Prolate spheroidal wave functions (PSWFs) provide a natural and effective tool for computing with bandlimited functions defined on an interval. As demonstrated by Slepian et al., the so called generalized prolate spheroidal functions (GPSFs) extend this apparatus to higher dimensions. While the analytical and numerical apparatus in one dimension is fairly complete, the situation in higher dimensions is less satisfactory. This report attempts to improve the situation by providing analytical and numerical tools for GPSFs, including the efficient evaluation of eigenvalues, the construction of quadratures, interpolation formulae, etc. Our results are illustrated with several numerical examples.

On generalized prolate spheroidal functions - Preliminary Report

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1 Introduction

Prolate spheroidal wave functions (PSWFs) are the natural basis for representing bandlimited functions on the interval. Much of the theory and numerical machinery for PSWFs in one dimension is fairly complete (see, for example, [21] and [15]). Slepian et al. showed in [1] that the so-called Generalized Prolate Spheroidal Functions (GPSFs) are the natural extension of PSWFs in higher dimensions. GPSFs are functions $\psi_j : \mathbb{R}^n \to \mathbb{C}$ satisfying

$$\lambda_j \psi_j(x) = \int_B \psi_j(t) e^{ic\langle x,t\rangle} dt \tag{1}$$

for some $\lambda_j \in \mathbb{C}$ where B denotes the unit ball in $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{C}$ is referred to as bandlimited with bandlimit $c > 0$ if

$$f(x) = \int_B \sigma(t) e^{ic\langle x,t\rangle} dt \tag{2}$$

where B denotes the unit ball in $\mathbb{R}^n$ and σ is a square-integrable function defined on B. Bandlimited functions are encountered in a variety of applications including in signal processing, antenna design, radar, etc.

Much of the theory and numerical machinery of GPSFs in two dimensions is described in [17]. In this report, we provide analytical and numerical tools for GPSFs in $\mathbb{R}^n$. We introduce algorithms for evaluating GPSFs, quadrature rules for integrating bandlimited functions, and numerical interpolation schemes for expanding bandlimited functions into GPSF expansions. We also provide numerical machinery for efficient evaluation of eigenvalues λ_j (see (1)).

The structure of this paper is as follows. In Section 2 we provide basic mathematical background that will be used throughout the remainder of the paper. Section 3 contains analytical facts related to the numerical evaluation of GPSFs that will be used in subsequent sections. In Section 4, we describe a numerical scheme for evaluating GPSFs. Section 5 contains a quadrature rule for integrating bandlimited functions. Section 6 includes a numerical scheme for expanding bandlimited functions into GPSFs. In Section 7, we provide the numerical results of implementing the quadrature and interpolation schemes as well as plots of GPSFs and their eigenvalues. In Section 8 we provide certain miscellaneous properties of GPSFs.

2 Mathematical and Numerical Preliminaries

In this section, we introduce notation and elementary mathematical and numerical facts which will be used in subsequent sections.

In accordance with standard practice, we define the Gamma function, $\Gamma(x)$, by the formula

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \tag{3}$$

where e will denote the base of the natural logarithm. We will be denoting by $\delta_{i,j}$ the function defined by the formula

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{4}$$

The following is a well-known technical lemma that will be used in Section 3.2.

Lemma 2.1 *For any real number $a > 0$ and for any integer $n > ae$,*

$$\frac{a^n \sqrt{n}}{\Gamma(n+1)} < 1 \tag{5}$$

where $\Gamma(n)$ is defined in (3).

The following lemma follows immediately from Formula 9.1.10 in [3].

Lemma 2.2 *For all real numbers $x \in [0, 1]$, and for all real numbers $\nu \geq -1/2$,*

$$|J_\nu(x)| \leq \frac{|x/2|^\nu}{\Gamma(\nu+1)} \tag{6}$$

where J_ν is a Bessel function of the first kind and $\Gamma(\nu)$ is defined in (3).

2.1 Jacobi Polynomials

In this section, we summarize some properties Jacobi polynomials.
Jacobi Polynomials, denoted $P_n^{(\alpha,\beta)}$, are orthogonal polynomials on the interval $(-1, 1)$ with respect to weight function

$$w(x) = (1-x)^\alpha (1+x)^\beta. \tag{7}$$

Specifically, for all non-negative integers n, m with $n \neq m$ and real numbers $\alpha, \beta > -1$,

$$\int_{-1}^1 P_n^{(\alpha,\beta)}(x) P_m^{(\alpha,\beta)}(x) (1-x)^\alpha (1+x)^\beta dx = 0 \tag{8}$$

The following lemma, provides a stable recurrence relation that can be used to evaluate a particular class of Jacobi Polynomials (see, for example, [3]).

Lemma 2.3 *For any integer $n \geq 1$ and $\alpha > -1$,*

$$P_{n+1}^{(\alpha,0)}(x) = \frac{(2n+\alpha+1)\alpha^2 + (2n+\alpha)(2n+\alpha+1)(2n+\alpha+2)x}{2(n+1)(n+\alpha+1)(2n+\alpha)} P_n^{(\alpha,0)}(x) - \frac{2(n+\alpha)(n)(2n+\alpha+2)}{2(n+1)(n+\alpha+1)(2n+\alpha)} P_{n-1}^{(\alpha,0)}(x), \tag{9}$$

where

$$P_0^{(\alpha,0)}(x) = 1 \tag{10}$$

and

$$P_1^{(\alpha,0)}(x) = \frac{\alpha + (\alpha+2)x}{2}. \tag{11}$$

The Jacobi Polynomial $P_n^{(\alpha,0)}$ is defined in (8).

The following lemma provides a stable recurrence relation that can be used to evaluate derivatives of a certain class of Jacobi Polynomials. It is readily obtained by differentiating (9) with respect to x,

Lemma 2.4 *For any integer $n \geq 1$ and $\alpha > -1$,*

$$P_{n+1}^{(\alpha,0)\prime}(x) = \frac{(2n+\alpha+1)\alpha^2 + (2n+\alpha)(2n+\alpha+1)(2n+\alpha+2)x}{2(n+1)(n+\alpha+1)(2n+\alpha)} P_n^{(\alpha,0)\prime}(x) - \frac{2(n+\alpha)(n)(2n+\alpha+2)}{2(n+1)(n+\alpha+1)(2n+\alpha)} P_{n-1}^{(\alpha,0)\prime}(x) + \frac{(2n+\alpha)(2n+\alpha+1)(2n+\alpha+2)}{2(n+1)(n+\alpha+1)(2n+\alpha)} P_n^{(\alpha,0)}(x), \tag{12}$$

where

$$P_0^{(\alpha,0)\prime}(x) = 0 \tag{13}$$

and

$$P_1^{(\alpha,0)\prime}(x) = \frac{\alpha+2}{2}. \tag{14}$$

The Jacobi Polynomial $P_n^{(\alpha,0)}$ is defined in (8) and $P_n^{(\alpha,0)\prime}(x)$ denotes the derivative of $P_n^{(\alpha,0)}(x)$ with respect to x.

The following two lemmas, which provide a differential equation and a recurrence relation for Jacobi polynomials, can be found in, for example, [3].

Lemma 2.5 *For any integer $n \geq 2$ and $\alpha > -1$,*

$$(1-x^2)P_n^{(\alpha,0)\prime\prime}(x)+(-\alpha-(\alpha+2)x)P_n^{(\alpha,0)\prime}(x)+n(n+\alpha+1)P_n^{(\alpha,0)}(x) = 0 \quad (15)$$

for all $x \in [0,1]$ where $P_n^{(\alpha,0)}$ is defined in (8).

Lemma 2.6 *For all $\alpha > -1$, $x \in (0,1)$, and any integer $n \geq 2$,*

$$a_{1n}P_{n+1}^{(\alpha,0)} = (a_{2n} + a_{3n}x)P_n^{(\alpha,0)}(x) - a_{4n}P_{n-1}^{(\alpha,0)}(x) \quad (16)$$

where

$$\begin{aligned} a_{1n} &= 2(n+1)(n+\alpha+1)(2n+\alpha) \\ a_{2n} &= (2n+\alpha+1)\alpha^2 \\ a_{3n} &= (2n+\alpha)(2n+\alpha+1)(2n+\alpha+2) \\ a_{4n} &= 2(n+\alpha)(n)(2n+\alpha+2) \end{aligned} \quad (17)$$

and

$$\begin{aligned} P_0^{(\alpha,0)}(x) &= 1 \\ P_1^{(\alpha,0)}(x) &= \frac{\alpha + (\alpha+2)x}{2}. \end{aligned} \quad (18)$$

2.2 Zernike Polynomials

In this section, we describe properties of Zernike polynomials, which are a family of orthogonal polynomials on the unit ball in $\mathbb{R}^{p+2}$. They are the natural basis for representing GPFS.

Zernike polynomials are defined via the formula

$$Z_{N,n}^{\ell}(x) = R_{N,n}(\|x\|)S_N^{\ell}(x/\|x\|), \quad (19)$$

for all $x \in \mathbb{R}^{p+2}$ such that $\|x\| \leq 1$, where N and n are nonnegative integers, S_N^{ℓ} are the orthonormal surface harmonics of degree N (see Section 2.7), and $R_{N,n}$ are polynomials of degree $2n+N$ defined via the formula

$$R_{N,n}(x) = x^N \sum_{m=0}^{n} (-1)^m \binom{n+N+\frac{p}{2}}{m} \binom{n}{m} (x^2)^{n-m}(1-x^2)^m, \quad (20)$$

for all $0 \le x \le 1$. The polynomials $R_{N,n}$ satisfy the relation

$$R_{N,n}(1) = 1, \tag{21}$$

and are orthogonal with respect to the weight function $w(x) = x^{p+1}$, so that

$$\int_0^1 R_{N,n}(x) R_{N,m}(x) x^{p+1}\,dx = \frac{\delta_{n,m}}{2(2n+N+\frac{p}{2}+1)}. \tag{22}$$

We define the polynomials $\overline{R}_{N,n}$ via the formula

$$\overline{R}_{N,n}(x) = \sqrt{2(2n+N+p/2+1)}\,R_{N,n}(x), \tag{23}$$

so that

$$\int_0^1 \big(\overline{R}_{N,n}(x)\big)^2 x^{p+1}\,dx = 1, \tag{24}$$

where N and n are nonnegative integers. In an abuse of notation, we refer to both the polynomials $Z_{N,n}^{\ell}$ and the radial polynomials $R_{N,n}$ as Zernike polynomials where the meaning is obvious.

Remark 2.1 *When $p = -1$, Zernike polynomials take the form*

$$\begin{aligned} Z_{0,n}^1(x) &= R_{0,n}(|x|) = P_{2n}(x), \\ Z_{1,n}^2(x) &= \operatorname{sgn}(x) \cdot R_{1,n}(|x|) = P_{2n+1}(x), \end{aligned} \tag{25}$$

for $-1 \le x \le 1$ and nonnegative integer n, where P_n denotes the Legendre polynomial of degree n and

$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x = 0, \\ -1 & \text{if } x < 0, \end{cases} \tag{26}$$

for all real x.

Remark 2.2 *When $p = 0$, Zernike polynomials take the form*

$$Z_{N,n}^1(x_1, x_2) = R_{N,n}(r)\cos(N\theta), \tag{27}$$

$$Z_{N,n}^2(x_1, x_2) = R_{N,n}(r)\sin(N\theta), \tag{28}$$

where $x_1 = r\cos(\theta)$, $x_2 = r\sin(\theta)$, and N and n are nonnegative integers.

The following lemma, which can be found in, for example, [3], shows how Zernike polynomials are related to Jacobi polynomials.

Lemma 2.7 *For all non-negative integers N, n,*

$$R_{N,n}(x) = (-1)^n x^N P_n^{(N+\frac{p}{2},0)}(1-2x^2), \tag{29}$$

where $0 \le x \le 1$, and $P_n^{(\alpha,0)}$, $\alpha > -1$, is defined in (20).

2.3 Numerical Evaluation of Zernike Polynomials

In this section, we provide a stable recurrence relation (see Lemma 2.8) that can be used to evaluate Zernike Polynomials.

The following lemma follows immediately from applying Lemma 2.7 to (16).

Lemma 2.8 *The polynomials $R_{N,n}$, defined in (20) satisfy the recurrence relation*

$$\begin{aligned} &R_{N,n+1}(x) = \\ &-\frac{((2n+N+1)N^2 + (2n+N)(2n+N+1)(2n+N+2)(1-2x^2))}{2(n+1)(n+N+1)(2n+N)} R_{N,n}(x) \\ &-\frac{2(n+N)(n)(2n+N+2)}{2(n+1)(n+N+1)(2n+N)} R_{N,n-1}(x) \end{aligned} \tag{30}$$

where $0 \le x \le 1$, N is a non-negative integer, n is a positive integer, and

$$R_{N,0}(x) = x^N \tag{31}$$

and

$$R_{N,1}(x) = -x^N \frac{N + (N+2)(1-2x^2)}{2}. \tag{32}$$

Remark 2.3 *The algorithm for evaluating Zernike polynomials using the recurrence relation in Lemma 2.8 is known as Kintner's method (see [12] and, for example, [8]).*

2.4 Modified Zernike polynomials, $\overline{T}_{N,n}$

In this section, we define the modified Zernike polynomials, $\overline{T}_{N,n}$ and provide some of their properties. This family of functions will be used in Section 4 for the numerical evaluation of GPSFs.

We define the function $T_{N,n}$ by the formula

$$T_{N,n}(r) = r^{\frac{p+1}{2}} R_{N,n}(r) \tag{33}$$

where N, n are non-negative integers. We define $\overline{T}_{N,n} : [0,1] \to \mathbb{R}$ by the formula,

$$\overline{T}_{N,n}(r) = r^{\frac{p+1}{2}} \overline{R}_{N,n}(r) \tag{34}$$

where N, n are non-negative integers and $\overline{R}_{N,n}$ is a normalized Zernike polynomial defined in (23), so that

$$\int_0^1 (\overline{T}_{N,n}(r))^2 dr = 1. \tag{35}$$

Lemma 2.9 *The functions $\overline{T}_{N,n}$ are orthonormal on the interval $(0,1)$ with respect to weight function $w(x)=1$. That is,*

$$\int_0^1 \overline{T}_{N,n}(r)\overline{T}_{N,m}(r)dr = \delta_{n,m}. \tag{36}$$

Proof. Using (34), (22) and (24), for all non-negative integers N, n, m,

$$\begin{aligned}\int_0^1 \overline{T}_{N,n}(r)\overline{T}_{N,m}(r)dr &= \int_0^1 r^{\frac{p+1}{2}}\overline{R}_{N,n}(r)r^{\frac{p+1}{2}}\overline{R}_{N,m}(r)dr \\ &= \int_0^1 \overline{R}_{N,n}(r)\overline{R}_{N,m}(r)r^{p+1}dr \\ &= \delta_{n,m}\end{aligned} \tag{37}$$

■

The following identity follows immediately from the combination of (34),(29), and (23).

Lemma 2.10 *For all non-negative integers N, n,*

$$\overline{T}_{N,n}(r) = P_n^{(N+p/2,0)}(1-2r^2)(-1)^n\sqrt{2(2n+N+p/2+1)}r^{\frac{p+1}{2}} \tag{38}$$

where $\overline{T}_{N,n}$ is defined in (34) and $P_n^{(N+p/2,0)}$ is a Zernike polynomial defined in (8).

The following lemma, which provides a differential equation for $\overline{T}_{N,n}$, follows immediately from substituting (38) into Lemma 40.

Lemma 2.11 *For all $r \in [0,1]$, non-negative integers N, n and real $p \geq -1$,*

$$(1-r^2)\overline{T}''_{N,n}(r) - 2r\overline{T}'_{N,n}(r) + \left(\chi_{N,n} + \frac{\frac{1}{4}-(N+\frac{p}{2})^2}{r^2}\right)\overline{T}_{N,n}(r) = 0 \tag{39}$$

where $\chi_{N,n}$ is defined by the formula

$$\chi_{N,n} = (N+p/2+2n+1/2)(N+p/2+2n+3/2). \tag{40}$$

The following lemma provides a recurrence relation satisfied by $\overline{T}_{N,n}$. It follows immediately from the combination of Lemma 2.10 and (9).

Lemma 2.12 *For any non-negative integers N, n and for all $r \in [0,1]$,*

$$\begin{aligned}r^2\overline{T}_{N,n}(r) =& \frac{\sqrt{2(2n+N+p/2+1)}}{\sqrt{2(2(n-1)+N+p/2+1)}}\frac{a_{4n}}{2a_{3n}}\overline{T}_{N,n-1}(r) \\ &+ \frac{a_{2n}+a_{3n}}{2a_{3n}}\overline{T}_{N,n}(r) \\ &+ \frac{\sqrt{2(2n+N+p/2+1)}}{\sqrt{2(2(n+1)+N+p/2+1)}}\frac{a_{1n}}{2a_{3n}}\overline{T}_{N,n+1}(r)\end{aligned} \tag{41}$$

where $\overline{T}_{N,n}$ is defined in (34) and

$$
\begin{aligned}
a_{1n} &= 2(n+1)(n+N+p/2+1)(2n+N+p/2)\\
a_{2n} &= (2n+N+p/2+1)N+p/2^2\\
a_{3n} &= (2n+N+p/2)(2n+N+p/2+1)(2n+N+p/2+2)\\
a_{4n} &= 2(n+N+p/2)(n)(2n+N+p/2+2).
\end{aligned}
\tag{42}
$$

Proof. Applying the change of variables $1-2r^2 = x$ to (16) and setting $\alpha = N + p/2$, we obtain

$$
\begin{aligned}
r^2 P_n^{(N+p/2,0)}(1-2r^2) =& \frac{a_{2n}}{2a_3n} P_n^{(N+p/2,0)}(1-2r^2) + \frac{1}{2} P_n^{(N+p/2,0)}(1-2r^2)\\
&- \frac{a_{4n}}{2a_3n} P_{n-1}^{(N+p/2,0)}(1-2r^2) - \frac{a_{1n}}{2a_3n} P_{n+1}^{(N+p/2,0)}(1-2r^2).
\end{aligned}
\tag{43}
$$

Identity (41) follows immediately m the combination of (43) with Lemma 2.10. ■

The following observation provides a scheme for computing $\overline{T}_{N,n}$.

Observation 2.4 *Combining (34), Lemma 2.8, and (23), we observe that the modified Zernike polynomial $\overline{T}_{N,n}(r)$ can be evaluated by first computing $P_n^{(N+p/2,0)}(1-2r^2)$ via recurrence relation (16) and then multiplying the resulting number by*

$$
r^N(-1)^n\sqrt{2(2n+N+p/2+1)}r^{\frac{p+1}{2}}. \tag{44}
$$

We define the function $\overline{T}^*_{N,n}$ by the formula

$$
\overline{T}^*_{N,n}(r) = \frac{\overline{T}_{N,n}(r)}{r^{N+\frac{p+1}{2}}}. \tag{45}
$$

where N, n are non-negative integers and $r \in (0,1)$. The following technical lemma involving $\overline{T}^*_{N,n}$ will be used in Section 3.3.

Lemma 2.13 *For all non-negative integers N, n,*

$$
\overline{T}^*_{N,n}(0) = \sqrt{2(2n+N+p/2+1)}(-1)^n \binom{n+N+p/2}{n}. \tag{46}
$$

Proof. Combining (34) and (20), we observe that

$$
\overline{T}_{N,n}(r) = \sum_{k=0}^{n} a_{N+k} r^{N+\frac{p+1}{2}+2k} \tag{47}
$$

where a_{N+k} is some real number for $k = 0, 1, ..., n$. In particular, using (20),

$$a_N = \sqrt{2(2n + N + p/2 + 1)}(-1)^n \binom{n + N + p/2}{n}. \tag{48}$$

Combining (45) and (48), we obtain (46). ■

The following lemma provides a relation that will be used in section 4.1 for the evaluation of certain eigenvalues.

Lemma 2.14 *Suppose that N is a nonnegative integer and that $n \geq 1$ is an integer. Then*

$$\begin{aligned} &\tilde{a}_n x T'_{N,n+1}(x) - \tilde{b}_n x T_{N,n+1}(x) + \tilde{c}_n x T'_{N,n-1}(x) \\ &= a_n T_{N,n+1}(x) - b_n T_{N,n}(x) + c_n T_{N,n-1}(x), \end{aligned} \tag{49}$$

for all $0 \leq x \leq 1$, where

$$\begin{aligned} \tilde{a}_n &= 2(n + N + 1)(2n + N), \\ \tilde{b}_n &= 2N(2n + N + 1), \\ \tilde{c}_n &= -2n(2n + N + 2), \\ a_n &= (2N + 4n + 5)(n + N + 1)(2n + N), \\ b_n &= N(2n + N + 1) - 2(2n + N)_3, \\ c_n &= n(2N + 4n - 1)(2n + N + 2), \end{aligned} \tag{50}$$

with $(\cdot)_k$ denoting the Pochhammer symbol or rising factorial.

2.5 Prüfer Transform

In this section, we describe the Prüfer Transform, which will be used in Section 5.1 in an algorithm for finding the roots of GPSFs. A more detailed description of the Prüfer Transform can be found in [9].

Lemma 2.15 (Prüfer Transform) *Suppose that the function $\phi : [a, b] \to \mathbb{R}$ satisfies the differential equation*

$$\phi''(x) + \alpha(x)\phi'(x) + \beta(x)\phi(x) = 0, \tag{51}$$

where $\alpha, \beta : (a, b) \to \mathbb{R}$ are differential functions. Then,

$$\frac{d\theta}{dx} = -\sqrt{\beta(x)} - \left(\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2}\right) sin(2\theta), \tag{52}$$

where the function $\theta : [a, b] \to \mathbb{R}$ is defined by the formula,

$$\frac{\phi'(x)}{\phi(x)} = \sqrt{\beta(x)} \tan(\theta(x)). \tag{53}$$

Proof. Introducing the notation

$$z(x) = \frac{\phi'(x)}{\phi(x)} \tag{54}$$

for all $x \in [a, b]$, and differentiating (54) with respect to x, we obtain the identity

$$\frac{\phi''}{\phi} = \frac{dz}{dx} + z^2(x). \tag{55}$$

Substituting (55) and (54) into (51), we obtain,

$$\frac{dz}{dx} = -(z^2(x) + \alpha(x)z(x) + \beta(x)). \tag{56}$$

Introducing the notation,

$$z(x) = \gamma(x)\tan(\theta(x)), \tag{57}$$

with θ, γ two unknown functions, we differentiate (57) and observe that,

$$\frac{dz}{dx} = \gamma(x)\frac{\theta'}{\cos^2(\theta)} + \gamma'(x)\tan(\theta(x)) \tag{58}$$

and squaring both sides of (57), we obtain

$$z(x)^2 = \tan^2(\theta(x))\gamma(x)^2. \tag{59}$$

Substituting (58) and (59) into (56) and choosing

$$\gamma(x) = \sqrt{\beta(x)} \tag{60}$$

we obtain

$$\frac{d\theta}{dx} = -\sqrt{\beta(x)} - \left(\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2}\right) sin(2\theta). \tag{61}$$

■

Remark 2.5 *The Prüfer Transform is often used in algorithms for finding the roots of oscillatory special functions. Suppose that $\phi : [a, b] \to \mathbb{R}$ is a special function satisfying differential equation (34). It turns out that in most cases, coefficient*

$$\beta(x) \tag{62}$$

in (51) is significantly larger than

$$\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2} \tag{63}$$

on the interval $[a,b]$*, where* α *and* β *are defined in (51).*

Under these conditions, the function θ *(see (53)), is monotone and its derivative neither approaches infinity nor* 0*. Furthermore, finding the roots of* ϕ *is equivalent to finding* $x \in [a,b]$ *such that*

$$\theta(x) = \pi/2 + k\pi \tag{64}$$

for some integer k*. Consequently, we can find the roots of* φ *by solving (61), a well-behaved differential equation.*

Remark 2.6 *If for all* $x \in [a,b]$*, the function* $\sqrt{\beta(x)}$ *satisfies*

$$\sqrt{\beta(x)} > \frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2}, \tag{65}$$

then, for all $x \in [a,b]$*, we have* $\frac{d\theta}{dx} < 0$ *(see (52)) and we can view* $x : [-\pi, \pi] \to \mathbb{R}$ *as a function of* θ *where* x *satisfies the first order differential equation*

$$\frac{dx}{d\theta} = \left(-\sqrt{\beta(x)} - \left(\frac{\beta'(x)}{4\beta(x)} + \frac{\alpha(x)}{2} \right) sin(2\theta) \right)^{-1}. \tag{66}$$

2.6 Miscellaneous Analytical Facts

In this section, we provide several facts from analysis that will by used in subsequent sections.

The following theorem is an identity involving the incomplete beta function.

Theorem 2.16 *Suppose that* $a, b > 0$ *are real numbers and* n *is a nonnegative integer. Then*

$$B_x(a+n,b) = \frac{\Gamma(a+n)}{\Gamma(a+b+n)} \left(\frac{\Gamma(a+b)}{\Gamma(a)} B_x(a,b) - (1-x)^b \sum_{k=1}^{n} \frac{\Gamma(a+b+k-1)}{\Gamma(a+k)} x^{a+k-1} \right) \tag{67}$$

for all $0 \le x \le 1$*, where* $B_x(a,b)$ *denotes the incomplete beta function.*

The following lemma is an identity involving the gamma function.

Lemma 2.17 *Suppose that n is a nonnegative integer. Then*

$$\sqrt{\pi} + \sum_{k=1}^{n} \frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)} = \frac{2\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}. \tag{68}$$

The following two lemmas are identities involving the incomplete beta function.

Lemma 2.18 *Suppose that $0 \le r \le 1$. Then*

$$B_{1-r^2}(1, \tfrac{1}{2}) = 2(1-r). \tag{69}$$

Lemma 2.19 *Suppose that $0 \le r \le 1$. Then*

$$B_{1-r^2}(\tfrac{1}{2}, \tfrac{1}{2}) = 2\arccos(r). \tag{70}$$

2.6.1 Bessel Functions

The primary analytical tool of this subsection is Theorem 2.25.

The following lemmas 2.20, 2.21, 2.22, 2.23, 2.24 describe the limiting behavior of certain integrals involving Bessel functions.

Lemma 2.20 *Suppose that $\nu > 0$. Then*

$$\int_0^1 (J_\nu(2cr))^2 \frac{1}{r}\, dr = \frac{1}{2\nu} + O\Big(\frac{1}{c}\Big), \tag{71}$$

as $c \to \infty$.

Lemma 2.21 *Suppose that $\nu > 0$. Then*

$$\int_0^1 (J_\nu(2cr))^2\, dr = \frac{1}{2\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big), \tag{72}$$

as $c \to \infty$.

Lemma 2.22 *Suppose that $\nu > 0$ is real and k is a positive integer. Then*

$$\int_0^1 (J_\nu(2cr))^2 r^k\, dr = O\Big(\frac{1}{c}\Big), \tag{73}$$

as $c \to \infty$.

Lemma 2.23 *Suppose that n is a positive integer. Then*

$$\int_0^1 \frac{(J_n(2cr))^2}{r} \arccos(r)\,dr = \frac{\pi}{4n} - \frac{1}{2\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big), \tag{74}$$

as $c \to \infty$.

Lemma 2.24 *Suppose that n and k are positive integers. Then*

$$\int_0^1 (J_n(2cr))^2(1-r^2)^{k-\frac{1}{2}}\,dr = \frac{1}{2\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big), \tag{75}$$

as $c \to \infty$.

The following theorem describes the limiting behavior of a certain integral involving a Bessel function and the incomplete beta function.

Theorem 2.25 *Suppose that $p \geq -1$ is an integer. Then*

$$\int_0^1 \frac{(J_{p/2+1}(2cr))^2}{r} B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,dr = \frac{\sqrt{\pi}\,\Gamma(\frac{p}{2}+\frac{3}{2})}{(p+2)\Gamma(\frac{p}{2}+2)} - \frac{1}{\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big) \tag{76}$$

as $c \to \infty$, where $B_x(a,b)$ denotes the incomplete beta function.

Proof. Suppose that $p \geq -1$ is an odd integer, and let $n = \frac{p}{2}+\frac{1}{2}$. Then

$$\int_0^1 \frac{\big(J_{p/2+1}(2cr)\big)^2}{r} B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,dr = \int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r} B_{1-r^2}(1+n,\tfrac{1}{2})\,dr. \tag{77}$$

By Theorem 2.16 and Lemma 2.18, we observe that

$$\begin{aligned}
&\int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r} B_{1-r^2}(1+n,\tfrac{1}{2})\,dr \\
&= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r}\Big(\frac{\sqrt{\pi}}{2}B_{1-r^2}(1,\tfrac{1}{2}) - r\sum_{k=1}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}(1-r^2)^k\Big)\,dr \\
&= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r}\Big(\sqrt{\pi}(1-r) - r\sum_{k=1}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}(1-r^2)^k\Big)\,dr, \qquad (78)
\end{aligned}$$

where $0 \leq r \leq 1$ and n is a nonnegative integer. By lemmas 2.20, 2.21, and 2.22, it follows that

$$\begin{aligned}
&\int_0^1 \frac{\big(J_{n+1/2}(2cr)\big)^2}{r} B_{1-r^2}(1+n,\tfrac{1}{2})\,dr \\
&= \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\Big(\frac{\sqrt{\pi}}{2n+1} - \frac{1}{2\pi}\Big(\sqrt{\pi}+\sum_{k=1}^{n}\frac{\Gamma(k+\frac{1}{2})}{\Gamma(k+1)}\Big)\frac{\log(c)}{c}\Big) + o\Big(\frac{\log(c)}{c}\Big), \qquad (79)
\end{aligned}$$

as $c \to \infty$, where $0 \le r \le 1$ and n is a nonnegative integer. Applying Lemma 2.17,

$$\begin{aligned}
&\int_0^1 (J_{n+1/2}(2cr))^2 B_{1-r^2}(1+n, \tfrac{1}{2})\,dr \\
&\quad = \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})}\left(\frac{\sqrt{\pi}}{2n+1} - \frac{1}{\pi}\frac{\Gamma(n+\frac{3}{2})}{\Gamma(n+1)}\frac{\log(c)}{c}\right) + o\left(\frac{\log(c)}{c}\right) \\
&\quad = \frac{\sqrt{\pi}\,\Gamma(n+1)}{2(n+\frac{1}{2})\Gamma(n+\frac{3}{2})} - \frac{1}{\pi}\frac{\log(c)}{c} + o\left(\frac{\log(c)}{c}\right),
\end{aligned} \tag{80}$$

as $c \to \infty$, where $0 \le r \le 1$ and n is a nonnegative integer. Therefore,

$$\int_0^1 \frac{(J_{p/2+1}(2cr))^2}{r} B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,dr = \frac{\sqrt{\pi}\,\Gamma(\frac{p}{2}+\frac{3}{2})}{(p+2)\Gamma(\frac{p}{2}+2)} - \frac{1}{\pi}\frac{\log(c)}{c} + o\left(\frac{\log(c)}{c}\right), \tag{81}$$

as $c \to \infty$, for all $0 \le r \le 1$ and odd integers $p \ge -1$.

The proof in the case when $p \ge 0$ is an even integer is essentially identical.

■

2.6.2 The Area and Volume of a Hypersphere

The following theorem provides well-known formulas for the volume and area of a $(p+2)$-dimensional hypersphere. The formulas can be found in, for example, [4].

Theorem 2.26 *Suppose that $S^{p+2}(r) = \{x \in \mathbb{R}^{p+2} : \|x\| = r\}$ denotes the $(p+2)$-dimensional hypersphere of radius $r > 0$. Suppose further that $A_{p+2}(r)$ denotes the area of $S^{p+2}(r)$ and $V_{p+2}(r)$ denotes the volume enclosed by $S^{p+2}(r)$. Then*

$$A_{p+2}(r) = \frac{2\pi^{p/2+1}}{\Gamma(\frac{p}{2}+1)} r^{p+1}, \tag{82}$$

and

$$V_{p+2}(r) = \frac{\pi^{p/2+1}}{\Gamma(\frac{p}{2}+2)} r^{p+2}. \tag{83}$$

The following theorem provides a formula for the volume of the intersection of two $(p+2)$-dimensional hyperspheres (see, for example, [4]).

Theorem 2.27 *Suppose that $p \ge -1$ is an integer, let B denote the closed unit ball in $\mathbb{R}^{p+2}$, and let $B(c)$ denote the set $\{x \in \mathbb{R}^{p+2} : \|x\| \le c\}$, where $c > 0$. Then*

$$\int_{\mathbb{R}^D} \mathbb{1}_B(u-t)\mathbb{1}_B(t)\,dt = V_{p+2}(1)\frac{B_{1-\|u\|^2/4}(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}, \tag{84}$$

for all $u \in B(2)$, where $B(a,b)$ denotes the beta function, $B_x(a,b)$ denotes the incomplete beta function, V_{p+2} is defined by (83), and $\mathbb{1}_A$ is defined via the formula

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{85}$$

2.7 Spherical Harmonics in $\mathbb{R}^{p+2}$

Suppose that S^{p+1} denotes the unit sphere in $\mathbb{R}^{p+2}$. The spherical harmonics are a set of real-valued continuous functions on S^{p+1}, which are orthonormal and complete in $L^2(S^{p+1})$. The spherical harmonics of degree $N \geq 0$ are denoted by $S_N^1, S_N^2, \ldots, S_N^\ell, \ldots, S_N^{h(N)} \colon S^{p+1} \to \mathbb{R}$, where

$$h(N) = (2N+p)\frac{(N+p-1)!}{p!\,N!}, \tag{86}$$

for all nonnegative integers N.

The following theorem defines the spherical harmonics as the values of certain harmonic, homogeneous polynomials on the sphere (see, for example, [2]).

Theorem 2.28 *For each spherical harmonic S_N^ℓ, where $N \geq 0$ and $1 \leq \ell \leq h(N)$ are integers, there exists a polynomial $K_N^\ell \colon \mathbb{R}^{p+2} \to \mathbb{R}$ which is harmonic, i.e.*

$$\nabla^2 K_N^\ell(x) = 0, \tag{87}$$

for all $x \in \mathbb{R}^{p+2}$, and homogenous of degree N, i.e.

$$K_N^\ell(\lambda x) = \lambda^N K_N^\ell(x), \tag{88}$$

for all $x \in \mathbb{R}^{p+2}$ and $\lambda \in \mathbb{R}$, such that

$$S_N^\ell(\xi) = K_N^\ell(\xi), \tag{89}$$

for all $\xi \in S^{p+1}$.

The following lemma follows immediately from the orthonormality of spherical harmonics and Theorem 2.28.

Lemma 2.29 *For all $N > 0$ and for all $1 \leq \ell \leq h(N)$,*

$$\int_{S^{p+1}} S_N^\ell(x)dx = 0. \tag{90}$$

For $N = 0$ and $\ell = 1$, S_N^ℓ is the constant function defined by the formula

$$S_0^1(x) = A_{p+2}(1)^{(-1/2)} \tag{91}$$

where A_{p+2} is defined in (82).

The following theorem is proved in, for example, [2].

Theorem 2.30 *Suppose that N is a nonnegative integer. Then there are exactly*

$$(2N+p)\frac{(N+p-1)!}{p!\,N!} \tag{92}$$

linearly independent, harmonic, homogenous polynomials of degree N in $\mathbb{R}^{p+2}$.

The following theorem states that for any orthogonal matrix U, the function $S_N^\ell(U\xi)$ is expressible as a linear combination of $S_N^1(\xi), S_N^2(\xi), \ldots, S_N^{h(N)}(\xi)$ (see, for example, [2]).

Theorem 2.31 *Suppose that N is a nonnegative integer, and that $S_N^1, S_N^2, \ldots, S_N^{h(N)} \colon S^{p+1} \to \mathbb{R}$ are a complete set of orthonormal spherical harmonics of degree N. Suppose further that U is a real orthogonal matrix of dimension $p+2 \times p+2$. Then, for each integer $1 \le \ell \le h(N)$, there exist real numbers $v_{\ell,1}, v_{\ell,2}, \ldots, v_{\ell,h(N)}$ such that*

$$S_N^\ell(U\xi) = \sum_{k=1}^{h(N)} v_{\ell,k} S_N^k(\xi), \tag{93}$$

for all $\xi \in S^{p+1}$. Furthermore, if V is the $h(N) \times h(N)$ real matrix with elements $v_{i,j}$ for all $1 \le i,j \le h(N)$, then V is also orthogonal.

Remark 2.7 *From Theorem (2.31), we observe that the space of linear combinations of functions S_N^ℓ is invariant under all rotations and reflections of S^{p+1}.*

The following theorem states that if an integral operator acting on the space of functions $S^{p+1} \to \mathbb{R}$ has a kernel depending only on the inner product, then the spherical harmonics are eigenfunctions of that operator (see, for example, [2]).

Theorem 2.32 (Funk-Hecke) *Suppose that $F \colon [-1,1] \to \mathbb{R}$ is a continuous function, and that $S_N \colon S^{p+1} \to \mathbb{R}$ is any spherical harmonic of degree N. Then*

$$\int_\Omega F(\langle \xi, \eta \rangle) S_N(\xi)\, d\Omega(\xi) = \lambda_N S_N(\eta), \tag{94}$$

for all $\eta \in S^{p+1}$, where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^{p+2}$, the integral is taken over the whole area of the hypersphere Ω, and λ_N depends only on the function F.

2.8 Generalized Prolate Spheroidal Functions

2.8.1 Basic Facts

In this section, we summarize several facts about generalized prolate spheroidal functions (GPSFs). Let B denote the closed unit ball in $\mathbb{R}^{p+2}$. Given a real number $c > 0$, we define the operator $F_c \colon L^2(B) \to L^2(B)$ via the formula

$$F_c[\psi](x) = \int_B \psi(t) e^{ic\langle x,t\rangle}\, dt, \tag{95}$$

for all $x \in B$, where $\langle \cdot, \cdot \rangle$ denotes the inner product on $\mathbb{R}^{p+2}$. Clearly, F_c is compact. Obviously, F_c is also normal, but not self-adjoint. We denote the eigenvalues of F_c by $\lambda_0, \lambda_1, \dots, \lambda_n, \dots$, and assume that $|\lambda_j| \ge |\lambda_{j+1}|$ for each non-negative integer j. For each non-negative integer j, we denote by ψ_j the eigenfunction corresponding to λ_j, so that

$$\lambda_j \psi_j(x) = \int_B \psi_j(t) e^{ic\langle x,t\rangle}\, dt, \tag{96}$$

for all $x \in B$. We assume that $\|\psi_j\|_{L^2(B)} = 1$ for each j. The following theorem is proved in [1] and describes the eigenfunctions and eigenvalues of F_c.

Theorem 2.33 *Suppose that $c > 0$ is a real number and that F_c is defined by (95). Then the eigenfunctions $\psi_0, \psi_1, \dots, \psi_n, \dots$ of F_c are real, orthonormal, and complete in $L^2(B)$. For each j, the eigenfunction ψ_j is either even, in the sense that $\psi_j(-x) = \psi_j(x)$ for all $x \in B$, or odd, in the sense that $\psi_j(-x) = -\psi_j(x)$ for all $x \in B$. The eigenvalues corresponding to even eigenfunctions are real, and the eigenvalues corresponding to odd eigenfunctions are purely imaginary. The domain on which the eigenfunctions are defined can be extended from B to $\mathbb{R}^{p+2}$ by requiring that (96) hold for all $x \in \mathbb{R}^{p+2}$; the eigenfunctions will then be orthogonal on $\mathbb{R}^{p+2}$ and complete in the class of band-limited functions with bandlimit c.*

We define the self-adjoint operator $Q_c \colon L^2(B) \to L^2(B)$ via the formula

$$Q_c = \left(\frac{c}{2\pi}\right)^{p+2} F_c^* \cdot F_c. \tag{97}$$

Since F_c is normal, it follows that Q_c has the same eigenfunctions as F_c, and that the jth eigenvalue μ_j of Q_c is connected to λ_j via the formula

$$\mu_j = \left(\frac{c}{2\pi}\right)^{p+2} |\lambda_j|^2. \tag{98}$$

We also observe that

$$Q_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p/2+1} \int_B \frac{J_{p/2+1}\big(c\|x-t\|\big)}{\|x-t\|^{p/2+1}} \psi(t)\, dt, \tag{99}$$

for all $x \in \mathbb{R}^{p+2}$, where J_ν denotes the Bessel functions of the first kind and $\|\cdot\|$ denotes Euclidean distance in $\mathbb{R}^{p+2}$ (see Appendix A for a proof).

We observe that

$$Q_c[\psi](x) = \mathbb{1}_B(x) \cdot \mathcal{F}^{-1}\big[\mathbb{1}_{B(c)}(t) \cdot \mathcal{F}[\psi](t)\big](x), \tag{100}$$

where $\mathcal{F}\colon L^2(\mathbb{R}^{p+2}) \to L^2(\mathbb{R}^{p+2})$ is the $(p+2)$-dimensional Fourier transform, $B(c)$ denotes the set $\{\, x \in \mathbb{R}^{p+2} : \|x\| \le c \,\}$, and $\mathbb{1}_A$ is defined via the formula

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \tag{101}$$

From (100) it follows that $\mu_j < 1$ for all j.

We observe further that Q_c is closely related to the operator $P_c\colon L^2(\mathbb{R}^{p+2}) \to L^2(\mathbb{R}^{p+2})$, defined via the formula

$$P_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p/2+1} \int_{\mathbb{R}^{p+2}} \frac{J_{p/2+1}\big(c\|x-t\|\big)}{\|x-t\|^{p/2+1}} \psi(t)\, dt, \tag{102}$$

which is the orthogonal projection onto the space of bandlimited functions on $\mathbb{R}^{p+2}$ with bandlimit $c > 0$.

2.8.2 Eigenfunctions and Eigenvalues of F_c

In this section we describe the eigenvectors and eigenvalues of the operator F_c, defined in (95). Suppose that ψ is some eigenfunction of the integral operator F_c, with corresponding complex eigenvalue λ, so that

$$\lambda\psi(x) = \int_B \psi(t) e^{ic\langle x,t\rangle}\, dt, \tag{103}$$

for all $x \in B$ (see Theorem 2.33).

Observation 2.8 *The operator F_c, defined by* (95)*, is spherically symmetric in the sense that, for any $(p+2) \times (p+2)$ orthogonal matrix U, F_c commutes with the operator $\hat{U}\colon L^2(B) \to L^2(B)$, defined via the formula*

$$\hat{U}[\psi](x) = \psi(Ux), \tag{104}$$

for all $x \in B$. Hence, the problem of finding the eigenfunctions and eigenvalues of F_c is amenable to the separation of variables.

Suppose that

$$\psi(x) = \Phi_N^\ell(\|x\|) S_N^\ell(x/\|x\|), \tag{105}$$

where S_N^ℓ, $\ell = 0, 1, \dots, h(N, p)$ denotes the spherical harmonics of degree N (see Section 2.7), and $\Phi_N^\ell(r)$ is a real-valued function defined on the interval $[0, 1]$. We observe that

$$e^{ic\langle x,t\rangle} = \sum_{N=0}^{\infty} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c\|x\|\|t\|)}{(c\|x\|\|t\|)^{p/2}} S_N^\ell(x/\|x\|) S_N^\ell(t/\|t\|), \tag{106}$$

where $x, t \in B$, and where J_ν denotes the Bessel functions of the first kind (see Section VII of [1] for a proof). Substituting (105) and (106) into (103), we find that

$$\lambda \Phi_N^\ell(r) = i^N (2\pi)^{p/2+1} \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \Phi_N^\ell(\rho) \rho^{p+1}\, d\rho, \tag{107}$$

for all $0 \le r \le 1$. We define the operator $H_{N,c} \colon L^2\big([0,1], \rho^{p+1}\, d\rho\big) \to L^2\big([0,1], \rho^{p+1}\, d\rho\big)$ via the formula

$$H_{N,c}[\Phi](r) = \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \Phi(\rho) \rho^{p+1}\, d\rho, \tag{108}$$

where $0 \le r \le 1$, and observe that $H_{N,c}$ is clearly compact and self-adjoint, and does not depend on ℓ. Dropping the index ℓ, we denote by $\beta_{N,0}, \beta_{N,1}, \dots, \beta_{N,n}, \dots$ the eigenvalues of $H_{N,c}$, and assume that $|\beta_{N,n}| \ge |\beta_{N,n+1}|$ for each nonnegative integer n. For each nonnegative integer n, we let $\Phi_{N,n}$ denote the eigenvector corresponding to eigenvalue $\beta_{N,n}$, so that

$$\beta_{N,n} \Phi_{N,n}(r) = \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \Phi_{N,n}(\rho) \rho^{p+1}\, d\rho, \tag{109}$$

for all $0 \le r \le 1$. Clearly, the eigenfunctions $\Phi_{N,n}$ are purely real. We assume that $\|\Phi_{N,n}\|_{L^2([0,1],\rho^{p+1}\, d\rho)} = 1$ and that $\Phi_{N,n}(1) > 0$ for each nonnegative integer N and n (see Theorem 8.6). It follows from (109) and (107) that the eigenvectors and eigenvalues of F_c are given by the formulas

$$\psi_{N,n}^\ell(x) = \Phi_{N,n}(\|x\|) S_N^\ell(x/\|x\|), \tag{110}$$

and

$$\lambda_{N,n}^\ell = i^N (2\pi)^{p/2+1} \beta_{N,n}, \tag{111}$$

respectively, where $x \in B$, N and n are nonnegative integers, and ℓ is an integer so that $1 \le \ell \le h(N, p)$ (see Section 2.7). We note in formula (111) the expected degeneracy of eigenvalues due to the spherical symmetry of the integral operator F_c (see Observation 2.8); we denote $\lambda_{N,n}^\ell$ by $\lambda_{N,n}$ where the meaning is clear. We also make the following observation.

Observation 2.9 *The domain on which the functions $\Phi_{N,n}$ are defined may be extended from the interval $[0,1]$ to the complex plane $\mathbb{C}$ by requiring that (103) hold for all $r \in \mathbb{C}$. Moreover, the functions $\Phi_{N,n}$, extended in this way, are entire.*

2.8.3 The Dual Nature of GPSFs

In this section, we observe that the eigenfunctions $\Phi_{N,0}, \Phi_{N,1}, \ldots, \Phi_{N,n}, \ldots$ of the integral operator $H_{N,c}$, defined in (108), are also the eigenfunctions of a certain differential operator.

Let $\beta_{N,n}$ denote the eigenvalue corresponding to the eigenfunction $\Phi_{N,n}$, for all nonnegative integers N and n, so that

$$\beta_{N,n}\Phi_{N,n}(r) = \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \Phi_{N,n}(\rho)\rho^{p+1}\, d\rho, \tag{112}$$

where $0 \le r \le 1$, N and n are nonnegative integers, and J_ν denotes the Bessel functions of the first kind (see (109)). Making the substitutions

$$\varphi_{N,n}(r) = r^{(p+1)/2}\Phi_{N,n}(r), \tag{113}$$

and

$$\gamma_{N,n} = c^{(p+1)/2}\beta_{N,n}, \tag{114}$$

we observe that

$$\gamma_{N,n}\varphi_{N,n}(r) = \int_0^1 J_{N+p/2}(cr\rho)\sqrt{cr\rho}\,\varphi_{N,n}(\rho)\, d\rho, \tag{115}$$

where $0 \le r \le 1$, and N and n are arbitrary nonnegative integers. We define the operator $M_{N,c} \colon L^2([0,1]) \to L^2([0,1])$ via the formula

$$M_{N,c}[\varphi](r) = \int_0^1 J_{N+p/2}(cr\rho)\sqrt{cr\rho}\,\varphi(\rho)\, d\rho, \tag{116}$$

where $0 \le r \le 1$, and N is an arbitrary nonnegative integer. Obviously, $M_{N,c}$ is compact and self-adjoint. Clearly, the eigenvalues of $M_{N,c}$ are $\gamma_{N,0}, \gamma_{N,1}, \ldots, \gamma_{N,n}, \ldots$, and $\varphi_{N,n}$ is the eigenfunction corresponding to eigenvalue $\gamma_{N,n}$, for each nonnegative integer n.

We define the differential operator $L_{N,c}$ via the formula

$$L_{N,c}[\varphi](x) = \frac{d}{dx}\left((1-x^2)\frac{d\varphi}{dx}(x)\right) + \left(\frac{\frac{1}{4} - (N+\frac{p}{2})^2}{x^2} - c^2x^2\right)\varphi(x), \tag{117}$$

where $0 < x < 1$, N is a nonnegative integer, and φ is twice continuously differentiable. Let C be the class of functions φ which are bounded and twice continuously differentiable on the interval $(0, 1)$, such that $\varphi'(0) = 0$ if $p = -1$ and $N = 0$, and $\varphi(0) = 0$ otherwise. Then it is easy to show that, operating on functions in class C, $L_{N,c}$ is self-adjoint. From Sturmian theory we obtain the following theorem (see [1]).

Theorem 2.34 *Suppose that $c > 0$, N is a nonnegative integer, and $L_{N,c}$ is defined via* (117)*. Then there exists a strictly increasing unbounded sequence of positive numbers $\chi_{N,0} < \chi_{N,1} < \ldots$ such that for each nonnegative integer n, the differential equation*

$$L_{N,c}[\varphi](x) + \chi_{N,n}\varphi(x) = 0 \tag{118}$$

has a solution which is bounded and twice continuously differentiable on the interval $(0, 1)$*, so that $\varphi'(0) = 0$ if $p = -1$ and $N = 0$, and $\varphi(0) = 0$ otherwise.*

The following theorem is proved in [1].

Theorem 2.35 *Suppose that $c > 0$, N is a nonnegative integer, and the operators $M_{N,c}$ and $L_{N,c}$ are defined via* (116) *and* (117) *respectively. Suppose also that $\varphi\colon (0, 1) \to \mathbb{R}$ is in $L^2([0, 1])$, is twice differentiable, and that $\varphi'(0) = 0$ if $p = -1$ and $N = 0$, and $\varphi(0) = 0$ otherwise. Then*

$$L_{N,c}\big[M_{N,c}[\varphi]\big](x) = M_{N,c}\big[L_{N,c}[\varphi]\big](x), \tag{119}$$

for all $0 < x < 1$.

Remark 2.10 *Since Theorem 2.34 shows that the eigenvalues of $L_{N,c}$ are not degenerate, Theorem 2.35 implies that $L_{N,c}$ and $M_{N,c}$ have the same eigenfunctions.*

2.8.4 Zernike Polynomials and GPSFs

In this section we describe the relationship between Zernike polynomials and GPSFs. We use $\varphi^c_{N,n}$, where $c > 0$ and N and n are arbitrary nonnegative integers, to denote the nth eigenfunction of $L_{N,c}$, defined in (117); we denote by $\chi_{N,n}(c)$ the eigenvalue corresponding to eigenfunction $\varphi^c_{N,n}$.

For $c = 0$, the eigenfunctions and eigenvalues of the differential operator $L_{N,c}$, defined in (117), are

$$\overline{T}_{N,n}(x) \tag{120}$$

and

$$\chi_{N,n}(0) = (N + \tfrac{p}{2} + 2n + \tfrac{1}{2})(N + \tfrac{p}{2} + 2n + \tfrac{3}{2}), \tag{121}$$

respectively, where $0 \le x \le 1$, N and n are arbitrary nonnegative integers, and $\overline{T}_{N,n}$ is defined in (34).

For small $c > 0$, the connection between Zernike polynomials and GPSFs is given by the formulas

$$\varphi^c_{N,n}(x) = \overline{T}_{N,n}(x) + o(c^2), \tag{122}$$

and

$$\chi_{N,n}(c) = \chi_{N,n}(0) + o(c^2), \tag{123}$$

as $c \to 0$, where $0 \le x \le 1$ and N and n are arbitrary nonnegative integers (see [1]).

For $c > 0$, the functions $T_{N,n}$ are also related to the integral operator $M_{N,c}$, defined in (116), via the formula

$$M_{N,c}\big[T_{N,n}\big](x) = \int_0^1 J_{N+p/2}(cxy)\sqrt{cxy}\, T_{N,n}(y)\, dy = \frac{(-1)^n J_{N+p/2+2n+1}(cx)}{\sqrt{cx}}, \tag{124}$$

where $x \ge 0$ and N and n are arbitrary nonnegative integers (see Equation (85) in [10]).

3 Analytical Apparatus

In this section, we provide analytical apparatus relating to GPSFs that will be used in numerical schemes in subsequent sections.

3.1 Properties of GPSFs

The following theorem provides a formula for ratios of eigenvalues $\beta_{N,n}$ (see (109)), and finds use in the numerical evaluation of $\beta_{N,n}$. A proof follows immediately from Theorem 7.1 of [15].

Theorem 3.1 *Suppose that N is a nonnegative integer. Then*

$$\frac{\beta_{N,m}}{\beta_{N,n}} = \frac{\int_0^1 x\Phi'_{N,n}(x)\Phi_{N,m}(x)x^{p+1}\, dx}{\int_0^1 x\Phi'_{N,m}(x)\Phi_{N,n}(x)x^{p+1}\, dx}, \tag{125}$$

for each nonnegative integers n and m.

3.2 Decay of the Expansion Coefficients of GPSFs into Zernike Polynomials

Since the functions $\Phi_{N,n}$ are analytic on $\mathbb{C}$ for all nonnegative integers N and n (see Observation 2.9), and $\Phi^{(k)}_{N,n}(0) = 0$ for $k = 0, 1, \ldots, N-1$ (see Theorem 8.5), the functions $\Phi_{N,n}$ are representable by a series of Zernike polynomials of the form

$$\Phi_{N,n}(r) = \sum_{k=0}^{\infty} a_{n,k} \overline{R}_{N,k}(r), \tag{126}$$

for all $0 \le r \le 1$, where $a_{n,0}, a_{n,1}, \ldots$ satisfy

$$a_{n,k} = \int_0^1 \overline{R}_{N,k}(r) \Phi_{N,n}(r) dr \tag{127}$$

where $\overline{R}_{N,n}$ is defined in (23). The following technical lemma will be used in the proof of Theorem 3.3.

Lemma 3.2 *For any integer $p \ge -1$, for all $c > 0$, and for all $\rho \in [0, 1]$,*

$$\left| \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \overline{R}_{N,k}(r) r^{p+1} dr \right| < \left(\frac{1}{2}\right)^{N+p/2+2k+1} \tag{128}$$

for any non-negative integers N, k such that $N + 2k \ge ec$ where $\overline{R}_{N,n}$ is defined in (23) and $J_{N+p/2}$ is a Bessel function of the first kind.

Proof. According to equation (85) in [10],

$$\int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} R_{N,k}(r) r^{p+1} dr = \frac{(-1)^n J_{N+p/2+2k+1}(c\rho)}{(c\rho)^{p/2+1}}, \tag{129}$$

where $J_{N+p/2}$ is a Bessel function of the first kind. Applying Lemma 2.2 to (129), we obtain

$$\left| \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \overline{R}_{N,k}(r) r^{p+1} dr \right| \le \frac{(c\rho/2)^{N+p/2+2k+1}}{(c\rho)^{p/2+1}} \frac{\sqrt{2(N + p/2 + 2k + 1)}}{\Gamma(N + p/2 + 2k + 2)}. \tag{130}$$

Combining Lemma 2.1 and (130), we have

$$\begin{aligned} \left| \int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}} \overline{R}_{N,k}(r) r^{p+1} dr \right| &\le \left(\frac{1}{2}\right)^{N+p/2+2k+1} (c\rho)^{N+2k} \frac{\sqrt{2(2k+N)}}{\Gamma(2k+N+1)} \\ &\le \left(\frac{1}{2}\right)^{N+p/2+2k+1} \end{aligned}$$

(131)

for $N + 2k \geq ec$. ■

The following theorem shows that the coefficients $a_{N,k}$ of GPSFs in a Zernike polynomial basis decay exponentially and establishes a bound for the decay rate.

Theorem 3.3 *For all non-negative integers N, n, k and for all $c > 0$,*

$$\int_0^1 \Phi_{N,n}(r)\overline{R}_{N,k}r^{p+1}dr < (p+2)^{-1/2}(\beta_{N,n})^{-1}\left(\frac{1}{2}\right)^{N+p/2+2k+1} \tag{132}$$

where $N + 2k \geq ec$.

Proof. Combining (112) and (127),

$$\int_0^1 \Phi_{N,n}(r)\overline{R}_{N,k}r^{p+1} = \int_0^1 (\beta_{N,n})^{-1}\left(\int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}}\Phi_{N,n}(\rho)\rho^{p+1}d\rho\right)\overline{R}_{N,k}(r)r^{p+1}dr. \tag{133}$$

Changing the order of integration of (133),

$$\int_0^1 \Phi_{N,n}(r)\overline{R}_{N,k}r^{p+1} = (\beta_{N,n})^{-1}\int_0^1 \Phi_{N,n}(\rho)\rho^{p+1}\int_0^1 \frac{J_{N+p/2}(cr\rho)}{(cr\rho)^{p/2}}\overline{R}_{N,k}(r)r^{p+1}drd\rho. \tag{134}$$

Applying Lemma 3.2 to (134) and applying Cauchy-Schwarz, we obtain

$$\begin{aligned}\int_0^1 \Phi_{N,n}(r)\overline{R}_{N,k}r^{p+1} &\leq (\beta_{N,n})^{-1}\left(\frac{1}{2}\right)^{N+p/2+2k+1}\int_0^1 \Phi_{N,n}(\rho)\rho^{p+1}d\rho \\ &\leq (p+2)^{-1/2}(\beta_{N,n})^{-1}\left(\frac{1}{2}\right)^{N+p/2+2k+1}.\end{aligned} \tag{135}$$

for $N + 2k \geq ec$. ■

3.3 Tridiagonal Nature of $L_{N,c}$

In this section, we show that in the basis of $\overline{T}_{N,n}$ (see (34)), the matrix representing differential operator $L_{N,c}$ (see (117)) is symmetric and tridiagonal.

The following lemma provides an identity relating the differential operator $L_{N,c}$ to $\overline{T}_{N,n}$.

Lemma 3.4 *For all non-negative integers N, n and real numbers $c > 0$*

$$L_{N,c}[\overline{T}_{N,n}] = -\chi_{N,n}\overline{T}_{N,n}(x) - c^2x^2\overline{T}_{N,n}(x) \tag{136}$$

for all $x \in [0,1]$ where $\chi_{N,n}$ is defined in (40) and $L_{N,c}$ is defined in (117).

Proof. Applying $L_{N,c}$ to $\overline{T}_{N,n}$, we obtain

$$L_{N,c}[\overline{T}_{N,n}](x) = (1-x^2)\overline{T}''_{N,n}(x) - 2x\overline{T}_{N,n}(x) + \left(\frac{\frac{1}{4} - (N+\frac{p}{2})^2}{x^2} - c^2x^2\right)\overline{T}_{N,n}(x). \tag{137}$$

Identity (136) follows immediately from the combination of (39) and (137). ■

The following theorem follows readily from the combination of Lemma 3.4 and Lemma 2.12.

Theorem 3.5 *For any non-negative integer N, any integer $n \geq 1$, and for all $r \in (0,1)$,*

$$L_{N,c}[\overline{T}_{N,n}] = a_n\overline{T}_{N,n-1}(r) + b_n\overline{T}_{N,n}(r) + c_n\overline{T}_{N,n+1}(r) \tag{138}$$

where

$$\begin{aligned} a_n &= \frac{-c^2(n+N+p/2)n}{(2n+N+p/2)\sqrt{2n+N+p/2+1}\sqrt{2n+N+p/2-1}} \\ b_n &= \frac{-c^2(N+p/2)^2}{2(2n+N+p/2)(2n+N+p/2+2)} - \frac{c^2}{2} + \chi_{N,n} \\ c_n &= \frac{-c^2(n+1+N+p/2)(n+1)}{(2n+N+p/2+2)\sqrt{2n+N+p/2+3}\sqrt{2n+N+p/2+1}} \end{aligned} \tag{139}$$

and $\chi_{N,n}$ is defined in (40).

Observation 3.1 *It follows immediately from Theorem 3.5 that the matrix corresponding to the differential operator $L_{N,c}$ acting on the $\overline{T}_{N,n}$ basis is symmetric and tridiagonal. Specifically, for any positive integer n and for all $r \in (0,1)$,*

$$\begin{bmatrix} b_0 & c_0 & & & & 0 \\ c_0 & b_1 & c_1 & & & \\ & c_1 & b_2 & c_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & c_{n-2} & b_{n-1} & c_{n-1} \\ 0 & & & & c_{n-1} & b_n \end{bmatrix} \begin{bmatrix} \overline{T}_{N,0}(r) \\ \\ \vdots \\ \\ \\ \overline{T}_{N,n}(r) \end{bmatrix} + \begin{bmatrix} 0 \\ \\ \vdots \\ \\ 0 \\ c_n\overline{T}_{N,n+1}(r) \end{bmatrix} = \begin{bmatrix} \overline{T}_{N,0}(r) \\ \\ \vdots \\ \\ \\ \overline{T}_{N,n}(r) \end{bmatrix} \tag{140}$$

where b_k and c_k are defined in (139) and $\overline{T}_{N,k}$ is defined in (34).

Observation 3.2 *Let A be the infinite symmetric tridiagonal matrix satisfying $A_{1,1} = b_0$, $A_{1,2} = c_0$ and for all integers $k \geq 2$,*

$$\begin{aligned} A_{k,k-1} &= c_{k-1} \\ A_{k,k} &= b_k \\ A_{k,k+1} &= c_k, \end{aligned} \tag{141}$$

where b_k, c_k are defined in (139). That is,

$$A = \begin{bmatrix} b_0 & c_0 & & & \\ c_0 & b_1 & c_1 & & \\ & c_1 & b_2 & c_2 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{142}$$

Suppose further that we define the infinite vector α_n by the equation

$$a_n = (a_{n,0}, a_{n,1}, ...)^T, \tag{143}$$

where $a_{n,k}$ is defined in (127). By the combination of Theorem 2.35 and Remark 23, we know that $\varphi_{N,n}$ is the eigenfunction corresponding to $\chi_{N,n}(c)$, the nth smallest eigenvalue of differential operator $L_{N,c}$. Therefore,

$$A\alpha_n = \chi_{N,n}(c)\alpha_n. \tag{144}$$

Furthermore, the $a_{n,k}$ decay exponentially fast in k (see Theorem 3.3).

Remark 3.3 *The eigenvalues $\chi_{N,n}$ of differential operator $L_{N,c}$ and the coefficients in the Zernike expansion of the eigenfunctions $\Phi_{N,n}$ can be computed numerically to high relative precision by the following process. First, we reduce the infinite dimensional matrix A (see (142)) to A_K, its upperleft $K \times K$ submatrix where K is chosen, using Theorem 3.3, so that $a_{n,K-1}$ is smaller than machine precision and is in the regime of exponential decay. We then use standard algorithms to find the eigenvalues and eigenvectors of matrix A_K. See Algorithm 4.1 for a more detailed description of the algorithm.*

4 Numerical Evaluation of GPSFs

In this section, we describe an algorithm (Algorithm 4.1) for the evaluation of $\Phi_{N,n}(r)$ (see (109)) for all $r \in [0, 1]$.

Algorithm 4.1

Step 1. Use Theorem 3.3 to determine how many terms are needed in a Zernike expansion of $\Phi_{N,n}$. We assume that we want a K term expansion.

Step 2. Generate A_K, the symmetric, tri-diagonal, upper-left $K \times K$ sub-matrix of A (see (142)).

Step 3. Use an eigensolver to find the eigenvector, $\tilde{a}_n$, corresponding to the $n+1^{\text{th}}$ largest eigenvalue, $\tilde{\chi}_{N,n}$. That is, find $\tilde{a}_n$ and $\tilde{\chi}_{N,n}$ such that

$$A_K \tilde{a}_n = \tilde{\chi}_{N,n} \tilde{a}_n \tag{145}$$

where we define the components of $\tilde{a}_{N,n}$ by the formula,

$$\tilde{a}_n = (a_{n,0}, a_{n,1}, ..., a_{n,K-1}). \tag{146}$$

Step 4. Evaluate $\Phi_{N,n}(r)$ by the expansion

$$\Phi_{N,n}(r) = \sum_{i=0}^{k} a_{n,i} \overline{R}_{N,i}(r) \tag{147}$$

where, $\overline{R}_{N,i}$ is evaluated via Lemma 2.8 and $a_{n,i}$ are the components of eigenvector (146) recovered in Step 3.

Remark 4.1 *It turns out that because of the structure of A_K, standard numerical algorithms will compute the components of eigenvector $\tilde{a}_n$ (see 146)), and thus the coefficients of a GPSF in a Zernike expansion, to high relative precision. In particular, the components of $\tilde{a}_n$ that are of magnitude far less than machine precision, are computed to high relative precision. For example, when using double-precision arithmetic, a component of $\tilde{a}_n$ of magnitude 10^{-100} will be computed in absolute precision to 116 digits. This fact is proved in a more general setting in [14].*

4.1 Numerical Evaluation of the Single Eigenvalue $\beta_{N,i}$

In this section, we describe a sum that can be used to evaluate the eigenvalue $\beta_{N,n}$ (see Theorem 109) for fixed n to high relative precision.

The following is a technical lemma will be used in the proof of Theorem 4.3.

Lemma 4.1 *For all non-negative integers N, k,*

$$\int_0^1 \rho^N \varphi_{N,k}(\rho) \rho^{\frac{p+1}{2}} d\rho = \frac{a_{k,0}}{\sqrt{2N+p+2}} \tag{148}$$

where $\varphi_{N,k}$ is defined in (113) and $p \geq -1$ is an integer.

Proof. Using (20),

$$\int_0^1 \rho^N \varphi_{N,k}(\rho)\rho^{\frac{p+1}{2}} d\rho = \int_0^1 R_{N,0}(\rho)\varphi_{N,k}(\rho)\rho^{\frac{p+1}{2}} d\rho \tag{149}$$

Applying (127) and (34) to (149), we obtain

$$\int_0^1 \rho^N \varphi_{N,k}(\rho)\rho^{\frac{p+1}{2}} d\rho = \frac{1}{\sqrt{2N+p+2}} \int_0^1 \overline{T}_{N,0}(\rho)\varphi_{N,k}(\rho) d\rho = \frac{a_{k,0}}{\sqrt{2N+p+2}}. \tag{150}$$

■

We will denote by $\varphi^*_{N,n}(r)$ the function on $[0,1]$ defined by the formula

$$\varphi^*_{N,n}(r) = \frac{\varphi_{N,n}(r)}{r^{N+\frac{p}{2}}} \tag{151}$$

where N, n are non-negative integers.

The following identity will be used in the proof of Theorem 4.3.

Lemma 4.2 *For all non-negative integers N, k,*

$$\varphi^*_{N,k}(0) = \sum_{i=0}^{\infty} a_{k,i} \sqrt{2(2i+N+p/2+1)} (-1)^i \binom{i+N+p/2}{i}. \tag{152}$$

where $\varphi^*_{N,k}$ is defined in (151) and $a_{k,i}$ is defined in (127).

Proof. Combining (151) and (45), we have

$$\varphi^*_{N,k}(r) = \frac{\varphi_{N,k}(r)}{r^{N+\frac{p}{2}}} = \sum_{i=0}^{\infty} a_{k,i} \frac{\overline{T}_{N,i}(r)}{r^{N+\frac{p}{2}}} = \sum_{i=0}^{\infty} a_{k,i} \overline{T}^*_{N,i}(r) \tag{153}$$

where $\overline{T}^*_{N,n}$ is defined in (45) and $\overline{T}_{N,n}$ is defined in (34). Identity (152) follows immediately from applying Lemma 2.13 to (153) and setting $r = 0$. ■

The following theorem provides a formula that can be used to compute $\beta_{N,n}$ (see (114)), an eigenvalue of integral operator $H_{N,c}$ (see (108).

Theorem 4.3 *For all non-negative integers N, k,*

$$\beta_{N,k} = \frac{a_{k,0} c^N (2^{N+p/2} \Gamma(N+p/2+1) \sqrt{2N+p+2})^{-1}}{\sum_{i=0}^{\infty} a_{k,i} \sqrt{2(2i+N+p/2+1)} (-1)^i \binom{i+N+p/2}{i}} \tag{154}$$

where $\beta_{N,k}$ is defined in (112) and $a_{k,i}$ are defined in (127).

Proof. It is well known that $J_{N+p/2}$, a Bessel Function of the first kind, satisfies the identity

$$J_{N+p/2} = \left(\frac{cr\rho}{2}\right)^{N+p/2} \sum_{k=0}^{\infty} \frac{(-(cr\rho)^2/4)^k}{k!\Gamma(N+p/2+k+1)} \tag{155}$$

where $\Gamma(n)$ is the gamma function. Dividing both sides of (115) by $r^{N+\frac{(p+1)}{2}}$, we obtain the equation

$$\gamma_{N,k}\varphi^*_{N,k}(r) = \int_0^1 \frac{J_{N+p/2}(cr\rho)}{r^{N+\frac{p}{2}}} \sqrt{c\rho}\varphi_{N,k}(\rho)d\rho \tag{156}$$

where $\varphi^*_{N,k}$ is defined in (151). Setting $r = 0$, in (156) and subsituting in (152) and (155), we obtain

$$\gamma_{N,k} = \int_0^1 \left(\frac{c\rho}{2}\right)^{N+p/2} \frac{(c\rho)^{1/2}}{\Gamma(N+p/2+1)} \varphi_{N,k}(\rho)d\rho \\ \left(\sum_{i=0}^{\infty} a_{k,i}\sqrt{2(2i+N+p/2+1)}(-1)^i \binom{i+N+p/2}{i}\right)^{-1}. \tag{157}$$

Equation (154) follows immediately from applying Lemma 4.1 and (114) to (157). ■

Remark 4.2 *For any non-negative integers N, k, the eigenvalue $\beta_{N,k}$ can be evaluated stably by first using Algorithm 4.1 to compute the eigenvector $\tilde{a}_k$ (see (146)), and then evaluating $\beta_{N,k}$ via sum (154) where $\tilde{a}_k$ are approximations to a_k. In (154), the sum*

$$\sum_{i=0}^{\infty} a_{k,i}\sqrt{2(2i+N+p/2+1)}(-1)^i \binom{i+N+p/2}{i} \tag{158}$$

can be computed to high relative precision by truncating the sum at a point when the partial sum up to that point is a factor of machine precision larger than the next term.

4.2 Numerical Evaluation of the Eigenvalues $\beta_{N,0}, \beta_{N,1}, ..., \beta_{N,k}$

In this section, we describe an algorithm for numerically evaluating the eigenvalues $\beta_{N,0}, \beta_{N,1}, ..., \beta_{N,k}$ (see (109)) for any non-negative integers N, k (see Algorithm 4.2).

In Observation 4.3, we describe a stable numerical scheme for converting an expansion of the form

$$\sum_{i=0}^{K} x_i r \overline{T}'_{N,i}(r), \tag{159}$$

where $x_0, ..., x_K$ are real numbers, into an expansion of the form

$$\sum_{i=0}^{K} \alpha_i \overline{T}_{N,i}(r) \tag{160}$$

where $\alpha_0, ..., \alpha_K$ are real numbers, $\overline{T}_{N,n}(r)$ is defined in (34), and $\overline{T}'_{N,n}(r)$ denotes the derivative of $\overline{T}_{N,n}(r)$ with respect to r.

Observation 4.3 *Fix $\epsilon > 0$ and let $x_0, ..., x_K$ be a sequence of real numbers such that*

$$\sum_{i=K_1+1}^{K} |x_k| < \epsilon \tag{161}$$

where $0 \le K_1 \le K$. Using (34), we have

$$\sum_{i=0}^{K} x_i \overline{T}_{N,n}(r) = \sum_{i=0}^{K} \alpha_i T_{N,n}(r) \tag{162}$$

where $x_0, ..., x_K$ are real numbers and α_i is defined by the formula

$$\alpha_i = x_i \sqrt{2(2i + N + p/2 + 1)}. \tag{163}$$

Scaling both sides of (49), we obtain

$$\begin{aligned} &\alpha_0 r T'_{N,0}(r) - \frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} r T'_{N,1}(r) + \frac{\alpha_0 \tilde{c}_1}{\tilde{a}_1} r T'_{N,2}(r) \\ &= \frac{\alpha_0 a_1}{\tilde{a}_1} T_{N,0}(r) - \frac{\alpha_0 b_1}{\tilde{a}_1} T_{N,1}(r) + \frac{\alpha_0 c_1}{\tilde{a}_1} T_{N,2}(r). \end{aligned} \tag{164}$$

where $a_i, b_i, c_i, \tilde{a}_i, \tilde{b}_i, \tilde{c}_i$ are defined in Lemma 2.14. Scaling (49) and adding the resulting *equation to (164), we obtain*

$$
\begin{aligned}
&\alpha_0 r T'_{N,0}(r) - \frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} r T'_{N,1}(r) + \frac{\alpha_0 \tilde{c}_1}{\tilde{a}_1} r T'_{N,2}(r) \\
&+ \left(\left(\frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} + \alpha_1 \right) \tilde{a}_2^{-1} \right) \left(\tilde{a}_2 r T'_{N,1}(r) - \tilde{b}_2 r T'_{N,2}(r) + \tilde{c}_2 r T'_{N,3}(r) \right) \\
&= \frac{\alpha_0 a_1}{\tilde{a}_1} T_{N,0}(r) - \frac{\alpha_0 b_1}{\tilde{a}_1} T_{N,1}(r) + \frac{\alpha_0 c_1}{\tilde{a}_1} T_{N,2}(r) \\
&+ \left(\left(\frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} + \alpha_1 \right) \tilde{a}_2^{-1} \right) \left(a_2 T_{N,1}(r) - b_2 T_{N,2}(r) + c_2 T_{N,3}(r) \right).
\end{aligned}
\tag{165}
$$

Simplying the left hand side of (165), we have

$$
\begin{aligned}
&\alpha_0 r T'_{N,0}(r) + \alpha_1 r T'_{N,1}(r) + \left(\frac{\alpha_0 \tilde{c}_1}{\tilde{a}_1} - \frac{\tilde{b}_2}{\tilde{a}_2} \left(\frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} + \alpha_1 \right) \right) r T'_{N,2}(r) \\
&+ \left(\left(\frac{\alpha_0 \tilde{b}_1}{\tilde{a}_1} + \alpha_1 \right) \tilde{a}_2^{-1} \right) \left(\tilde{c}_2 r T'_{N,3}(r) \right).
\end{aligned}
\tag{166}
$$

We continue by adding scaled versions of (49) to (165) until the expansion on the left hand side of (165) approximates (162). After $K_1 + 1$ steps, the new expansion will be accurate to approximately ϵ precision. Specifically, at the start of step k, for $2 \le k \le K_1 + 1$, we have

$$
\begin{aligned}
&\alpha_0 r T'_{N,0}(r) + \alpha_1 r T'_{N,1}(r) + ... + \alpha_{k-2} r T'_{N,k-2}(r) + x_{k-1} r T'_{N,k-1}(r) + x_k r T'_{N,k}(r) \\
&= y_0 T_{N,0} + y_1 T_{N,1} + ... + y_k T_{N,k}
\end{aligned}
\tag{167}
$$

where $x_{k-1}, x_k, x_{k+1}, y_0, y_1, ..., y_k$ are some real numbers. Scaling both sides of (49) and adding the resulting equation to (167), we obtain

$$
\begin{aligned}
&\alpha_0 r T'_{N,0}(r) + \alpha_1 r T'_{N,1}(r) + ... + \alpha_{k-2} r T'_{N,k-2}(r) + x_{k-1} r T'_{N,k-1}(r) + x_k r T'_{N,k}(r) \\
&+ \left(\frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k} \right) \left(\tilde{a}_k r T'_{N,k-1}(r) - \tilde{b}_k r T'_{N,k}(r) + \tilde{c}_k r T'_{N,k+1}(r) \right) \\
&= y_0 T_{N,0} + y_1 T_{N,1} + ... + y_k T_{N,k} \\
&+ \left(\frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k} \right) \left(a_k T_{N,k-1}(r) - b_k T_{N,k}(r) + c_k T_{N,k+1}(r) \right).
\end{aligned}
\tag{168}
$$

Simplifying both sides of (168), we have

$$\begin{aligned}
&\alpha_0 rT'_{N,0}(r) + \alpha_1 rT'_{N,1}(r) + ... + \alpha_{k-2} rT'_{N,k-2}(r) + \alpha_{k-1} rT'_{N,k-1}(r) \\
&+ \left(\frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k}(-\tilde{b}_k) + x_k \right) rT'_{N,k}(r) + \left(\frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k}\tilde{c}_k \right) rT'_{N,k+1}(r) \\
&= y_0 T_{N,0} + y_1 T_{N,1} + ... + \left(\frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k} a_k + y_{k-1} \right) T_{N,k-1}(r) \\
&+ \left(\frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k}(-b_k) + y_k \right) T_{N,k}(r) + \left(\frac{-x_{k-1} + \alpha_{k-1}}{\tilde{a}_k} c_k \right) T_{N,k+1}(r).
\end{aligned} \tag{169}$$

We then scale back each term in the new expansion in $T_{N,n}$ *to get an expansion in* $\overline{T}_{N,n}$*. That is, we scale the* i^{th} *term in the new expansion by*

$$(2(2i + N + p/2 + 1))^{-(1/2)}. \tag{170}$$

The following observation, when combined with Observation 4.3, provides a numerical scheme for evaluating integrals of the form

$$\int_0^1 r\Phi'_{N,n}(r)\Phi_{N,m}(r)r^{p+1}dr. \tag{171}$$

This scheme will be used in Algorithm 4.2.

Observation 4.4 *Suppose that*

$$r\Phi'_{N,n}(r) = \sum_{i=0}^{K} x_i \overline{R}_{N,i}(r) \tag{172}$$

and

$$\Phi_{N,m}(r) = \sum_{i=0}^{K} y_i \overline{R}_{N,i}(r). \tag{173}$$

where x_i, y_i *are real numbers. Then, combining (22) with (24), we have,*

$$\int_0^1 r\Phi'_{N,n}(r)\Phi_{N,m}(r)r^{p+1}dr = \int_0^1 \sum_{i=0}^{K} x_i \overline{R}_{N,i}(r) \sum_{i=0}^{K} y_i \overline{R}_{N,i}(r) r^{p+1}dr = \sum_{i=0}^{K} x_i y_i. \tag{174}$$

We now describe an algorithm for evaluating the eigenvalues $\beta_{N,0}, \beta_{N,1}, ..., \beta_{N,k}$ for any non-negative integers N, k.

Algorithm 4.2

Step 1. Use Algorithm 4.1 to recover the Zernike expansions of the GPSFs

$$\Phi_{N,0}, \Phi_{N,1}, ..., \Phi_{N,n}. \tag{175}$$

Step 2. Compute the eigenvalue $\beta_{N,0}$ (see (109)) using Remark 4.2.

Step 3. Use Observation 4.3 to evaluate the $\overline{R}_{N,n}$ expansion of $r\Phi'_{N,0}$ and $r\Phi'_{N,1}$.

Step 4. Use Observation 4.4 to compute the integrals

$$\int_0^1 r\Phi'_{N,1}(r)\Phi_{N,0}(r)r^{p+1}dr \tag{176}$$

and

$$\int_0^1 r\Phi'_{N,0}(r)\Phi_{N,1}(r)r^{p+1}dr \tag{177}$$

where the Zernike expansions of $\Phi_{N,0}(r), \Phi_{N,1}(r)$ were computed in Step 1 and the Zernike expansions of $r\Phi'_{N,0}(r), \Phi'_{N,1}(r)$ were computed in Step 3.

Step 5. Using Theorem 3.1, evaluate $\beta_{N,1}$ using the formula

$$\beta_{N,1} = \beta_{N,0}\frac{\int_0^1 r\Phi'_{N,1}(r)\Phi_{N,0}(r)r^{p+1}\,dr}{\int_0^1 r\Phi'_{N,0}(r)\Phi_{N,1}(r)r^{p+1}\,dr}. \tag{178}$$

where $\beta_{N,0}$ was obtained in Step 2 and the numerator and denominator of (178) were evaluated in Step 4.

Step 6. Repeat Steps 3-5 k times, each time computing the next eigenvalue, $\beta_{N,i+1}$ via the formula

$$\beta_{N,i+1} = \beta_{N,i}\frac{\int_0^1 r\Phi'_{N,i+1}(r)\Phi_{N,i}(r)r^{p+1}\,dr}{\int_0^1 r\Phi'_{N,i}(r)\Phi_{N,i+1}(r)r^{p+1}\,dr}. \tag{179}$$

5 Quadratures for Band-limited Functions

In this section, we describe a quadrature scheme for bandlimited functions using nodes that are a tensor product of roots of GPSFs in the radial direction and nodes that integrate spherical harmonics in the angular direction.

The following lemma shows that a quadrature rule that accurately integrates complex exponentials, also integrates bandlimited functions accurately.

Lemma 5.1 *Let $\xi_1, ..., \xi_n \in B$ and $w_1, ..., w_n \in \mathbb{R}$ weights such that*

$$\left| \int_B e^{ic\langle x,t\rangle} dt - \sum_{i=1}^{n} w_i e^{ic\langle x,\xi_i\rangle} \right| < \epsilon \tag{180}$$

for all $x \in B$ where B denotes the unit ball in $\mathbb{R}^n$ for any non-negative integer n and $\epsilon > 0$ is fixed. Then, for all $f : B \to \mathbb{C}$ such that

$$f(x) = \int_B \sigma(t) e^{ic\langle x,t\rangle} dt \tag{181}$$

where $\sigma \in L^2(B)$, we have

$$\left| \int_B f(x)dx - \sum_{i=1}^{n} w_i f(\xi_i) \right| < \epsilon \int_B |\sigma(t)|dt \tag{182}$$

Proof. Clearly,

$$\begin{aligned} \left| \int_B f(t)dt - \sum_{i=1}^{n} w_i f(\xi_i) \right| &= \left| \int_B \int_B \sigma(t) e^{ic\langle x,t\rangle} dt dx - \sum_{i=0}^{n} w_i \int_B \sigma(t) e^{ic\langle \xi_i,t\rangle} dt \right| \\ &= \left| \int_B \sigma(t) \left(\int_B e^{ic\langle x,t\rangle} dx - \sum_{i=0}^{n} w_i e^{ic\langle \xi_i,t\rangle} \right) dt \right|. \end{aligned} \tag{183}$$

Applying (180) to (183), we obtain

$$\begin{aligned} \left| \int_B f(t)dt - \sum_{i=1}^{n} w_i f(\xi_i) \right| &\le \int_B |\sigma(t)| \left| \int_B e^{ic\langle x,t\rangle} dx - \sum_{i=0}^{n} w_i e^{ic\langle \xi_i,t\rangle} \right| dt \\ &< \epsilon \int_B |\sigma(t)|dt. \end{aligned} \tag{184}$$

■

The following technical lemma will be used in the construction of quadratures for bandlimited functions.

Lemma 5.2 *For any positive integer K and any integer $p \ge -1$,*

$$\begin{aligned} &\left| \int_B e^{ic\langle x,t\rangle} dt - \int_B \sum_{N=0}^{K} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c\|x\|\|t\|)}{(c\|x\|\|t\|)^{p/2}} S_N^\ell(x/\|x\|) S_N^\ell(t/\|t\|) dt \right| \\ &\le (2\pi)^{p/2+1} \sum_{N=K+1}^{\infty} \frac{c^{2N}(1/2)^{2N+p}}{\Gamma(N+p/2+1)^2} \frac{\pi^{p/2+1}}{\Gamma(p/2+2)} \left(\sum_{\ell=1}^{h(N,p)} |S_N^\ell(x/\|x\|)| \right) \end{aligned}$$

$$(185)$$

for all $x \in B$ and $c > 0$.

Proof. It follows immediately from (106) that for any integer $p \geq -1$ and for all $x \in \mathbb{R}^{p+2}$,

$$\left| \int_B e^{ic\langle x,t\rangle} dt - \int_B \sum_{N=0}^{K} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c\|x\|\|t\|)}{(c\|x\|\|t\|)^{p/2}} S_N^\ell(x/\|x\|) S_N^\ell(t/\|t\|) dt \right| \leq (2\pi)^{p/2+1} \sum_{N=K+1}^{\infty} \sum_{\ell=1}^{h(N,p)} \left| S_N^\ell(x/\|x\|) \right| \int_B \left| \frac{J_{N+p/2}(c\|x\|\|t\|)}{(c\|x\|\|t\|)^{p/2}} S_N^\ell(t/\|t\|) \right| dt \tag{186}$$

where $r = \|x\|$, B denotes the unit ball in $\mathbb{R}^{p+1}$, and S_N^ℓ is defined in (89). Applying Cauchy-Schwarz and Lemma 2.2 to (186) and using the fact that Spherical Harmonics have L^2 norm of 1, we obtain,

$$\left| \int_B e^{ic\langle x,t\rangle} dt - \int_B \sum_{N=0}^{K} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c\|x\|\|t\|)}{(c\|x\|\|t\|)^{p/2}} S_N^\ell(x/\|x\|) S_N^\ell(t/\|t\|) dt \right| \leq (2\pi)^{p/2+1} \sum_{N=K+1}^{\infty} \sum_{\ell=1}^{h(N,p)} \left| S_N^\ell(x/\|x\|) \right| \int_B \left| \frac{(c\|x\|\|t\|)^N (1/2)^{N+p/2}}{\Gamma(N+p/2+1)} \right|^2 dt. \tag{187}$$

Equation (185) follows immediately from applying (83) and (86) to (187). ■

Remark 5.1 *Lemma 5.2 shows that a complex exponential on the unit ball is well approximated by a function of the form*

$$\sum_{N=0}^{K} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c\|x\|\|t\|)}{(c\|x\|\|t\|)^{p/2}} S_N^\ell(x/\|x\|) S_N^\ell(t/\|t\|) dt \tag{188}$$

where the error of the approximation decays super-exponentially in K. Furthermore, the spherical harmonics S_N^ℓ integrate to 0 for all $N \geq 1$ (see Lemma 5.3). Combining these facts, we observe that in order to integrate a complex exponential on the unit ball, it is sufficient to use a quadrature rule that is the tensor product of nodes in the angular *direction that integrate all spherical harmonics* S_N^ℓ *for sufficiently large* N *and nodes in the radial direction that integrate functions of the form*

$$\frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}}\rho^{p+1}. \tag{189}$$

We will show in Remark 5.2 that accurately computing functions of the form of (189) can be done using a quadrature rule that integrates GPSFs.

The following lemma shows that (189) is well represented by an expansion in GPSFs. This lemma will be used to construct quadrature nodes for bandlimited functions.

Lemma 5.3 *For all real numbers* $r, \rho \in (0, 1)$,

$$\frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}}\rho^{p+1} = \sum_{i=0}^{\infty} \beta_{0,i}\Phi_{0,i}(r)\Phi_{0,i}(\rho) \tag{190}$$

where $J_{p/2}$ *is a Bessel function,* $\Phi_{0,n}$ *is defined in (109) and* $\beta_{0,i}$ *is defined in (112).*

Proof. Since $\Phi_{0,i}$ are complete in $L^2[0,1]_{r^{p+1}}$,

$$\frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}}\rho^{p+1} = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \alpha_{i,j}\Phi_{0,i}(r)\Phi_{0,j}(\rho) \tag{191}$$

where $\alpha_{i,j}$ is defined by the formula

$$\alpha_{i,j} = \int_0^1\int_0^1 \frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} r^{p+1}\Phi_{0,i}(r)\Phi_{0,j}(\rho) dr \rho^{p+1} d\rho. \tag{192}$$

Changing the order of integration of (192) and substituting in (112), we obtain,

$$\begin{aligned}\alpha_{i,j} &= \int_0^1 \Phi_{0,j}(r)\int_0^1 \frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}}\rho^{p+1}\Phi_{0,i}(\rho)d\rho r^{p+1}dr \\ &= \beta_{0,i}\int_0^1 \Phi_{0,j}(r)\Phi_{0,i}(r)r^{p+1}dr \\ &= \delta_{i,j}\beta_{0,i}\end{aligned} \tag{193}$$

where $\beta_{0,i}$ is defined in (112). Identity (190) follows immediately from the combination of (191) and (193). ■

The following remark shows that a quadrature rule that correctly integrates certain GPSFs also integrates certain Bessel functions.

Remark 5.2 *Let $\rho_1, ..., \rho_n$ be the n roots of $\Phi_{0,n}$ and $w_1, ..., w_n \in \mathbb{R}$ the n weights such that*

$$\int_0^1 \Phi_{0,k}(r) r^{p+1} dr = \sum_{i=0}^{n} \Phi_{0,k}(\rho_i) w_i \tag{194}$$

for $k = 0, 1, ..., K$. By Lemma 5.3,

$$\begin{aligned} &\left| \int_0^1 \frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} \rho^{p+1} d\rho - \sum_{i=1}^{n} \frac{J_{p/2}(cr\rho_i)}{(cr\rho_i)^{p/2}} w_i \right| \\ &= \left| \int_0^1 \left(\sum_{j=0}^{\infty} \beta_{0,j} \Phi_{0,j}(r) \Phi_{0,j}(\rho) \right) d\rho - \sum_{i=1}^{n} w_i \left(\sum_{j=0}^{\infty} \beta_{0,j} \Phi_{0,j}(r) \Phi_{0,j}(\rho_i) \right) \right| \end{aligned} \tag{195}$$

where $\beta_{0,j}$ is defined in (112). Applying (194) to (195), we obtain

$$\begin{aligned} &\left| \int_0^1 \frac{J_{p/2}(cr\rho)}{(cr\rho)^{p/2}} \rho^{p+1} d\rho - \sum_{i=1}^{n} \frac{J_{p/2}(cr\rho_i)}{(cr\rho_i)^{p/2}} \rho_i^{p+1} w_i \right| \\ &= \left| \int_0^1 \left(\sum_{j=K+1}^{\infty} \beta_{0,j} \Phi_{0,j}(r) \Phi_{0,j}(\rho) \right) d\rho - \sum_{i=1}^{n} w_i \left(\sum_{j=K+1}^{\infty} \beta_{0,j} \Phi_{0,j}(r) \Phi_{0,j}(\rho_i) \right) \right|. \end{aligned} \tag{196}$$

Clearly, as long as $\beta_{0,K+1}$ is in the regime of exponential decay, (196) is of magnitude approximately $\beta_{0,K+1}$.

We now describe a quadrature rule that correctly integrates functions of the form of (188). This quadrature rule uses nodes that are a tensor product of roots of $\Phi_{0,n}$ in the radial direction and nodes that integrate spherical harmonics in the angular direction.

Observation 5.3 *Suppose that $r_1, ..., r_n \in (0, 1)$ and weights $w_1, ..., w_n \in \mathbb{R}$ satisfy*

$$\int_0^1 \Phi_{0,k}(r) r^{p+1} dr = \sum_{i=1}^{n} w_i \Phi_{0,k}(r_i) \tag{197}$$

for $k = 0, 1, ..., K_1$. Suppose further that $x_1, ..., x_m \in S^{p+1}$ are nodes and $v_1, ..., v_m \in \mathbb{R}$ are weights such that

$$\int_{S^{p+1}} S_N^{\ell}(x) dx = \sum_{i=1}^{m} v_i S_N^{\ell}(x_i) \tag{198}$$

for all $N \le K_2$ and for all $\ell \in \{1,2,...,h(N,p)\}$. Then it follows immediately from Remark 5.1 and Remark 5.2 that

$$\left| \int_B e^{ic\langle x,t\rangle} dt - \sum_{i=0}^{m} v_i \sum_{j=1}^{n} w_j e^{ic\langle x, r_j x_i\rangle} \right| \tag{199}$$

will be exponentially small for large enough n,m. Lemma 5.1 shows that quadrature (199) will also accurately integrate functions of the form

$$\int_B \sigma(t) e^{ic\langle x,t\rangle} dt \tag{200}$$

where σ is in $L^2(B)$.

Remark 5.4 *A Chebyshev quadrature of the form (5.1) can be generated by first computing the n roots of $\Phi_{0,n}$ (see Section 5.1) and then solving the $n \times n$ linear system of equations*

$$\int_0^1 \Phi_{0,k}(r) r^{p+1} dr = \sum_{i=1}^{n} w_i \Phi_{0,k}(r_i) \tag{201}$$

for $w_1, ..., w_n$ where $r_1, ..., r_n$ are the n roots of $\Phi_{0,n}$. Section 5.2 contains a description of an algorithm for generating Gaussian quadratures for GPSFs.

5.1 Roots of $\Phi_{0,n}$

In this section, we describe an algorithm for finding the roots of $\Phi_{N,n}$. These roots will be used in the design of quadratures for GPSFs.

The following lemma provides a differential equation satisfied by $\varphi_{0,n}$. It will be used in the evaluation of roots of $\varphi_{0,n}$ later in this section.

Lemma 5.4 *For all non-negative integers n,*

$$\varphi_{0,n}''(r) + \alpha(r)\varphi_{0,n}'(r) + \beta(r)\varphi_{0,n}(r) = 0, \tag{202}$$

where

$$\alpha(r) = \frac{-2r}{1-r^2} \tag{203}$$

and

$$\beta(r) = \frac{1/4 - (N+p/2)^2}{(1-r^2)r^2} - \frac{c^2 r^2 + \chi_{N,n}}{1-r^2} \tag{204}$$

where $\varphi_{0,n}$ is defined in (113) and $\chi_{N,n}$ is defined in (118).

The following lemma is obtained by applying the Prufer Transform (see Lemma 2.15) to (202).

Lemma 5.5 *For all non-negative integers* n*, real numbers* $k > -1$*, and* $r \in (0,1)$*,*

$$\frac{d\theta}{dr} = -\sqrt{\beta(r)} - \left(\frac{\beta'(r)}{4\beta(r)} + \frac{\alpha(r)}{2}\right)\sin(2\theta(r)), \tag{205}$$

where the function $\theta : (0,1) \to \mathbb{R}$ *is defined by the formula*

$$\frac{\varphi_{N,n}(r)}{\varphi'_{N,n}(r)} = \sqrt{\beta(r)}\tan(\theta(r)), \tag{206}$$

and $\beta'(r)$*, the derivative of* $\beta(r)$ *with respect to* r*, is defined by the formula*

$$\beta'(r) = \frac{-2(1/4-(N+p/2)^2)(1-2r^2)}{(1-r^2)r^3} + \frac{-2rc^2(1-r^2)+2r(-c^2r^2-\chi_{N,n})}{(1-r^2)^2} \tag{207}$$

and where $\alpha(r)$*,* $\beta(r)$ *are defined in (203) and (204),* $\varphi_{N,n}$ *is defined in (113) and* $\chi_{N,n}$ *is defined in (118).*

Remark 5.5 *For any non-negative integer* n*,*

$$\frac{d\theta}{dr} < 0 \tag{208}$$

for all $r \in (r_1, r_n)$ *where* r_1 *and* r_n *are the smallest and largest roots of* $\varphi_{N,n}$ *respectively. Therefore, applying Remark 2.6 to (205), we can view* r *as a function of* θ *where* r *satisfies the differential equation*

$$\frac{dr}{d\theta} = \left(-\sqrt{\beta(r)} - \left(\frac{\beta'(r)}{4\beta(r)} + \frac{\alpha(r)}{2}\right)\sin(2\theta(r))\right)^{-1} \tag{209}$$

where α*,* β*, and* β' *are defined in (203), (204) and (207).*

The following is a description of an algorithm for the evaluation of the n roots of $\Phi_{N,n}$. We denote the n roots of $\Phi_{N,n}$ by $r_1 < r_2 < ... < r_n$.

Algorithm 5.1

Step 0. Compute the $\overline{T}_{N,n}$ expansion of $\varphi_{N,n}$ using Algorithm 4.1.

Step 1. Use bisection to find the root in $x_0 \in (0,1)$ of $\beta(r)$ where $\beta(r)$ is defined in (204). If β has no root on $(0,1)$, then set $x_0 = 1$.

Step 2. If $\chi_{0,n}(c) > 1/\sqrt{c}$, place Chebyshev nodes (order $5n$, for example) on the interval $(0, x_0)$ and check, starting at x_0 and moving in the negative direction, for a sign change. Once a sign change has occured, use Newton to find an accurate approximation to the root.

If $\chi_{0,n}(c) \leq 1/\sqrt{c}$, then use three steps of Mueller's method starting at x_0, using the first and second derivatives of $\varphi_{0,n}$ followed by Newton's method.

Step 3. Defining θ_0 by the formula

$$\theta_0 = \theta(x_0), \tag{210}$$

where θ is defined in (137), solve the ordinary differential equation $\frac{dr}{d\theta}$ (see (209)) on the interval $(\pi/2, \theta_0)$, with the initial condition $r(\theta_0) = x_0$. To solve the differential equation, it is sufficient to use, for example, second order Runge Kutta with 100 steps (independent of n). We denote by $\tilde{r}_n$ the approximation to $r(\pi/2)$ obtained by this process. It follows immediately from (64) that $\tilde{r}_n$ is an approximation to r_n, the largest root of $\varphi_{N,n}$.

Step 4. Use Newton's method with $\tilde{r}_n$ as an initial guess to find r_n to high precision. The GPSF $\varphi_{N,n}$ and its derivative $\varphi'_{N,n}$ can be evaluated using the expansion evaluated in Step 0.

Step 5. With initial condition

$$x(\pi/2) = r_n, \tag{211}$$

solve differential equation $\frac{dr}{d\theta}$ (see (209)) on the interval

$$(-\pi/2, \pi/2) \tag{212}$$

using, for example, second order Runge Kuta with 100 steps. We denote by $\tilde{r}_{n-1}$ the approximation to

$$r(-\pi/2) \tag{213}$$

obtained by this process.

Step 6. Use Newton's method, with initial guess $\tilde{r}_{n-1}$, to find to high precision the second largest root, r_{n-1}.

Step 7. For $k = \{1, 2, ..., n-1\}$, repeat Step 4 on the interval

$$(-\pi/2 - k\pi, -\pi/2 - (k-1)\pi) \tag{214}$$

with intial condition

$$x(-\pi/2-(k-1)\pi)=r_{n-k+1} \tag{215}$$

and repeat Step 5 on $\tilde{r}_{n-k}$.

5.2 Gaussian Quadratures for $\Phi_{0,n}$

In this section, we describe an algorithm for generating Gaussian quadratures for the GPSFs $\Phi_{0,0},\Phi_{0,1},...,\Phi_{0,n}$.

Definition 5.1 *A Gaussian Quadrature with respect to a set of functions $f_1,...,f_{2n-1}:[a,b]\to\mathbb{R}$ and non-negative weight function $w:[a,b]\to\mathbb{R}$ is a set of n nodes, $x_1,...,x_n\in[a,b]$, and n weights, $\omega_1,...,\omega_n\in\mathbb{R}$, such that, for any integer $j\le 2n-1$,*

$$\int_a^b f_j(x)w(x)dx=\sum_{i=0}^n \omega_i f_j(x_i). \tag{216}$$

Remark 5.6 *In order to generate a Gaussian quadrature for GPSFs with bandlimit $c>0$, we first generate a Chebyshev quadrature for GPSFs with bandlimit $c/2$ and then, using those nodes and weights as a starting point, we use Newton's method with step-length control to find nodes and weights that form a Gaussian quadrature for GPSFs with bandlimit c.*

The following is a description of an algorithm for generating Gaussian quadratures for the GPSFs

$$\Phi^c_{0,0},...,\Phi^c_{0,2n-1}. \tag{217}$$

Algorithm 5.2

Step 1. Using Algorithm 5.1, generate a Chebyshev quadrature for the functions

$$\Phi^{c/2}_{0,0},...,\Phi^{c/2}_{0,n-1}. \tag{218}$$

That is, find, $r_1,...,r_n$, the n roots of $\Phi_{0,n}$ and weights $w_1,...,w_n$ such that

$$\int_0^1 \Phi^{c/2}_{0,k}(r)dr=\sum_{i=1}^n w_i\Phi^{c/2}_{0,k}(r_i) \tag{219}$$

for $k=0,1,...,n-1$.

Step 2. Evaluate the vector $d = (d_0, d_1, ..., d_{2n-1})$ of discrepencies where d_k is defined by the formula

$$d_k = \int_0^1 \Phi^c_{0,k}(r)dr - \sum_{i=1}^{n} w_i \Phi^c_{0,k}(r_i) \tag{220}$$

for $k = 0, 1, ..., 2n - 1$.

Step 3. Generate A, the $2n \times 2n$ matrix of partial derivatives of d the n nodes and and n weights. Specifically, for $i = 1, ..., 2n$, the matrix A is defined by the formula

$$A_{i,j} = \begin{cases} \Phi^c_{0,j}(r_i) & \text{for } i = 1, ..., n, \\ w_i \Phi^{c\prime}_{0,j}(r_i) & \text{for } i = n + 1, ..., 2n. \end{cases} \tag{221}$$

where $\Phi^{c\prime}_{0,k}(r)$ denotes the derivative of $\Phi^c_{0,k}(r)$ with respect to r.

Step 4. Solve for $x \in \mathbb{R}^{2n}$ the $2n \times 2n$ linear system of equations

$$Ax = -d \tag{222}$$

where A is defined in (221) and d is defined in (220).

Step 5. Update nodes and weights correspondingly. That is, defining $r \in \mathbb{R}^{2n}$ to be the vector of nodes and weights

$$r = (r_1, r_2, ..., r_n, w_1, w_2, ..., w_n)^T, \tag{223}$$

we construct the updated vector of nodes and weights $\tilde{r}$ so that

$$\tilde{r} = r + \langle r, x \rangle r \tag{224}$$

Step 6. Check that the l^2 norm of $\tilde{r}$ is less than the l^2 norm of r. If now, then go back to Step 5 and divide the steplength by 2. That is, define $\tilde{r}$ by the formula,

$$\tilde{r} = r + \frac{1}{2}\langle r, x \rangle r. \tag{225}$$

Continue dividing the steplength by 2 until $\|\tilde{r}\|_2 < \|r\|_2$.

Step 7. Repeat steps 2-6 until the discrepencies, d_i for $i = 0, 1, ..., 2n - 1$ (see (220)) are approximately machine precision.

6 Interpolation via GPSFs

In this section, we describe a numerical scheme for representing a bandlimited function as an expansion in GPSFs.

In general, the interpolation problem is formulated as follows. Suppose that f is defined by the formula

$$f(x) = a_1 g_1(x) + a_2 g_2(x) + ... + a_n g_n(x) \tag{226}$$

where $g_1, ..., g_n$ are some fixed basis functions. The interpolation problem is to recover the coefficients $a_1, ..., a_n$. This is generally done by solving the $n \times n$ linear system of equations obtained from evaluating f at certain interpolation nodes. As long as f is well-represented by the interpolation nodes, then the procedure is accurate.

As shown in the context of quadrature (see Section 5), GPSFs are a natural basis for representing bandlimited functions. We formulate the interpolation problem for GPSFs as recovering the coefficients of a bandlimited function f in its GPSF expansion. That is, suppose that f is of the form

$$f(x) = \int_B \sigma(t) e^{ic\langle x,t\rangle} dt. \tag{227}$$

where $\sigma \in L^2(B)$. Then, f is representable in the form

$$f(x) = \sum_{i=1}^{N} a_i \psi_i(x) \tag{228}$$

where $\psi_j(x)$ is a GPSF defined in (96) and a_i satisfies

$$a_i = \int_B \psi_i(t) f(t) dt. \tag{229}$$

The problem is the recover the coefficients a_j.

The following lemma shows that a quadrature rule that recovers the coefficients of the expansion in GPSFs of a complex exponential will also recover the coefficients in a GPSF expansion of a bandlimited function.

Lemma 6.1 *Suppose that for all $t \in B$,*

$$\left| \int_B \psi_j(x) e^{ic\langle x,t\rangle} dx - \sum_{k=1} w_k \psi_j(x_k) e^{ic\langle x_k,t\rangle} \right| < \epsilon \tag{230}$$

where B denotes the unit ball in $\mathbb{R}^{p+2}$ and ψ_j is defined in (96). Then,

$$\left| \int_B \psi_j(x) f(x) dx - \sum_{k=1} w_k \psi_j(x_k) f(x_k) \right| < \epsilon \tag{231}$$

where

$$f(x) = \int_B \sigma(t) e^{ic\langle x,t\rangle} dt. \tag{232}$$

The following lemma shows that the product of a complex exponential with a GPSF of bandlimit $c > 0$ is a bandlimited function with bandlimit $2c$. The proof is a slight modification of Lemma 5.3 in [17].

Lemma 6.2 *For all $x \in B$ where B denotes the unit ball in $\mathbb{R}^{p+2}$ and for all $c > 0$,*

$$e^{ic\langle\omega,x\rangle} \psi_j(x) = \int_B \sigma(\xi) e^{i2c\langle\xi,x\rangle} d\xi \tag{233}$$

where ψ_j is defined in (96) and σ satisfies

$$\left| \int_B \sigma(t)^2 \right| \le 4/|\lambda_j|^2. \tag{234}$$

where λ_j is defined in (96).

Proof. Using (96),

$$\psi_j(x) e^{ic\langle\omega,x\rangle} = \frac{1}{\lambda_j} \int_B e^{ic\langle\omega+t,x\rangle} \psi_j(t) dt. \tag{235}$$

Applying the change of variables $\xi = (t+\omega)/2$ to (ref3440), we obtain

$$\psi_j(x) e^{ic\langle\omega,x\rangle} = \frac{1}{\lambda_j} \int_{B_\omega} e^{i2c\langle\xi,x\rangle} 2\psi_j(2\xi - \omega) d\xi \tag{236}$$

where B_ω is the ball of radius $1/2$ centered at $\omega/2$. It follows immediately from (236) that

$$\psi_j(x) e^{ic\langle\omega,x\rangle} = \frac{1}{\lambda_j} \int_{B_\omega} e^{i2c\langle\xi,x\rangle} \mu(\xi) d\xi. \tag{237}$$

where

$$\mu(\xi) = \begin{cases} \frac{2\psi_j(2\xi-\omega)}{\lambda_j} & \text{if } \xi \in B_\omega, \\ 0 & \text{otherwise.} \end{cases} \tag{238}$$

Inequality (234) follows immediately from the combination of (238) with the fact that ψ_j is L^2 normalized. ■

The following observation describes a numerical scheme for recovering the coefficients in a GPSF expansion of a bandlimited function.

Observation 6.1 *Suppose that f is defined by the formula*

$$f(x) = \int_B \sigma(t) e^{ic\langle x,t\rangle} dt \tag{239}$$

where σ is some function in $L^2(B)$. Then, f is representable in the form

$$f(x) = \sum_{k=1}^{\infty} a_k \psi_k(x) \tag{240}$$

where

$$a_k = \int_B f(x)\psi_k(x)dx. \tag{241}$$

It follows immediately from the combination of Lemma 6.2 and Lemma 6.1 that using quadrature rule (199) with bandlimit $2c$ will integrate a_k accurately. That is, following the notation of Observation 5.1,

$$\left| a_k - \sum_{i=0}^{n} w_i \sum_{j=1}^{m} v_j f(r_i x_j)\psi_k(r_i x_j) \right| \tag{242}$$

is exponentially small for large enough m, n.

Remark 6.2 *When integrating a function of the form of (240) on the unit disk in R^2, the v_j in (242) are defined by the formula*

$$v_j = j\frac{2\pi}{2m-1} \tag{243}$$

for $j = 1, 2, ..., 2m-1$ and the sums

$$\sum_{j=1}^{m} v_j f(r_i x_j)\psi_k(r_i x_j) \tag{244}$$

for each i can be computed using an FFT.

The following lemma bounds the magnitudes of the coefficients of the GPSF expansion of a bandlimited function.

Lemma 6.3 *Suppose that f is defined by the formula*

$$f(x) = \int_B \sigma(t) e^{ic\langle x,t\rangle} dt. \tag{245}$$

Then,

$$f(x) = \sum_{i=1}^{N} a_i \psi_i(x) \tag{246}$$

where $\psi_j(x)$ is a GPSF defined in (96) and a_i satisfies

$$|a_i| \le |\lambda_i| \int_B |\sigma(t)|^2 dt \tag{247}$$

where λ_i is defined in (96).

Proof. Since ψ_j form an orthonormal basis for $L^2[B]$, f is representable in the form of (246) and for all positive integers i,

$$a_i = \int_B f(t)\psi_i(t) dt = \int_B \left(\int_B \sigma(\xi) e^{ic\langle t,\xi\rangle} d\xi \right) \psi_i(t) dt. \tag{248}$$

Combining (248) and (96) and using Cauchy-Schwarz, we obtain

$$|a_i| = |\lambda_i \int_B \sigma(t)\psi_i(t)dt| \le |\lambda_i| \int_B |\sigma(t)|^2 dt \int_B |\psi_j(t)|^2 dt = |\lambda_i| \int_B |\sigma(t)|^2 dt \tag{249}$$

■

Remark 6.3 *Lemma 6.3 shows that in order to accurately represent a bandlimited function, f, it is sufficient to find the projection of f onto all GPSFs with corresponding eigenvalue above machine precision. In Section 6.1, we approximate the number of GPSFs with large corresponding eigenvalues.*

6.1 Dimension of the Class of Bandlimited Functions

In this section, we investigate the properties of the eigenvalues $\mu_0, \mu_1, \ldots, \mu_j, \ldots$ of the operator Q_c, defined via formula (97). We denote by λ_j the eigenvalues of operator F_c, defined via formula (95), and let ψ_j denote the eigenfunctions corresponding to λ_j, for each nonnegative integer j.

The following two theorems evaluate the sums $\sum_{j=0}^{\infty} \mu_j$ and $\sum_{j=0}^{\infty} \mu_j^2$ respectively.

Theorem 6.4 *Suppose that $c > 0$. Then*

$$\sum_{j=0}^{\infty} \mu_j = \frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2}. \tag{250}$$

Proof. From (96), we observe the identity

$$\sum_{j=0}^{\infty} \lambda_j \psi_j(x)\psi_j(t) = e^{ic\langle x,t\rangle}, \tag{251}$$

for all $x, t \in B$, where B is the closed unit ball in $\mathbb{R}^{p+2}$, and the sum on the left hand side converges in the sense of $L^2(B) \otimes L^2(B)$. By taking the squared $L^2(B) \otimes L^2(B)$ norm of both sides and using (83), we obtain the formula

$$\sum_{j=0}^{\infty} |\lambda_j|^2 = \frac{\pi^{p+2}}{\Gamma(\frac{p}{2}+2)^2}. \tag{252}$$

Since

$$\mu_j = \left(\frac{c}{2\pi}\right)^{p+2} |\lambda_j|^2, \tag{253}$$

for all nonnegative integer j (see (96)), it follows that

$$\sum_{j=0}^{\infty} \mu_j = \frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2}. \tag{254}$$

■

Theorem 6.5 *Suppose that $c > 0$. Then*

$$\sum_{j=0}^{\infty} \mu_j^2 = \frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2} - \frac{c^{p+1}\log(c)}{\pi^2\Gamma(p+2)} + o\big(c^{p+1}\log(c)\big), \tag{255}$$

as $c \to \infty$.

Proof. By (99),

$$\sum_{j=0}^{\infty} \mu_j \psi_j(x)\psi_j(t) = \left(\frac{c}{2\pi}\right)^{p/2+1} \frac{J_{p/2+1}(c\|x-t\|)}{\|x-t\|^{p/2+1}}, \tag{256}$$

for all $x, t \in B$, where the sum on the left hand side converges in the sense of $L^2(B) \otimes L^2(B)$, and where J_ν denotes the Bessel functions of the first kind. Taking the squared $L^2(B) \otimes L^2(B)$ norm of both sides, we obtain the formula

$$\begin{aligned}
\sum_{j=0}^{\infty} \mu_j^2 &= \left(\frac{c}{2\pi}\right)^{p+2} \int_B \int_B \frac{\big(J_{p/2+1}(c\|x-t\|)\big)^2}{\|x-t\|^{p+2}}\, dx\, dt \\
&= \left(\frac{c}{2\pi}\right)^{p+2} \int_B \int_B \frac{\big(J_{p/2+1}(c\|x+t\|)\big)^2}{\|x+t\|^{p+2}}\, dx\, dt \\
&= \left(\frac{c}{2\pi}\right)^{p+2} \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \frac{\big(J_{p/2+1}(c\|x+t\|)\big)^2}{\|x+t\|^{p+2}} \mathbb{1}_B(x)\mathbb{1}_B(t)\, dx\, dt, \qquad (257)
\end{aligned}$$

where $\mathbb{1}_A$ is defined via the formula

$$\mathbb{1}_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases} \qquad (258)$$

Letting $u = x + t$, we observe that

$$\begin{aligned}
\sum_{j=0}^{\infty} \mu_j^2 &= \left(\frac{c}{2\pi}\right)^{p+2} \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}} \mathbb{1}_B(u-t)\mathbb{1}_B(t)\, du\, dt \\
&= \left(\frac{c}{2\pi}\right)^{p+2} \int_{\mathbb{R}^D} \int_{\mathbb{R}^D} \frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}} \mathbb{1}_{B(2)}(u)\mathbb{1}_B(u-t)\mathbb{1}_B(t)\, du\, dt \\
&= \left(\frac{c}{2\pi}\right)^{p+2} \int_{B(2)} \frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}} \int_{\mathbb{R}^D} \mathbb{1}_B(u-t)\mathbb{1}_B(t)\, dt\, du. \qquad (259)
\end{aligned}$$

Combining (259) and (84),

$$\begin{aligned}
\sum_{j=0}^{\infty} \mu_j^2 &= \left(\frac{c}{2\pi}\right)^{p+2} \int_{B(2)} \frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}} \cdot V_{p+2}(1) \frac{B_{1-\|u\|^2/4}(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})}\, du \\
&= \left(\frac{c}{2\pi}\right)^{p+2} \frac{V_{p+2}(1)}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})} \int_{B(2)} \frac{\big(J_{p/2+1}(c\|u\|)\big)^2}{\|u\|^{p+2}} B_{1-\|u\|^2/4}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\, du \\
&= \left(\frac{c}{2\pi}\right)^{p+2} \frac{V_{p+2}(1)A_{p+2}(1)}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})} \int_0^2 \frac{\big(J_{p/2+1}(cr)\big)^2}{r} B_{1-r^2/4}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\, dr \\
&= \left(\frac{c}{2\pi}\right)^{p+2} \frac{V_{p+2}(1)A_{p+2}(1)}{B(\frac{p}{2}+\frac{3}{2},\frac{1}{2})} \int_0^1 \frac{\big(J_{p/2+1}(2cr)\big)^2}{r} B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\, dr, \qquad (260)
\end{aligned}$$

where $V_{p+2}(1)$ denotes the volume of the unit ball in $\mathbb{R}^{p+2}$, $A_{p+2}(1)$ denotes the area of the unit sphere in $\mathbb{R}^{p+2}$, $B(a,b)$ denotes the beta function, and $B_x(a,b)$ denotes the incomplete beta function. Applying Theorem 2.26 to (260),

$$\begin{aligned}\sum_{j=0}^{\infty}\mu_j^2 &= \frac{c^{p+2}}{2^{p+1}\sqrt{\pi}\Gamma(\frac{p}{2}+1)\Gamma(\frac{p}{2}+\frac{3}{2})}\int_0^1 \frac{\big(J_{p/2+1}(2cr)\big)^2}{r}B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,dr \\ &= \frac{c^{p+2}}{\pi\Gamma(p+2)}\int_0^1 \frac{\big(J_{p/2+1}(2cr)\big)^2}{r}B_{1-r^2}(\tfrac{p}{2}+\tfrac{3}{2},\tfrac{1}{2})\,dr. \qquad (261)\end{aligned}$$

Combining (261) and (76),

$$\begin{aligned}\sum_{j=0}^{\infty}\mu_j^2 &= \frac{c^{p+2}}{\pi\Gamma(p+2)}\left(\frac{\sqrt{\pi}\,\Gamma(\frac{p}{2}+\frac{3}{2})}{(p+2)\Gamma(\frac{p}{2}+2)} - \frac{1}{\pi}\frac{\log(c)}{c} + o\Big(\frac{\log(c)}{c}\Big)\right) \\ &= \frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2} - \frac{c^{p+1}\log(c)}{\pi^2\Gamma(p+2)} + o\big(c^{p+1}\log(c)\big), \qquad (262)\end{aligned}$$

as $c \to \infty$.

■

The following corollary follows immediately from theorems 6.4 and 6.5.

Corollary 6.6 *Suppose that $c > 0$. Then*

$$\sum_{j=0}^{\infty}\mu_j(1-\mu_j) = \frac{c^{p+1}\log(c)}{\pi^2\Gamma(p+2)} + o\big(c^{p+1}\log(c)\big), \qquad (263)$$

as $c \to \infty$.

From (250) and (263) we observe that the spectrum of Q_c consists of three parts:

$$\frac{c^{p+2}}{2^{p+2}\Gamma(\frac{p}{2}+2)^2} \qquad (264)$$

eigenvalues close to 1;

$$\frac{c^{p+1}\log(c)}{\pi^2\Gamma(p+2)} \qquad (265)$$

eigenvalues in the transition region; and the rest close to 0.

7 Numerical Experiments

The quadrature and interpolation formulas described in Sections 5 and 6 were implemented in Fortran 77. We used the Lahey/Fujitsu compiler on a 2.9 GHz Intel i7-3520M Lenovo laptop. All examples in this section were run in double precision arithmetic.

In Figure 1 and Figure 2 we plot the eigenvalues $|\lambda_{N,n}|$ of integral operator F_c (see (95)) for different N and different c.

In Figures 3, 4, 5, and 6 we plot the GPSFs $\Phi_{N,n}(r)$ (see (109)) for different N, n, and c.

In Tables 1-6, we provide the performance of quadrature rule (199) in integrating the function

$$e^{ic\langle x,t\rangle} \tag{266}$$

over the unit disk where $x = (0.9, 0.2)$. We provide the results for $c = 20$ and $c = 100$ using both Chebyshev and Gaussian quadratures in the radial direction (see Remark 5.4).

In Tables 7, 8, and 9, we provide magnitudes of coefficients of the GPSF expansion of the function on the unit disk $e^{ic\langle x,t\rangle}$ where $x = (0.3, 0.4)$. These coefficients were obtained via interpolation scheme (242).

In each table in this section, the column labeled "c" denotes the value of c in (266). The column labeled "radial nodes" denotes the number of nodes in the radial direction. These nodes integrate GPSFs. The column labeled "angular nodes" gives the number of equispaced nodes used in the angular direction. The column labeled "N" denotes the N of $\Phi_{N,n}$ (see 109). The column labeled "n" denotes the n of $\Phi_{N,n}$. The column labeled "integral via quadrature" denotes value of the integral obtained via quadrature rule (199) The column labeled "relative error" denotes the relative error of the integral obtained via quadrature to the true value of the integral. The true value of the integral was obtained by a calculation in extended precision. In Tables 7, 8, and 9, the column labeled $|\alpha_{N,n}|$ denotes the coefficient of $\Phi_{N,n}(r)sin(\theta)$ in the GPSF expansion of (266). These coefficients were obtained via formula (242).

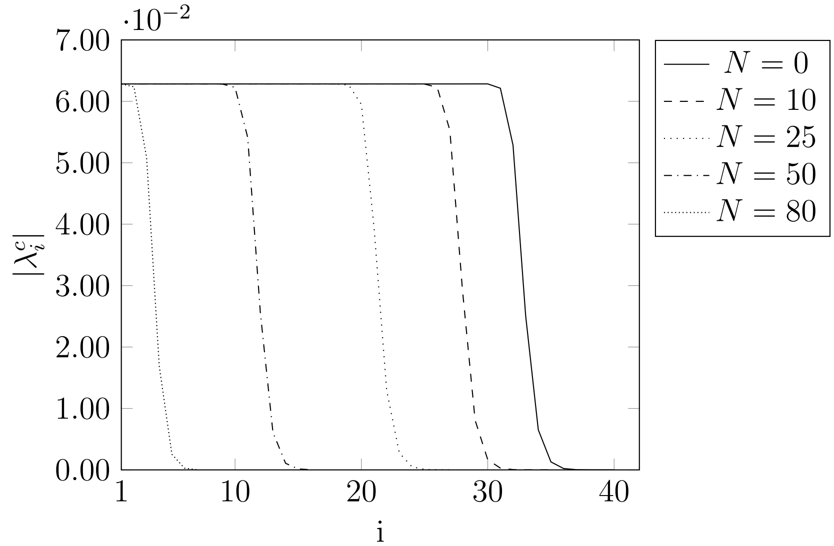

Figure 1: Eigenvalues of F_c (see (95)) for $c = 100$ and $p = 0$

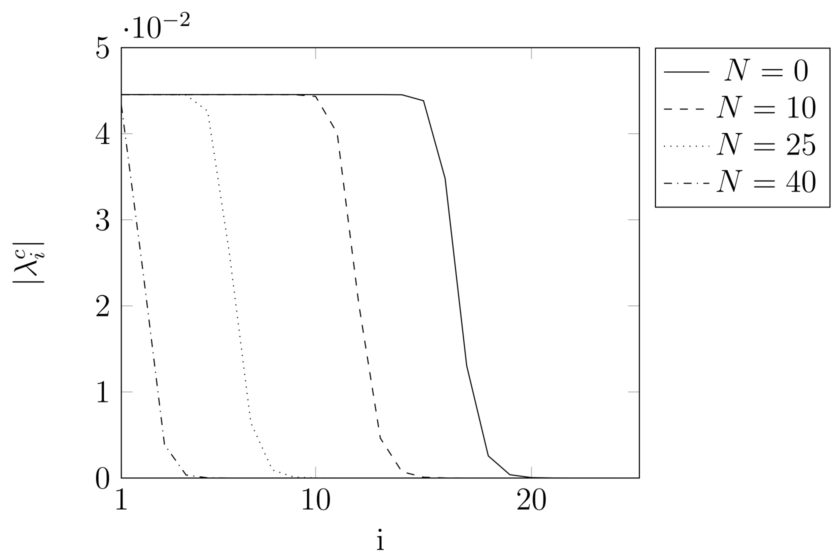

Figure 2: Eigenvalues of F_c (see (95)) for $c = 50$ and $p = 1$

c	radial nodes	angular nodes	integral via quadrature	relative error
20	6	50	$-0.1076416394449520 + i0.13791 \times 10^{-15}$	0.84109×10^{0}
20	8	50	$-0.0584248723305745 + i0.31659 \times 10^{-15}$	0.70864×10^{-3}
20	10	50	$-0.0584663050529888 + i0.26671 \times 10^{-15}$	0.15834×10^{-7}
20	12	50	$-0.0584663041272412 + i0.27929 \times 10^{-15}$	0.75601×10^{-13}
20	14	50	$-0.0584663041272372 + i0.16220 \times 10^{-15}$	0.68485×10^{-14}
20	16	50	$-0.0584663041272371 + i0.27777 \times 10^{-15}$	0.29262×10^{-14}
20	18	50	$-0.0584663041272375 + i0.23191 \times 10^{-15}$	0.75991×10^{-14}

Table 1: Quadratures for $e^{ic\langle x,t\rangle}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of radial nodes for $c = 20$. Chebyshev quadratures are used in the radial direction.

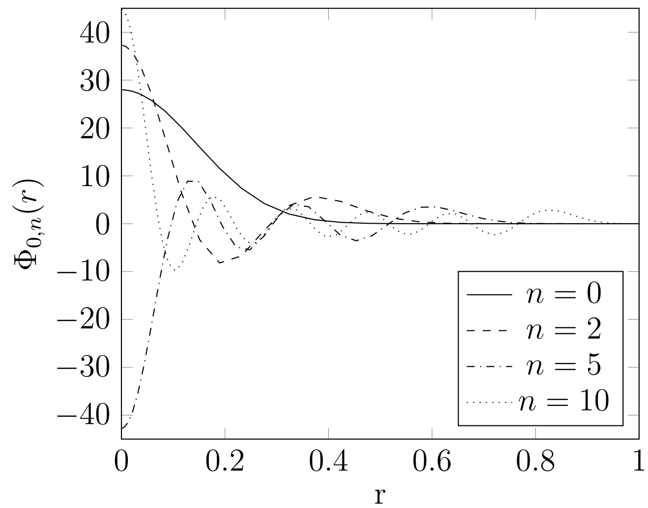

Figure 3: Plots of GPSFs $\Phi_{0,n}$ (see (109)) with $c = 50$ and $p = 1$

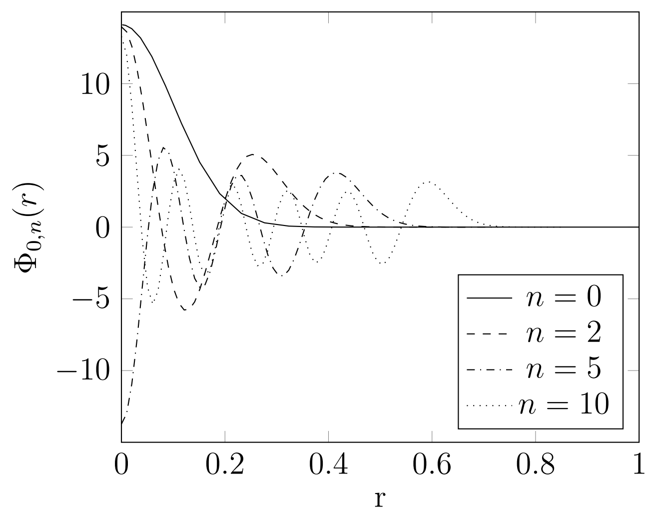

Figure 4: Plots of GPSFs $\Phi_{0,n}$ (see (109)) with $c = 100$ and $p = 0$

c	radial nodes	angular nodes	integral via quadrature	relative error
20	14	20	$-0.0856165805088149 - i0.57734 \times 10^{-16}$	0.46437×10^{0}
20	14	25	$-0.0584663041272373 + i0.10816 \times 10^{-2}$	0.18500×10^{-1}
20	14	30	$-0.0584748094426783 - i0.18258 \times 10^{-15}$	0.14547×10^{-3}
20	14	35	$-0.0584663041272371 - i0.37973 \times 10^{-8}$	0.64949×10^{-7}
20	14	40	$-0.0584663041418621 + i0.14875 \times 10^{-15}$	0.25015×10^{-9}
20	14	45	$-0.0584663041272375 - i0.94777 \times 10^{-14}$	0.16653×10^{-12}
20	14	50	$-0.0584663041272372 + i0.16220 \times 10^{-15}$	0.51483×10^{-14}
20	14	55	$-0.0584663041272368 + i0.39248 \times 10^{-15}$	0.30672×10^{-14}
20	14	60	$-0.0584663041272371 - i0.13661 \times 10^{-16}$	0.53592×10^{-14}

Table 2: Quadratures for $e^{ic\langle x,t\rangle}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of angular nodes for $c = 20$. Chebyshev quadratures are used in the radial direction.

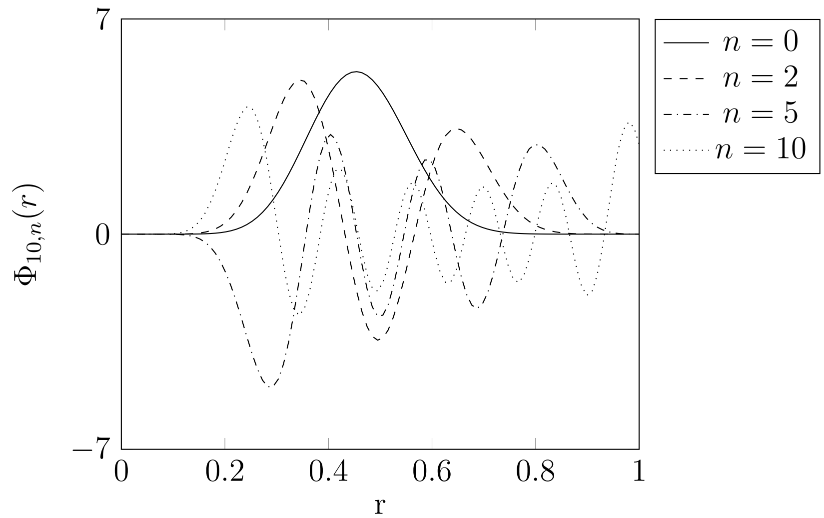

Figure 5: Plots of GPSFs $\Phi_{10,n}$ (see (109)) with $c = 50$ and $p = 1$

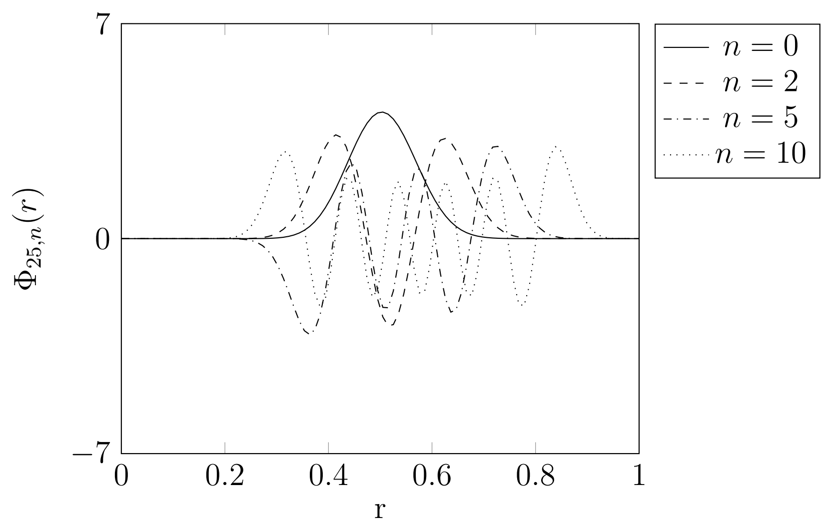

Figure 6: Plots of GPSFs $\Phi_{25,n}$ (see (109)) with $c = 100$ and $p = 0$

c	radial nodes	angular nodes	integral via quadrature	relative error
20	4	50	$-0.0510613892349747 + i0.37123 \times 10^{-15}$	0.12603×10^{0}
20	6	50	$-0.0584663254751910 + i0.11623 \times 10^{-15}$	0.36513×10^{-6}
20	8	50	$-0.0584663041272613 + i0.37340 \times 10^{-15}$	0.41931×10^{-12}
20	10	50	$-0.0584663041272369 + i0.20903 \times 10^{-15}$	0.15463×10^{-14}
20	12	50	$-0.0584663041272371 + i0.34694 \times 10^{-15}$	0.35160×10^{-14}

Table 3: Quadratures for $e^{ic\langle x,t\rangle}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of radial nodes for $c = 20$. Gaussian quadratures generated via Algorithm 5.2 are used in the radial direction.

c	radial nodes	angular nodes	integral via quadrature	relative error
100	30	140	$0.0164989321769857 - i0.50090 \times 10^{-16}$	0.10612×10^{2}
100	32	140	$-0.0019104800874610 - i0.65264 \times 10^{-15}$	0.11305×10^{0}
100	34	140	$-0.0017165140985462 - i0.21673 \times 10^{-15}$	0.45510×10^{-4}
100	36	140	$-0.0017164370759186 - i0.27856 \times 10^{-15}$	0.63672×10^{-6}
100	38	140	$-0.0017164359820963 - i0.30840 \times 10^{-15}$	0.54009×10^{-9}
100	40	140	$-0.0017164359830235 - i0.30398 \times 10^{-15}$	0.94943×10^{-13}

Table 4: Quadratures for $e^{ic\langle x,t\rangle}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of radial nodes for $c = 100$. Chebyshev quadratures are used in the radial direction.

c	radial nodes	angular nodes	integral via quadrature	relative error
100	40	115	$-0.0017164359830236 - i0.21183 \times 10^{-6}$	0.12341×10^{-3}
100	40	120	$-0.0017164338146549 - i0.44658 \times 10^{-15}$	0.12633×10^{-5}
100	40	125	$-0.0017164359830231 - i0.48252 \times 10^{-10}$	0.28112×10^{-7}
100	40	130	$-0.0017164359819925 - i0.11947 \times 10^{-14}$	0.60096×10^{-9}
100	40	135	$-0.0017164359830233 + i0.24522 \times 10^{-14}$	0.13296×10^{-11}
100	40	140	$-0.0017164359830235 - i0.30398 \times 10^{-15}$	0.94943×10^{-13}
100	40	145	$-0.0017164359830231 - i0.78770 \times 10^{-15}$	0.23749×10^{-12}
100	40	150	$-0.0017164359830231 - i0.31466 \times 10^{-15}$	0.16075×10^{-12}

Table 5: Quadratures for $e^{ic\langle x,t\rangle}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of angular nodes for $c = 100$. Chebyshev quadratures are used in the radial direction.

c	radial nodes	angular nodes	integral via quadrature	relative error
100	20	150	$-0.0017164492038983 - i0.52665 \times 10^{-15}$	0.77025×10^{-5}
100	22	150	$-0.0017164359833709 - i0.60352 \times 10^{-15}$	0.20280×10^{-9}
100	24	150	$-0.0017164359830226 - i0.45244 \times 10^{-15}$	0.28465×10^{-12}
100	26	150	$-0.0017164359830229 - i0.35123 \times 10^{-15}$	0.50904×10^{-13}
100	28	150	$-0.0017164359830224 - i0.55262 \times 10^{-15}$	0.35430×10^{-12}
100	30	150	$-0.0017164359830228 - i0.63794 \times 10^{-15}$	0.39846×10^{-12}

Table 6: Quadratures for $e^{ic\langle x,t\rangle}$ where $x = (0.9, 0.2)$ over the unit disk using several different numbers of radial nodes for $c = 100$. Gaussian quadratures generated using Algorithm 5.2 are used in the radial direction.

radial nodes	angular nodes	c	N	n	$\lvert\alpha_{N,n}\rvert$
40	140	50	1	0	$0.5331000423667240 \times 10^{-2}$
40	140	50	1	1	$0.4428631717847083 \times 10^{-1}$
40	140	50	1	2	$0.1658210569373790 \times 10^{0}$
40	140	50	1	3	$0.3007289752527894 \times 10^{0}$
40	140	50	1	4	$0.1775918995268194 \times 10^{0}$
40	140	50	1	5	$0.1698366869978232 \times 10^{0}$
40	140	50	1	6	$0.1326556850627168 \times 10^{0}$
40	140	50	1	7	$0.1913962335203701 \times 10^{0}$
40	140	50	1	8	$0.1031820332780429 \times 10^{-1}$
40	140	50	1	9	$0.1525659498901890 \times 10^{0}$
40	140	50	1	10	$0.1596240985391338 \times 10^{0}$
40	140	50	1	11	$0.5077661980005956 \times 10^{-1}$
40	140	50	1	12	$0.7004482833257132 \times 10^{-1}$
40	140	50	1	13	$0.1328923889087414 \times 10^{0}$
40	140	50	1	14	$0.1238722286983581 \times 10^{0}$
40	140	50	1	15	$0.6158313809902630 \times 10^{-1}$
40	140	50	1	16	$0.9273653953916678 \times 10^{-2}$
40	140	50	1	17	$0.1222486302912020 \times 10^{-2}$
40	140	50	1	18	$0.5966018610435559 \times 10^{-3}$
40	140	50	1	19	$0.9457503976218055 \times 10^{-4}$
40	140	50	1	20	$0.7272803775518590 \times 10^{-5}$
40	140	50	1	21	$0.2471737500102828 \times 10^{-7}$
40	140	50	1	22	$0.5697214169860662 \times 10^{-7}$
40	140	50	1	23	$0.6261378248559833 \times 10^{-8}$
40	140	50	1	24	$0.2876620855784414 \times 10^{-9}$
40	140	50	1	25	$0.3487372839216281 \times 10^{-11}$
40	140	50	1	26	$0.1344784001636234 \times 10^{-11}$
40	140	50	1	27	$0.8389389113185264 \times 10^{-13}$
40	140	50	1	28	$0.3090050472181085 \times 10^{-14}$
40	140	50	1	29	$0.5438594709432636 \times 10^{-15}$

Table 7: Coefficients, obtained via formula (242), of the GPSF expansion of the function on the unit disk $e^{ic\langle x,t\rangle}$ where $x = (0.3, 0.4)$.

radial nodes	angular nodes	c	N	n	$\|\alpha_{N,n}\|$
40	140	50	10	0	$0.6083490415455435 \times 10^{-1}$
40	140	50	10	1	$0.2656230046895768 \times 10^{-1}$
40	140	50	10	2	$0.4475286860599875 \times 10^{-1}$
40	140	50	10	3	$0.4833769722091774 \times 10^{-2}$
40	140	50	10	4	$0.3152644364681537 \times 10^{-1}$
40	140	50	10	5	$0.3440099078209665 \times 10^{-1}$
40	140	50	10	6	$0.1216643028774711 \times 10^{-1}$
40	140	50	10	7	$0.1348650121618380 \times 10^{-1}$
40	140	50	10	8	$0.2761069443786074 \times 10^{-1}$
40	140	50	10	9	$0.2729520957518510 \times 10^{-1}$
40	140	50	10	10	$0.1713503999971936 \times 10^{-1}$
40	140	50	10	11	$0.4647646609621038 \times 10^{-2}$
40	140	50	10	12	$0.5498106244002701 \times 10^{-3}$
40	140	50	10	13	$0.4531628449744277 \times 10^{-3}$
40	140	50	10	14	$0.9388943333348342 \times 10^{-4}$
40	140	50	10	15	$0.1018790565231280 \times 10^{-4}$
40	140	50	10	16	$0.4628420439758330 \times 10^{-6}$
40	140	50	10	17	$0.3302969345113099 \times 10^{-7}$
40	140	50	10	18	$0.7386880328505609 \times 10^{-8}$
40	140	50	10	19	$0.5793842432833322 \times 10^{-9}$
40	140	50	10	20	$0.1808244166685658 \times 10^{-10}$
40	140	50	10	21	$0.8331243140844428 \times 10^{-12}$
40	140	50	10	22	$0.1247624356690115 \times 10^{-12}$
40	140	50	10	23	$0.6402836746674745 \times 10^{-14}$
40	140	50	10	24	$0.3219490617035674 \times 10^{-15}$
40	140	50	10	25	$0.4392156715211933 \times 10^{-16}$
40	140	50	10	26	$0.4216375878565715 \times 10^{-16}$
40	140	50	10	27	$0.1192730971046164 \times 10^{-15}$
40	140	50	10	28	$0.5964172072581517 \times 10^{-16}$
40	140	50	10	29	$0.9795188267888765 \times 10^{-16}$

Table 8: Coefficients, obtained via formula (242), of the GPSF expansion of the function on the unit disk $e^{ic\langle x,t\rangle}$ where $x = (0.3, 0.4)$.

radial nodes	angular nodes	c	N	n	$\|\alpha_{N,n}\|$
40	140	50	30	0	$0.4972797526740737 \times 10^{-3}$
40	140	50	30	1	$0.1401428942935588 \times 10^{-2}$
40	140	50	30	2	$0.2710925506457800 \times 10^{-2}$
40	140	50	30	3	$0.3545718524468668 \times 10^{-2}$
40	140	50	30	4	$0.2241476750854641 \times 10^{-2}$
40	140	50	30	5	$0.6682792235496368 \times 10^{-3}$
40	140	50	30	6	$0.1339565034261751 \times 10^{-3}$
40	140	50	30	7	$0.2092420216819930 \times 10^{-4}$
40	140	50	30	8	$0.2648137075865133 \times 10^{-5}$
40	140	50	30	9	$0.2763313112747597 \times 10^{-6}$
40	140	50	30	10	$0.2398228591769509 \times 10^{-7}$
40	140	50	30	11	$0.1734961623216772 \times 10^{-8}$
40	140	50	30	12	$0.1041121888882874 \times 10^{-9}$
40	140	50	30	13	$0.5099613478241473 \times 10^{-11}$
40	140	50	30	14	$0.1958321703329898 \times 10^{-12}$
40	140	50	30	15	$0.5129356249335817 \times 10^{-14}$
40	140	50	30	16	$0.1790936343596214 \times 10^{-15}$
40	140	50	30	17	$0.2114685083013237 \times 10^{-15}$
40	140	50	30	18	$0.1221114171480004 \times 10^{-15}$
40	140	50	30	19	$0.1775028830408308 \times 10^{-15}$
40	140	50	30	20	$0.9115790774963023 \times 10^{-16}$
40	140	50	30	21	$0.7676533284257323 \times 10^{-16}$
40	140	50	30	22	$0.1056865232130847 \times 10^{-15}$
40	140	50	30	23	$0.1282851493246300 \times 10^{-15}$
40	140	50	30	24	$0.1301036117017623 \times 10^{-15}$
40	140	50	30	25	$0.6302967734899496 \times 10^{-16}$
40	140	50	30	26	$0.7542252119317336 \times 10^{-16}$
40	140	50	30	27	$0.6734661033178358 \times 10^{-16}$
40	140	50	30	28	$0.7752009233608849 \times 10^{-16}$
40	140	50	30	29	$0.1184019207536341 \times 10^{-15}$

Table 9: Coefficients, obtained via formula (242), of the GPSF expansion of the function on the unit disk $e^{ic\langle x,t\rangle}$ where $x = (0.3, 0.4)$.

8 Miscellaneous Properties of GPSFs

8.1 Properties of the Derivatives of GPSFs

The following theorem follows immediately from (113) and (117).

Theorem 8.1 *Let $c > 0$. Then*

$$\frac{d}{dx}\Big((x^{p+1} - x^{p+3})\frac{d\Phi_{N,n}}{dx}(x)\Big) + \Big(\chi_{N,n}x^{p+1} - \frac{(p+1)(p+3)}{4}x^{p+1} - N(N+p)x^{p-1} - c^2x^{p+3}\Big)\Phi_{N,n}(x) = 0, \tag{267}$$

where $0 < x < 1$ and N and n are arbitrary nonnegative integers.

Corollary 8.2 *Let $c > 0$. Then*

$$x^2(1-x^2)\Phi''_{N,n}(x) + \big((p+1)x - (p+3)x^3\big)\Phi'_{N,n}(x) + \Big(\chi_{N,n}x^2 - \frac{(p+1)(p+3)}{4}x^2 - N(N+p) - c^2x^4\Big)\Phi_{N,n}(x) = 0, \tag{268}$$

where $0 < x < 1$ and N and n are arbitrary nonnegative integers.

The following lemma connects the values of the $(k+2)$nd derivative of the function $\Phi_{N,n}$ with its derivatives of orders $k-4, k-3, \dots, k+1$, and is obtained by repeated differentiation of (268).

Lemma 8.3 *Let $c > 0$. Then*

$$\begin{aligned}
&(x^2 - x^4)\Phi^{(k+2)}_{N,n}(x) + \big((2k+1+p)x - (4k+3+p)x^3\big)\Phi^{(k+1)}_{N,n}(x) \\
&\quad + \Big(k(k+p) - N(N+p) + \big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3) \\
&\qquad - 3k(2k+1+p)\big]x^2 - c^2x^4\Big)\Phi^{(k)}_{N,n}(x) \\
&\quad + \Big(\big[2k\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - k(k-1)(4k+1+3p)\big]x - 4kc^2x^3\Big)\Phi^{(k-1)}_{N,n}(x) \\
&\quad + \Big(k(k-1)\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - k(k-1)(k-2)(k+p) - 6k(k-1)c^2x^2\Big)\Phi^{(k-2)}_{N,n}(x) \\
&\qquad - 4k(k-1)(k-2)c^2x\Phi^{(k-3)}_{N,n}(x) - k(k-1)(k-2)(k-3)c^2\Phi^{(k-4)}_{N,n}(x) = 0,
\end{aligned} \tag{269}$$

where $0 < x < 1$, N and n are arbitrary nonnegative integers, and k is an arbitrary integer so that $k \geq 4$. Also,

$$(x^2 - x^4)\Phi''_{N,n}(x) + \big((p+1)x - (p+3)x^3\big)\Phi'_{N,n}(x) + \Big(-N(N+p) + \big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+2)\big]x^2 - c^2x^4\Big)\Phi_{N,n}(x) = 0, \quad (270)$$

and

$$(x^2 - x^4)\Phi^{(3)}_{N,n}(x) + \big((p+3)x - (p+7)x^3\big)\Phi''_{N,n}(x) + \Big((p+1) - N(N+p) + \big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3) - 3(p+3)\big]x^2 - c^2x^4\Big)\Phi'_{N,n}(x) + \Big(2\big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big]x - 4c^2x^3\Big)\Phi_{N,n}(x) = 0, \quad (271)$$

and

$$(x^2 - x^4)\Phi^{(4)}_{N,n}(x) + \big((p+5)x - (p+11)x^3\big)\Phi^{(3)}_{N,n}(x) + \Big(2(p+2) - N(N+p) + \big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3) - 6(p+5)\big]x^2 - c^2x^4\Big)\Phi''_{N,n}(x) + \Big(\big[4\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - 6(p+3)\big]x - 8c^2x^3\Big)\Phi'_{N,n}(x) + \Big(2\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - 12c^2x^2\Big)\Phi_{N,n}(x) = 0, \quad (272)$$

and

$$(x^2 - x^4)\Phi^{(5)}_{N,n}(x) + \big((p+7)x - (p+15)x^3\big)\Phi^{(4)}_{N,n}(x) + \Big(3(p+3) - N(N+p) + \big[\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3) - 9(p+7)\big]x^2 - c^2x^4\Big)\Phi^{(3)}_{N,n}(x) + \Big(\big[6\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - 6(3p+13)\big]x - 12c^2x^3\Big)\Phi''_{N,n}(x) + \Big(6\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - 6(p+3) - 36c^2x^2\Big)\Phi'_{N,n}(x) - 24c^2x\Phi_{N,n}(x) = 0, \quad (273)$$

where $0 < x < 1$ and N and n are arbitrary nonnegative integers.

The following corollary and theorem are obtained immediately from Lemma 8.3.

Corollary 8.4 *Let $c > 0$. Then*

$$\big(k(k+p) - N(N+p)\big)\Phi^{(k)}_{N,n}(0) + \Big(k(k-1)\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - k(k-1)(k-2)(k+p)\Big)\Phi^{(k-2)}_{N,n}(0) - k(k-1)(k-2)(k-3)c^2\Phi^{(k-4)}_{N,n}(0) = 0, \quad (274)$$

where N and n are arbitrary nonnegative integers, and k is an arbitrary integer so that $k \geq 4$. Also,

$$N(N+p)\Phi_{N,n}(0) = 0, \tag{275}$$

and

$$\big((p+1) - N(N+p)\big)\Phi'_{N,n}(0) = 0, \tag{276}$$

and

$$\big(2(p+2) - N(N+p)\big)\Phi''_{N,n}(0) + 2\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big)\Phi_{N,n}(0) = 0, \tag{277}$$

and

$$\begin{aligned}&\big(3(p+3) - N(N+p)\big)\Phi^{(3)}_{N,n}(0)\\ &\qquad + \Big(6\big(\chi_{N,n} - \tfrac{1}{4}(p+1)(p+3)\big) - 6(p+3)\Big)\Phi'_{N,n}(0) = 0,\end{aligned} \tag{278}$$

where N and n are arbitrary nonnegative integers.

Theorem 8.5 *If $N = 0$, then*

$$\Phi_{N,n}(0) \neq 0, \tag{279}$$

where n is an arbitrary nonnegative integer. If $N \geq 1$, then

$$\Phi^{(k)}_{N,n}(0) = 0 \quad \text{for } k = 0, 1, \ldots, N-1, \tag{280}$$

and

$$\Phi^{(N)}_{N,n}(0) \neq 0, \tag{281}$$

where n is an arbitrary nonnegative integer.

The following theorem follows directly from Theorem ??.

Theorem 8.6 *Suppose that N and n are nonnegative integers. Then*

$$\Phi_{N,n}(1) \neq 0. \tag{282}$$

8.2 Derivatives of GPSFs and Corresponding Eigenvalues With Respect to c

The following two theorems establish formulas for the derivatives of the eigenvalues $\mu_{N,n}$ (see (98)) and $\beta_{N,n}$ (see (109)) with respect to c.

Theorem 8.7 *Suppose that $c > 0$ is real and that N and n are nonnegative integers. Then*

$$\frac{\partial \beta_{N,n}}{\partial c} = \beta_{N,n} \frac{(\Phi_{N,n}(1))^2 - (p+2)}{2c}, \tag{283}$$

and

$$\frac{\partial \mu_{N,n}}{\partial c} = \frac{\mu_{N,n}}{c}((\Phi_{N,n}(1))^2 - (p+1)). \tag{284}$$

8.3 Integrals of Products of GPSFs and Their Derivatives

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10 Appendix A

10.1 Derivation of the Integral Operator Q_c

In this section we derive an explicit formula for the integral operator Q_c, defined in (97).

Suppose that B denotes the closed unit ball in $\mathbb{R}^{p+2}$. From (97),

$$Q_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p+2} \int_B \int_B e^{ic\langle x-t,u\rangle} \psi(t)\, du\, dt, \tag{285}$$

for all $x \in B$. We observe that

$$e^{ic\langle v,u\rangle} = \sum_{N=0}^{\infty} \sum_{\ell=1}^{h(N,p)} i^N (2\pi)^{p/2+1} \frac{J_{N+p/2}(c\|u\|\|v\|)}{(c\|u\|\|v\|)^{p/2}} S_N^\ell(u/\|u\|) S_N^\ell(v/\|v\|), \tag{286}$$

for all $u, v \in B$, where S_N^ℓ denotes the spherical harmonics of degree N, and J_ν denotes Bessel functions of the first kind (see Section VII of [1]). Therefore,

$$\begin{aligned}\int_B e^{ic\langle v,u\rangle}\,du &= (2\pi)^{p/2+1}\int_0^1 \frac{J_{p/2}(c\|v\|\rho)}{(c\|v\|\rho)^{p/2}}\rho^{p+1}\,d\rho \\ &= \frac{(2\pi)^{p/2+1}}{(c\|v\|)^{p/2}}\int_0^1 \rho^{p/2+1}J_{p/2}(c\|v\|\rho)\,d\rho \\ &= \left(\frac{2\pi}{c}\right)^{p/2+1}\frac{J_{p/2+1}(c\|v\|)}{\|v\|^{p/2+1}}, \qquad (287)\end{aligned}$$

for all $v \in \mathbb{R}^{p+2}$, where the last equality follows from formula 6.561(5) in [5]. Combining (285) and (287),

$$Q_c[\psi](x) = \left(\frac{c}{2\pi}\right)^{p/2+1}\int_B \frac{J_{p/2+1}\big(c\|x-t\|\big)}{\|x-t\|^{p/2+1}}\psi(t)\,dt, \qquad (288)$$

for all $x \in \mathbb{R}^{p+2}$.

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