) and that $\{\phi_1(z), \phi_2(z), \dots, \phi_n(z)\}$ is a set of complex-valued orthonormal functions in $L^2(X)$. The eigenvalues of the $n \times n$ matrix $A_{i,j} = \langle z\phi_i(z), \phi_j(z)\rangle$, which is the restriction of the multiplication operator to the subspace $\operatorname{span}(\{\phi_i\})$, turn out to be stable interpolation nodes on X for the functions $\{\phi_i\}$. The principal analytical observation of [1] is that the restriction of the multiplication operator on X to any linear subspace of $L^2(X)$ has its spectrum contained entirely in X, allowing the eigenvalues to serve as interpolation nodes. In this note, we describe an extremely brief alternative proof of this theorem.

Suppose that X is a convex, closed, and bounded subset of $\mathbb{C}$, and let U be an arbitrary finite dimensional linear subspace of $L^2(X)$. Suppose further that $M\colon L^2(X) \to L^2(X)$ denotes the multiplication operator, defined by

$$M[f](z) = zf(z), \tag{1}$$

for all $z \in X$, and suppose that $P_U\colon L^2(X) \to L^2(X)$ denotes the orthogonal projection onto the subspace U.

Definition 0.1. Suppose that V is a linear subspace of $L^2(X)$. The *field of values* of an operator $T\colon V \to V$, denoted by $F(T)$, is the set of all inner products $\langle T[\varphi], \varphi\rangle$, where φ is a unit vector in V. In other words,

$$F(T) = \{\langle T[\varphi], \varphi\rangle : \varphi \in V, \|\varphi\| = 1\}. \tag{2}$$

Clearly, the field of values of an operator contains its spectrum.

Definition 0.2. The *convex hull* of a set $X \subset \mathbb{C}$ is the smallest convex set which contains X, and is denoted by $\operatorname{Conv}(X)$. It is equal to the set of all expected values of z over all probability distributions on X. In other words, the convex hull is the set of all integrals of the form $\int_X zp(z)\,dz$, where $p\colon X \to [0, \infty)$ and $\int_X p(z)\,dz = 1$.

The following is the principal lemma underlying this note.

Lemma 0.1. *Suppose that $M\colon L^2(X) \to L^2(X)$ is the multiplication operator, defined by (1). Then $F(M) = \operatorname{Conv}(X)$. In other words, the field of values of M is equal to the convex hull of the set X.*

Proof. We first observe that $F(M)$ is the set of all inner products of the form $\langle z\varphi(z), \varphi(z)\rangle$, where φ is a unit vector in $L^2(X)$. In other words, we observe that $F(M)$ is the set of all integrals of the form $\int_X z|\varphi(z)|^2\,dz$, where φ is a vector in $L^2(X)$ and $\int_X |\varphi(z)|^2\,dz = 1$. We observe then that this set is identical to the set of all integrals of the form $\int_X zp(z)\,dz$, where $p\colon X \to [0, \infty)$ and $\int_X p(z)\,dz = 1$. Recalling the definition of $\operatorname{Conv}(X)$, we have that $F(M) = \operatorname{Conv}(X)$.

■

The following theorem is the principal analytical tool of [1], and states that the eigenvalues of the multiplication operator restricted to U are in the set X.

Theorem 0.2. *All eigenvalues of the operator $P_U \circ M\colon U \to U$ fall inside X.*

Proof. First, let $\sigma(P_U \circ M)$ denote the spectrum of $P_U \circ M$. Clearly, $\sigma(P_U \circ M) \subset F(P_U \circ M)$. Next we observe that, trivially, $F(P_U \circ M) \subset F(M)$. Finally, by Lemma 0.1, we have that $F(M)$ is equal to the convex hull of X, which we denote by $\mathrm{Conv}(X)$. Since X is itself convex, $\mathrm{Conv}(X)$ is equal to X. Putting all this together, we have

$$\sigma(P_U \circ M) \subset F(P_U \circ M) \subset F(M) = \mathrm{Conv}(X) = X. \tag{3}$$

■

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References

[1] Vioreanu, Bogdan and Vladimir Rokhlin. "Spectra of multiplication operators as a numerical tool." *SIAM J. Sci. Comput.* 36.1 (2014): A267–A288.