

In the present paper we describe a method for solving inverse problems for the Helmholtz equation in radially-symmetric domains given multi-frequency data. Our approach is based on the construction of suitable trace formulas which relate the impedance of the total field at multiple frequencies to derivatives of the potential. Using this trace formula we obtain a system of coupled differential equations which can be solved to obtain the potential in a stable manner. Finally, the performance of the reconstruction algorithm is illustrated with several numerical examples.

**On the inverse scattering problem for radially-symmetric
domains in two dimensions**

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Technical Report YALEU/DCS/TR-1538

February 13, 2019

◇ This author was supported in part by AFOSR FA9550-16-1-0175 and by the ONR (award no. N00014-14-1-0797)

Keywords: *Inverse scattering, trace formula, partial differential equations, inverse problems*

1 Introduction

Inverse scattering problems for the Helmholtz equation are ubiquitous in both pure mathematics and applied sciences. Such problems arise *inter alia* in medical imaging, ocean acoustics, remote sensing, geophysical exploration, non-destructive testing, radar, etc. Despite the obvious utility of numerical methods for inverse scattering, the construction of stable numerical inversion algorithms poses several difficulties. First and foremost is the inherent non-linearity of the problem. For the one-dimensional problem, it can be mapped to a linear problem [8, 16], though the mapping is not always numerically stable. In two and three dimensions no such mappings are known. This makes the construction of mathematically-rigorous inversion approaches difficult except in low-contrast regimes. Even when a mathematically-rigorous inversion approach exists the construction of a numerically stable scheme often adds significant challenges. Finally, given a stable numerical algorithm the computational cost is an additional factor; many algorithms can become prohibitively slow when the contrast is large or high accuracy is required.

The simplest approach is to linearize the nonlinear problem; specifically, the nonlinear problem in question is replaced by its linearization about an approximation to the true material parameters (most frequently assuming them to be constant). The result is a linear system which can be solved using standard techniques (see [29] for example).

Two classes of algorithms for solving the fully nonlinear problem are nonlinear optimization methods and direct methods. Broadly speaking, nonlinear optimization methods seek to recover material parameters in an iterative fashion by solving a sequence of (forward or inverse) scattering problems (see for example [32, 25, 27, 2, 3, 5, 4, 20, 17, 18, 19, 23, 24, 6]), while direct methods rely on a variety techniques originating from signal and image processing (see for example [11, 12, 28, 21]).

A frequently encountered third approach, particularly in one dimension, is based on *trace formulae* (for example [14],[13],[30], [31]) which relate data in a range of frequencies to local material parameters. In [9] an algorithm based on trace formulas was introduced for solving the inverse scattering problem for the Helmholtz equation in one dimension

with multifrequency data in a numerically stable and computationally efficient manner. This paper extends this approach to the case of radially-symmetric problems in two dimensions, though the apparatus described here can be immediately applied to higher dimensions.

The remainder of the paper is organized as follows. In Section 2 we give a detailed description of the problem and review necessary mathematical preliminaries. In Section 3 we summarize the analytical apparatus necessary for the remainder of the paper. In Section 4 we state and prove a trace formula for two-dimensional radially-symmetric scattering. Finally, Section 5 describes a numerical algorithm based on the constructed trace formula and provides numerical illustrations demonstrating the reconstructions it produces.

2 Preliminaries

2.1 Formulation of the problem

In this paper we consider the inverse scattering problem for acoustic waves in radially-symmetric annuli in two dimensions. Let Ω denote the annulus centered at the origin with inner radius a and outer radius b . Let B_a denote the ball of radius a centered at the origin. At a single frequency $k \in \mathbb{C}$ such that $\text{Im } k \geq 0$, the time-harmonic acoustic wave equation is

$$\Delta u(\mathbf{x}) + k^2 (1 + Q(\mathbf{x})) u(\mathbf{x}) = f(\mathbf{x}) \tag{1}$$

subject to the Sommerfeld radiation condition

$$\lim_{r \rightarrow \infty} \left(\frac{\partial u}{\partial r} + iku \right) = 0.$$

Here we assume the source $f \in L^2$ is a function supported on B_a and the potential Q is a continuous compactly-supported radially-symmetric function. In particular, we assume that $Q(\mathbf{r}) = q(\|\mathbf{r}\|)$ for some continuous function $q : \mathbb{R} \rightarrow \mathbb{R}$ supported on

the interval $[a, b]$ with $0 < a < b < \infty$. Here $\|\cdot\|$ denotes the standard Euclidean norm in \mathbb{R}^2 . Moreover, we assume that there exist two constants q_0 and q_1 such that $-1 < q_0 < q(r) < q_1 < \infty$ for all $a < r < b$.

2.2 Reduction to the radial problem

In this section we reduce the radially-symmetric acoustic scattering problem to a set of decoupled one-dimensional scattering problems. Let $u_n : [0, \infty) \rightarrow \mathbb{C}$ be the Fourier coefficient of u with respect to the angle θ , i.e.

$$u_n(r) = \int_0^{2\pi} e^{-in\theta} u(r \cos \theta, r \sin \theta) d\theta. \quad (2)$$

For any integer n the function u_n satisfies the differential equation

$$u_n''(r) + \frac{1}{r}u_n'(r) + k^2[1 + q(r)]u_n(r) - \frac{n^2}{r^2}u_n(r) = f_n(r), \quad (3)$$

where

$$f_n(r) = \int_0^{2\pi} e^{-in\theta} f(r \cos \theta, r \sin \theta) d\theta, \quad (4)$$

is the Fourier coefficient of f .

Remark 2.1. *On any interval $c < x < d$ on which the source f and potential q are identically zero the solutions to equation (3) are a linear combination of the Bessel function $J_n(kr)$ and the Hankel function $H_n(kr)$. Specifically, if f_n is supported on the interval $[0, R]$ and q is supported on the interval $[a, b]$ then for all $A \in \mathbb{C}$ there exist constants α, μ depending only on the source f_n and potential q such that for all $R < r < a$*

$$u_n(r) = AH_n(kr) + \alpha J_n(kr) \quad (5)$$

and for all $r > b$,

$$u_n(r) = \mu H_n(kr). \quad (6)$$

Remark 2.2. *The data required by the recovery algorithm presented in this paper depends only on the quantity $u'_n(r)/u_n(r)$. Hence in the remainder of the paper we will assume that solution $u_n(r)$ is scaled so that $\mu = 1$.*

In the following it will be convenient to rescale u_n by \sqrt{r} . This new quantity, $\sqrt{r}u_n(r)$, also satisfies a differential equation which can be readily obtained from equation (3).

Lemma 2.1. *Let u_n be a solution to the differential equation*

$$u_n''(r) + \frac{1}{r}u_n'(r) + k^2[1 + q(r)]u_n(r) - \frac{n^2}{r^2}u_n(r) = 0, \quad (7)$$

and define the function $\psi_n : [a, b] \rightarrow \mathbb{C}$ by

$$\psi_n(r) = \sqrt{r} u_n(r). \quad (8)$$

Then ψ_n satisfies the equation

$$\psi_n''(r) + k^2(1 + q(r))\psi_n(r) - \frac{n^2 - \frac{1}{4}}{r^2}\psi_n = 0, \quad (9)$$

with the boundary conditions

$$\psi_n(b) = \sqrt{b} H_n(kb), \quad (10)$$

$$\psi_n'(b) = k\sqrt{b} H_n'(kb) + \frac{1}{2b} H_n(kb). \quad (11)$$

Remark 2.3. The function $\psi_n : [a, b] \rightarrow \mathbb{C}$ can be extended to a function defined on $(0, \infty)$. Specifically, for $r > b$ we set

$$\psi_n(r) = \sqrt{r} H_n(kr) \quad (12)$$

and for $r < a$ we set

$$\psi_n(r) = \alpha\sqrt{r}H_n(kr) + \beta\sqrt{r}J_n(kr) \quad (13)$$

where the coefficients α and β are chosen so that $\psi_n(r)$ and $\psi'_n(r)$ are continuous at a .

2.3 Impedance

In this section we introduce the concept of impedance (see, for example, [31]) and summarize its properties which are relevant to the subsequent analysis.

Definition 2.1. *Given a solution ψ_n of (8) the impedance $\phi_n : [a, b] \times \{z \in \mathbb{C} : z \neq 0, \text{Im}z \geq 0\} \rightarrow \mathbb{C}$ is the function defined by the formula*

$$\phi_n(r, k) = \frac{\psi'_n(r)}{ik\psi_n(r)}. \quad (14)$$

Remark 2.4. The impedance ϕ_n corresponds to a scattering problem in which the source is located inside an annulus and propagates outward to infinity. One could in principle define an inward impedance corresponding to an incoming wave impinging upon an annulus and reflecting outward. For the inward impedance interference from waves passing through the scatterer from opposite directions produces poles which necessitate a different approach. A detailed analysis of the trace formula for the inward impedance will be published at a later date.

The definition of the impedance and equation (9) immediately imply the following lemma.

Lemma 2.2. *For all $a \leq r \leq b$, the impedance ϕ_n satisfies the Riccati equation*

$$\frac{\partial}{\partial r}\phi_n(r, k) = -ik\phi_n^2(r, k) - ik(1 + q(r)) + i\frac{n^2 - \frac{1}{4}}{kr^2}, \quad (15)$$

together with the boundary condition

$$\phi_n(b, k) = \frac{H'_n(kb)}{iH_n(kb)} + \frac{1}{2ibk}. \quad (16)$$

Remark 2.5. In a mild abuse of notation unless otherwise stated we will denote derivatives of $\phi_n(r, k)$ with respect to r by $\phi'_n(r, k)$.

Corollary 2.3. Suppose n is a non-negative integer and consider the function

$$w : [a, b] \times \{k \in \mathbb{C} : k \neq 0, \text{Im } k \geq 0\} \rightarrow \mathbb{C}$$

defined by

$$w(r, k) = \phi_n(r, k) - \frac{(\sqrt{r}H_n(kr))'}{ik\sqrt{r}H_n(kr)}, \quad (17)$$

where ϕ_n is the impedance defined in (14). Then for all $a \leq r \leq b$

$$w'(r, k) = -ikw(r, k) \left(w(r, k) + 2 \frac{(\sqrt{r}H_n(kr))'}{ik\sqrt{r}H_n(kr)} \right) - ikq(r), \quad (18)$$

and $w(b, k) = 0$ for all non-zero $k \in \mathbb{C}$ with non-negative imaginary part. Obviously, the differential equation (18) is equivalent to the integral equation

$$w(r, k) = ik \int_r^b \left(w(x, k) \left(w(x, k) + 2 \frac{(\sqrt{x}H_n(kx))'}{ik\sqrt{x}H_n(kx)} \right) + q(x) \right) dx, \quad a \leq r \leq b. \quad (19)$$

We conclude this section with the following lemma which characterizes the symmetry of the impedance in frequency.

Lemma 2.4. Let k be a non-zero real number and $0 < r < \infty$. Then for all non-negative integers n

$$\phi_n(r, k) = \overline{\phi_n(r, -k)}. \quad (20)$$

2.4 Properties of Bessel functions

In this section we list certain properties of Bessel and Hankel functions which will be used in the subsequent analysis.

Proposition 2.5. *Let n be a non-negative integer and z be a non-zero complex number with a non-negative imaginary part. The Bessel functions of the first and second kind have the following expansions about $z = 0$*

$$\begin{aligned} J_n(z) &= \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(n+k+1)}, \\ Y_n(z) &= -\frac{\left(\frac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{1}{4}z^2\right)^k + \frac{2}{\pi} \log\left(\frac{1}{2}z\right) J_n(z) - \\ &\quad \frac{\left(\frac{1}{2}z\right)^n}{\pi} \sum_{k=0}^{\infty} (\Psi(k+1) + \Psi(n+k+1)) \frac{\left(-\frac{1}{4}z^2\right)^k}{k!(n+k)!}, \end{aligned} \quad (21)$$

where $\Psi(z) = \Gamma'(z)/\Gamma(z)$ and we take the branch cut of \log to lie along the negative imaginary axis.

Moreover, if $H_n(z)$ denotes the n th order Hankel function then

$$\begin{aligned} H_n(z) &= \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{1}{4}z^2\right)^k}{k! \Gamma(n+k+1)} \\ &\quad - i \frac{\left(\frac{1}{2}z\right)^{-n}}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{1}{4}z^2\right)^k + i \frac{2}{\pi} \log\left(\frac{1}{2}z\right) J_n(z) - \\ &\quad i \frac{\left(\frac{1}{2}z\right)^n}{\pi} \sum_{k=0}^{\infty} (\Psi(k+1) + \Psi(n+k+1)) \frac{\left(-\frac{1}{4}z^2\right)^k}{k!(n+k)!}. \end{aligned} \quad (22)$$

Remark 2.6. It follows immediately from (22) that for all $n = 0, 1, 2, \dots$ there exists a constant C_n depending only on n such that if $|z| \leq C_n$ then

$$\left| \frac{H'_n(z)}{H_n(z)} \right| \leq \frac{4(n+1)}{|z|} \quad (23)$$

Hankel functions also have the following asymptotic expansions valid for large arguments.

Proposition 2.6. *Let n be a non-negative integer and $z \in \mathbb{C}$ such that $\text{Im}(z) \geq 0$. Then*

$$H_n(z) \sim \frac{2}{\pi i^{n+1}} \sqrt{\frac{\pi}{2z}} e^{iz + \frac{i\pi}{4}} \left(1 - \frac{n^2 - 1}{-8iz} + \frac{(4n^2 - 1)(4n^2 - 9)}{2(-8iz)^2} + \dots \right) \quad (24)$$

Corollary 2.7. *Let n be a non-negative integer and z a complex number of magnitude one with non-negative imaginary part. Then for all $\lambda \in \mathbb{R}$ with $\lambda > 0$,*

$$\left| \frac{H'_n(\lambda z)}{H_n(\lambda z)} + \frac{1}{2\lambda z} \right| = O(\lambda^{-2}) \quad (25)$$

as $\lambda \rightarrow \infty$.

The following proposition gives a formula for the Wronskian of J_n and Y_n and can be found, for example, in [1].

Proposition 2.8. *Let n be an integer and z be a complex number which is not a non-positive purely-imaginary number. Then*

$$J_n(z)Y'_n(z) - J'_n(z)Y_n(z) = \frac{2}{\pi z}. \quad (26)$$

A similar result holds for the Wronskian of J_n and H_n . Its proof is an immediate consequence of the definition of H_n and the preceding proposition.

Corollary 2.9. *Let n be an integer and z be any complex number which is not a non-positive purely-imaginary number. Then*

$$J_n(z)H'_n(z) - J'_n(z)H_n(z) = \frac{2i}{\pi z}. \quad (27)$$

2.5 Basic lemmas

Lemma 2.10 provides a variant of Gronwall's inequality (see, for example, [26]).

Lemma 2.10. *Suppose that $A, B \in \mathbb{C}$ and F, G are two positive real numbers. Suppose further that $f : [a, b] \rightarrow \mathbb{C}$ and $g : [a, b] \rightarrow \mathbb{C}$ are two functions such that $|f(r)| \leq F < \infty$*

and $|g(r)| \leq G < \infty$ for all $a \leq r \leq b$ and that $w : [a, b] \rightarrow \mathbb{C}$ is the function defined by

$$w(r) = A \int_r^b (w(x) + f(x)) w(x) dx + B \int_r^b g(x) dx. \quad (28)$$

If the constants A, B, F , and G are such that

$$\left(1 + \frac{F}{|B|G}\right) > \frac{1}{2} e^{\frac{|A|}{F}(b-a)}, \quad (29)$$

then for all $a \leq r \leq b$,

$$|w(r)| \leq \frac{4}{F}. \quad (30)$$

The following lemma provides a bound on the solutions to a certain initial value problem arising in the WKB approximation of solutions to inhomogeneous Helmholtz equations in one dimension (for proofs see, for example, [15, 9]).

Lemma 2.11. *Suppose that T and M are positive constants and let K be the set defined by*

$$K = \{k \neq 0 \in \mathbb{C} : \text{Im } k \geq -M, |k| \geq 1\}.$$

Suppose further that $\eta : [0, T] \times K \rightarrow \mathbb{R}$ is an absolutely continuous function uniformly bounded on $[0, T] \times K$. Let $w : [0, T] \times K \rightarrow \mathbb{C}$ be the solution to the following initial value problem

$$w''(t, k) - 2ikw'(t, k) = \eta(t, k)w(t, k) \quad (31)$$

$$w(0, k) = 1 \quad (32)$$

$$w'(0, k) = 0, \quad (33)$$

where $'$ denotes differentiation with respect to t . Then there exist constants C_1 and C_2

depending on η , M and T but which are independent of t and k such that

$$\left| w(t, k) - 1 + \frac{1}{2ik} \int_0^t \eta(\tau, k) \right| \leq \frac{C_1}{|k|^2}, \quad (34)$$

$$|w'(t, k)| \leq \frac{C_2}{|k|}, \quad (35)$$

for all $t \in [0, T]$.

3 Mathematical apparatus

In this section we establish properties of the impedance used in the construction of the trace formula.

Proposition 3.1. *Let n be a non-negative integer, and k be a non-zero complex number with non-negative imaginary part. Then the function u_n (see (3)) has no zeros on the interval $0 < r < \infty$ and hence neither does ψ_n defined in (8).*

Proof. First suppose that $\text{Im } k^2 \neq 0$. Note that $\lim_{r \rightarrow \infty} u_n(r) = 0$ and hence

$$ru_n(r)\bar{u}'_n(r) - ru'_n(r)\bar{u}_n(r) = \int_r^\infty (\bar{u}_n(x)(xu'_n(x))' - u_n(x)(x\bar{u}'_n(x))') dx. \quad (36)$$

The substitution of (7) into the right-hand side of the previous equation yields

$$ru_n(r)\bar{u}'_n(r) - ru'_n(r)\bar{u}_n(r) = \text{Im } k^2 \int_r^\infty x|u_n(x)|^2 dx, \quad (37)$$

and hence clearly u_n cannot vanish for any $r > 0$.

Next suppose $k = i\kappa$, for $\kappa \in \mathbb{R}^+$. Then u_n is real and

$$\int_r^\infty u_n(x)(xu'_n(x))' - \left(\kappa^2(1+q) + \frac{n^2}{x^2} \right) xu_n^2(x) dx = 0. \quad (38)$$

Integrating (38) by parts gives

$$ru_n(r)u_n'(r) = - \int_r^\infty x(u_n'(x))^2 + \left(\kappa^2(1+q) + \frac{n^2}{x^2} \right) xu_n^2(x) dx \quad (39)$$

which implies that $u_n(r) \neq 0$ for all $r > 0$.

Finally, suppose that $k \in \mathbb{R}$. By Remark 2.1 there exists a constant β such that for all $r > b$, $u_n(r) = \beta H_n(kr)$. If $k > 0$ then $J_n(kr)$ and $Y_n(kr)$ are real in which case $\operatorname{Re}(u_n(r)/\beta) = J_n(kr)$ and $\operatorname{Im}(u_n(r)/\beta) = Y_n(kr)$. Substituting these expressions into (26), we obtain

$$\operatorname{Re} \left(\frac{u_n'(r)}{\beta} \right) \operatorname{Im} \left(\frac{u_n(r)}{\beta} \right) - \operatorname{Re} \left(\frac{u_n(r)}{\beta} \right) \operatorname{Im} \left(\frac{u_n'(r)}{\beta} \right) = \frac{2}{\pi k r} \quad (40)$$

and hence $u_n(r) \neq 0$ for all $r > 0$. An almost identical argument applies to the case where $k < 0$. ■

Theorem 3.2. *Let $q : (0, \infty) \rightarrow [q_0, q_1]$ with $-1 < q_0 \leq q_1 < \infty$ be a continuous function supported on the interval $[a, b]$ with $0 < a < b < \infty$. For all non-negative integers n and for all real numbers $r > 0$ the impedance $\phi_n(r, k)$ is an analytic function of k everywhere in the complex upper-half plane.*

Proof. By Proposition 3.1 the impedance is well-defined for all non-zero k with non-negative imaginary part. Theorem 3.2 follows from the analytic dependence on parameters of solutions to ordinary differential equations (see, for example, [10]). ■

The following theorem describes the behaviour of the impedance in the vicinity of $k = 0$.

Theorem 3.3. *Let $0 < a < b < \infty$, and n be any non-negative integer. Then*

$$\phi_n(r, k) = \frac{(\sqrt{r}H_n(kr))'}{ik\sqrt{r}H_n(kr)} + O(k), \quad (41)$$

as $k \rightarrow 0$ in the complex upper half-plane (including the real axis).

Proof. Let n be a non-negative integer and k a non-zero complex number with non-negative imaginary part. Consider the function $w : [a, b] \times \{k \neq 0 \in \mathbb{C} : \text{Im } k \geq 0\} \rightarrow \mathbb{C}$ defined by

$$w(r) = \phi_n(r, k) - \frac{(\sqrt{r}H_n(kr))'}{ik\sqrt{r}H_n(kr)}. \quad (42)$$

We begin by observing that by Corollary 2.3, w satisfies the integral equation

$$w(r, k) = ik \int_r^b \left(w(x, k) \left(w(x, k) + 2 \frac{(\sqrt{r}H_n(kx))'}{ik\sqrt{r}H_n(kx)} \right) + q(x) \right) dx, \quad (43)$$

for all $a \leq r \leq b$. Next we note that by Remark 2.6, if $|k| < C_n/b$ then

$$\begin{aligned} \left| \frac{(\sqrt{r}H_n(kr))'}{ik\sqrt{r}H_n(kr)} \right| &\leq \frac{4(n+1)}{|k|a} + \frac{1}{2|k|a} \\ &\leq \frac{4(n+2)}{|k|a} \end{aligned} \quad (44)$$

for all $a \leq r \leq b$. Applying Lemma 2.10 to the integral equation (43) with $|A| = |B| = |k|$, $F = 8(n+2)/(|k|a)$ and $G = |q_0| + |q_1|$, and using the bound (44) we obtain

$$|w(r)| \leq \frac{|k|a}{2(n+2)} \quad (45)$$

for all $a \leq r \leq b$ provided that

$$1 + \frac{8(n+2)}{|k|^2 a (|q_0| + |q_1|)} \geq \frac{1}{2} e^{\frac{a|k|^2}{8(n+2)}(b-a)} \quad (46)$$

and $|k| \leq C_n/b$. Substituting $|k| < C_n/b$ into the right-hand side of (46) and rearranging yields

$$|k| \leq \min \left\{ \frac{C_n}{b}, \sqrt{\frac{16(n+2)}{a(|q_0| + |q_1|)}} \sqrt{\frac{1}{|e^{C_n^2/(8(n+2))} - 2|}} \right\}. \quad (47)$$

Combining (45), (47) and the definition of w in (42) we see that for all $r \in [a, b]$,

$$\phi_n(r, k) = \frac{(\sqrt{r}H_n(kr))'}{ik\sqrt{r}H_n(kr)} + O(k) \quad (48)$$

as $k \rightarrow 0$ in the complex upper half-plane (including the real axis). ■

The following theorem describes the behaviour of the impedance at large frequencies.

Theorem 3.4. *Suppose $q \in C^2(\mathbb{R})$ is a compactly supported function on the interval $[a, b]$. Moreover, suppose that there exist constants q_0 and q_1 such that $-1 < q_0 \leq q(r) \leq q_1 < \infty$ for all $r \in [a, b]$. Let ϕ_n be the impedance defined in (14). Then*

$$\phi_n(r, k) = \sqrt{1 + q(r)} - \frac{1}{4ik} \frac{q'(r)}{1 + q(r)} + O\left(\frac{1}{k^2}\right), \quad (49)$$

as $k \rightarrow \infty$, $\text{Im}(k) \geq 0$.

Proof. The proof is a slight modification of the standard analysis of the WKB approximation applied to equation (9) (see [9, 15] for example). Indeed, let $s(r) = \sqrt{1 + q(r)}$ and define $t : [a, b] \rightarrow [0, \infty)$ by

$$t(r) = \int_r^b n(s) ds, \quad (50)$$

observing that

$$t(a) \leq \sqrt{1 + q_1}(b - a). \quad (51)$$

We set $T = t(a)$ and define $\tilde{k} \in \mathbb{C}$ by

$$\tilde{k} = k \left(\frac{H'_n(kb)}{H_n(kb)} + \frac{1}{2kb} \right). \quad (52)$$

It follows from Corollary 2.7 that $|\tilde{k} - k| = O(|k|^{-1})$ as $|k| \rightarrow \infty$ anywhere in the upper half-plane.

Next we define $w(t, k)$ implicitly by

$$\psi_n(r(t), k) = \sqrt{\tilde{k}b} H_n(kb) \frac{e^{-i\tilde{k}t} w(t, k)}{\sqrt{s(r(t))}}. \quad (53)$$

For notational convenience in the following we will suppress the dependence of w on k and write $w(t)$ in place of $w(t, k)$.

After inserting (53) into (9), clearly w satisfies the following initial value problem

$$w''(t) - 2i\tilde{k}w'(t) = \eta(t)w(t), \quad (54)$$

$$w(0) = 1,$$

$$w'(0) = 0,$$

where

$$\eta(t) = \left(\tilde{k}^2 - k^2 + \frac{n^2 - \frac{1}{4}}{r^2(t) s^2(r(t))} + \frac{s''(r(t))}{2s^2(r(t))} - \frac{3(s'(r(t)))^2}{4s^3(r(t))} \right). \quad (55)$$

We note that $\tilde{k}^2 - k^2 = O(1)$ as $k \rightarrow \infty$ in the upper half-plane and thus that η is an absolutely continuous function on $[0, T]$ and is bounded uniformly in k and t for all $\text{Im } k \geq 0, |k| > 1$ and $t \in [0, T]$. Moreover, since $k - \tilde{k} = O(|k|^{-1})$, there exists some constant M such that $\text{Im } \tilde{k} \geq -M$ for all $k \in \mathbb{C}$ such that $\text{Im } k \geq 0, |k| > 1$.

Applying Lemma 2.11 to the initial value problem (54) we obtain

$$\left| w(t) - 1 + \frac{1}{2i\tilde{k}} \int_0^t \eta(\tau) d\tau \right| = O(|\tilde{k}|^{-2}), \quad (56)$$

$$|w'(t)| = O(|\tilde{k}|^{-1}), \quad (57)$$

for all $t \in [0, T]$.

Finally, it follows from the definition of w , see equation (53), that

$$\frac{\psi'_n(r)}{\psi_n(r)} = i\tilde{k}\sqrt{1+q(r)} + \frac{n'(r)}{2s(r)} + O(|\tilde{k}|^{-1}) \quad (58)$$

as $\tilde{k} \rightarrow \infty$ anywhere in the upper half plane. ■

4 The trace formula

In this section we present a trace formula for the impedance which is the principle analytic tool used in the inversion algorithm.

Theorem 4.1. *Suppose that $q \in C^2(\mathbb{R})$ is a compactly supported function on the interval $[a, b]$ and that there exist constants q_0 and q_1 such that $-1 < q_0 \leq q(r) \leq q_1 < \infty$ for all $r \in [a, b]$. Let ϕ_n be the impedance defined in Definition 2.1. Then*

$$\frac{q'(r)}{1+q(r)} = \frac{4}{\pi} \int_{-\infty}^{\infty} \left(\phi_n(r, k) - \frac{(\sqrt{r}H_n(kr))'}{ikH_n(kr)} + 1 - \sqrt{1+q(r)} \right) dk. \quad (59)$$

Proof. For $a \leq r \leq b$ define the function $f_r : \{k \neq 0 \in \mathbb{C} : \text{Im } k \geq 0\} \rightarrow \mathbb{C}$ by

$$f_r(k) = \phi_n(r, k) - \frac{(\sqrt{r}H_n(kr))'}{ikH_n(kr)} + 1 - \sqrt{1+q(r)}. \quad (60)$$

By Theorem 3.2, for all $a \leq r \leq b$ the function f_r is analytic in the upper half-plane and hence if Ω is any positive real number then

$$\int_{-\Omega}^{\Omega} f_r(k) dk = -i\Omega \int_0^{\pi} f_r(e^{i\theta}\Omega) e^{i\theta} d\theta. \quad (61)$$

Substituting the asymptotic expansion of ϕ_n from equation (49) into the previous expression yields

$$\int_{-\Omega}^{\Omega} f_r(k) dk = i\Omega \int_0^{\pi} \frac{1}{4i\Omega e^{i\theta}} \frac{q'(r)}{1+q(r)} e^{i\theta} d\theta + O\left(\frac{1}{\Omega}\right). \quad (62)$$

Taking the limit as $\Omega \rightarrow \infty$ completes the proof. ■

The following corollary is an immediate consequence of Theorem 4.1 and Lemma 2.2, and is the basis for the reconstruction algorithm described in Section 5.1.

Corollary 4.2. *Suppose that $q \in C^2(\mathbb{R})$ is a compactly supported function on the interval $[a, b]$ and that there exist constants q_0 and q_1 such that $-1 < q_0 \leq q(r) \leq q_1 < \infty$ for all $r \in [a, b]$. Then ϕ_n and q satisfy the following system of integro-differential equations*

$$\phi'_n(r, k) = -ik\phi_n^2(r, k) - ik(1 + q) + i\frac{n^2 - \frac{1}{4}}{kr^2}, \quad k \in \mathbb{R} \quad (63)$$

$$\frac{q'(r)}{1 + q(r)} = \frac{4}{\pi} \int_{-\infty}^{\infty} \left(\phi_n(r, k) - \frac{(\sqrt{r}H_n(kr))'}{ikH_n(kr)} + 1 - \sqrt{1 + q(r)} \right) dk \quad (64)$$

for all $r \in [a, b]$ together with the initial conditions

$$\phi_n(a, k) = k \left(\frac{H'_n(ka)}{iH_n(ka)} + \frac{1}{2ika} \right), \quad k \in \mathbb{R}, \quad (65)$$

$$q(a) = 0. \quad (66)$$

5 Numerical algorithm and results

5.1 The reconstruction algorithm

In this section we describe a reconstruction algorithm based on the trace formula derived in Theorem 4.1. As input it takes a non-negative integer n , an interval $[a, b]$ with $0 < a < b < \infty$, a spatial step size h , a bandlimit Ω , and the number of frequency samples N to use. As output the algorithm produces an approximation to the potential q on the interval $[a, b]$.

Step 1. Initialization: For $j = 1, \dots, N$ let $f_j = 2\Omega(j - 1)/(N - 1) - \Omega$ and $w_1 = \frac{\Omega}{N}$, $w_N = \frac{\Omega}{N}$ and $w_j = \frac{2\Omega}{N}$, $j = 2, \dots, N - 1$. We note that this corresponds to an N -point trapezoidal quadrature rule on the interval $[-\Omega, \Omega]$. Set $r_0 = a$ and $q_0 = q(r_0) = 0$. Set $\phi_{0,j} = \phi_m(a, f_j)$, for $j = 1, \dots, N$.

For $\ell = 0, \dots, (b - a)/h - 1$

Step 2. Obtain $q'(r_\ell)$ via the formula

$$q'(r_\ell) = \frac{\pi(1+q_\ell)}{4} \sum_{j=1}^N \left(\phi_{\ell,j} - \frac{H'_n(f_j r_\ell)}{iH_n(f_j r_\ell)} - \frac{1}{2if_j r_\ell} + 1 - \sqrt{1+q_\ell} \right) w_j, \quad (67)$$

and compute $q_{\ell+1}$ via the formula

$$q_{\ell+1} = q_\ell + hq'(r_\ell). \quad (68)$$

Step 3. For $j = 1, \dots, N$ set

$$\phi_{\ell+1,j} = \phi_{\ell,j} + h \left(-if_j \phi_{\ell,j}^2 - if_j(1+q_\ell) + i \frac{n^2 - \frac{1}{4}}{f_j r_\ell^2} \right).$$

Remark 5.1. The above algorithm is first-order in $1/\Omega$ and h and $1/N$. In the next section we discuss modifications which improve its rate of convergence with respect to these parameters.

5.2 Numerical acceleration of convergence

The algorithm presented in the previous section is first-order in the bandlimit Ω , the number of frequency samples N , and the spatial step size h , and is suitable for situations in which a few digits of relative precision are required for the reconstructions. If higher-precision reconstructions are required then the number of samples, the bandlimit and the number of spatial discretization points can become prohibitively large. In this section we outline straightforward modifications to the above algorithm which increase the rate of its convergence with respect to Ω , h , and N .

Dependence on N

As written the algorithm uses the trapezoidal rule to approximate the integral appearing in the trace formula (59) over a truncated interval $[-\Omega, \Omega]$. Theorem 3.3 guarantees that

computing the integral over this interval using trapezoid rule will result in an error that decays like $1/N$ where N is the number of frequencies used. For $k \in \mathbb{R}$ away from zero the integrand is smooth and hence any smooth quadrature rule such as Gauss-Legendre quadratures or nested Gauss-Legendre quadratures can be used to obtain arbitrarily high accuracy. Near $k = 0$ the presence of terms depending on $\log(k)$ cause singularities in the higher derivatives of the integrand which necessitate the use of a different quadrature rule. In particular, using generalized Gaussian quadratures [7] we produced a 35-point quadrature rule which integrates all functions of the form

$$f_{m,n}(k) = k^m \log^n(k)$$

on the interval $0 \leq k \leq 1/2$ for $m = 1, 2, \dots, 18$ and $n = -10, \dots, 4$ to a relative precision of 10^{-16} . The resulting quadrature rule can be used to perform the integrals in the neighborhood of $k = 0$. Alternatively, one could use an endpoint corrected trapezoid rule [22] to evaluate the contribution of the integral in the vicinity of the origin.

Using this quadrature method, for any $\Omega > 0$ and $0 < r < \infty$, integrals of the form

$$\frac{4}{\pi} \int_{-\Omega}^{\Omega} \left(\phi_n(r, k) - \frac{(\sqrt{r}H_n(kr))'}{ikH_n(kr)} + 1 - \sqrt{1 + q(r)} \right) dk \quad (69)$$

can be computed numerically to full machine precision with relatively few quadrature nodes (typically no more than 500 and often significantly fewer).

Dependence on Ω

The method outlined in the previous section allows one to compute integrals of the form

$$\frac{4}{\pi} \int_{-\Omega}^{\Omega} \left(\phi_n(r, k) - \frac{(\sqrt{r}H_n(kr))'}{ikH_n(kr)} + 1 - \sqrt{1 + q(r)} \right) dk \quad (70)$$

accurately and with relatively few quadrature nodes. It does not, however, eliminate the truncation error introduced by replacing the integral over the entire real line in the system (63) by the integral over the finite interval $[-\Omega, \Omega]$. From Theorem 3.4 it can be

observed that the resulting error due to this truncation will decay like $1/\Omega$. In this section we describe a modification to the inversion algorithm described above which produces faster convergence in Ω . The principle tool is Richardson extrapolation.

For notational convenience we denote the real part of the integrand appearing in the trace formula (59) by $F(r, k)$, noting that for any $0 < r < \infty$, $F(r, k) = F(r, -k)$. Specifically, $F : (0, \infty) \times \mathbb{R} \rightarrow \mathbb{R}$ is defined via the formula

$$F(r, k) = \left(\phi_n(r, k) - \frac{(\sqrt{r}H_n(kr))'}{ikH_n(kr)} + 1 - \sqrt{1 + q(r)} \right). \quad (71)$$

We observe that the imaginary part can be neglected since by Lemma 2.4 the integral of the imaginary part vanishes provided the endpoints of integration are symmetric about $k = 0$. Additionally, Theorem 4.1 guarantees that $F(r, k) = O(k^{-2})$ for large k . In fact, for any fixed r it has an asymptotic expansion in k valid in the limit as k goes to infinity; namely, there exist coefficients $A_2(r), A_4(r), \dots$ depending on the potential q and the point r , such that

$$F(r, k) = \frac{A_2(r)}{k^2} + \frac{A_4(r)}{k^4} + \dots + \frac{A_{2m}(r)}{k^{2m}} + O(k^{-2m-2}) \quad (72)$$

for any $m \geq 1$.

Thus

$$2 \int_{-2\Omega}^{2\Omega} F(r, k) dk - \int_{-\Omega}^{\Omega} F(r, k) dk = O(\Omega^{-3}). \quad (73)$$

Rather than compute both integrals, this extrapolation can be performed by adjusting the frequency quadrature weights w_j , $j = 1, \dots, N$. In addition, this extrapolation can be performed multiple times, each time increasing the rate of convergence by a factor of Ω^{-2} . Finally, we remark that it is not necessary to double the bounds of integration for each step of Richardson extrapolation: smaller ratios can be used at the expense of increasing the coefficients multiplying the integrals.

5.3 Dependence on h

The recovery algorithm described in Section 5.1 uses the forward Euler method to evolve both the impedance ϕ_n and the potential q from the inner radius of the annulus a to the outer radius b , which produces an error decaying linearly in the step size h . If one instead uses Heun's method for the evolution of the potential q followed by the Crank-Nicholson method to evolve the equations for the impedance ϕ_n , the result is a second-order method in h .

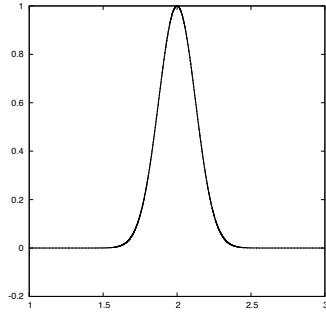
Remark 5.2. The above algorithm is second-order accurate in the step size h both for the evolution of the impedance as well as for the evolution of the potential q ; using Richardson extrapolation it is easy to obtain higher-order convergence in h .

5.4 Numerical results

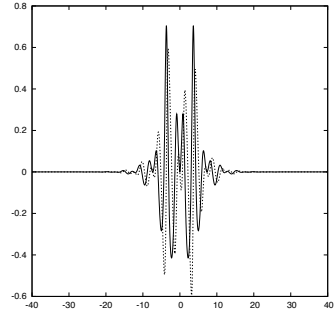
The algorithm described above, together with the modifications, was implemented in Fortran and the results are summarized below. All code was compiled in GFortran and run on a 2.7 GHz Apple laptop with 8 Gb of memory. To avoid so-called *inverse crimes* the forward data was obtained by solving the equation for the field u_n given in equation (3) using a fourth-order Runge-Kutta method. We show both the effect of increasing the order n (Figure 3) as well as changing the distance of the annulus from the origin (Figure 4). Finally, in Figure 5 we show recovery for a discontinuous potential.

6 Conclusions and discussion

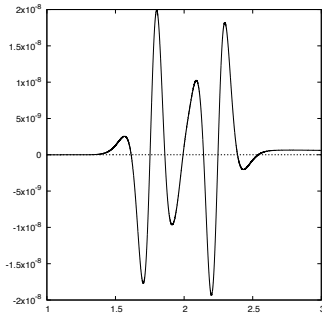
In this paper we present an algorithm for solving the interior inverse-scattering problem for radially-symmetric domains in \mathbb{R}^2 given multifrequency reflection data. The source used to generate the measurements is assumed to be inside the object and the measurements are taken at the center of the domain. The approach is based on the construction of a trace formula for the impedance of the angular moments of the field. As in [9], this leads to a system of coupled equations for the impedance and unknown potential which can be evolved starting from the measured data in the interior of the scatterer



(a) The exact and recovered potential

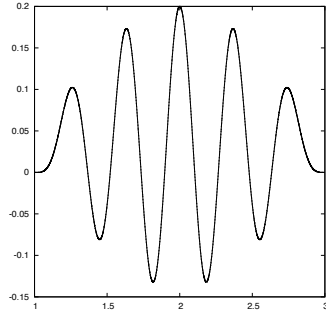


(b) The initial data
 $\phi_0 - (\sqrt{r}H_0(kr))' / (H_0(kr))$

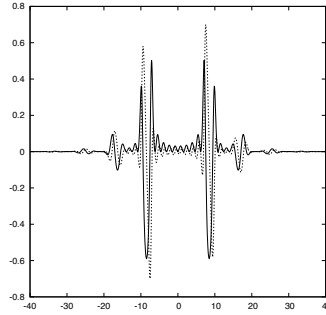


(c) The recovery error

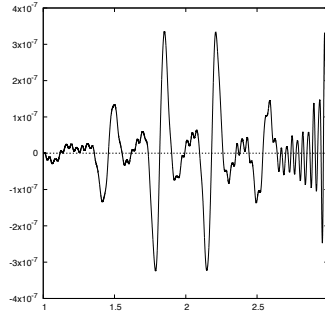
Figure 1: Numerical results for a Gaussian bump $q(r) = e^{-32(x-2)^2}$ with $n = 0$. The time to generate the data was 54 seconds, and the time to solve was 140 seconds. The solve was done using 270 frequencies in the range $[-160, 160]$, and a spatial step size of $1/20000$.



(a) The exact and recovered potential

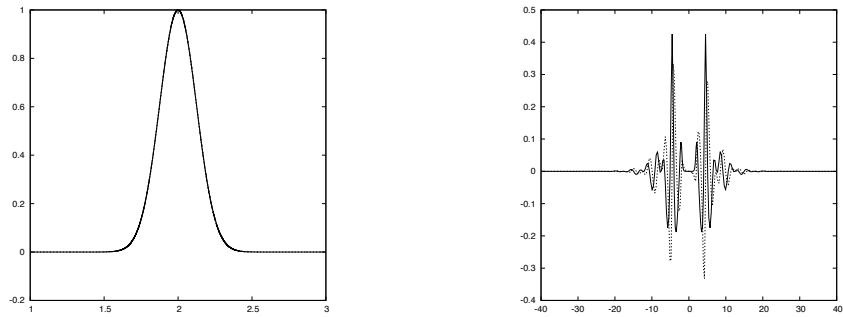


(b) The initial data
 $\phi_0 - (\sqrt{r}H_0(kr))' / (H_0(kr))$



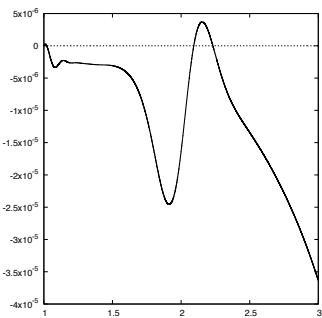
(c) The recovery error

Figure 2: Numerical results for the potential
 $q(r) = \frac{1}{10} [\cos(a(r-2)\pi) + 1] - \frac{a^2}{10b^2} [1 - \cos(b(r-2)\pi)]$ with $a = 5, b = 6$ and $n = 0$.
 The time to generate the data was 67 seconds, and the time to solve was 141 seconds.
 The solve was done using 270 frequencies in the range $[-160, 160]$ and a spatial step size of $1/20000$.



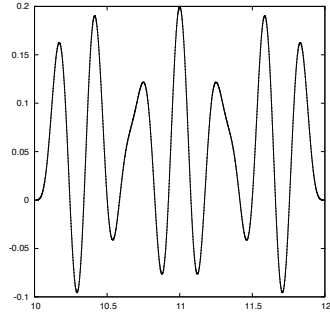
(a) The exact and recovered potential

(b) The initial data
 $\phi_4 - (\sqrt{r}H_4(kr))' / (H_0(kr))$

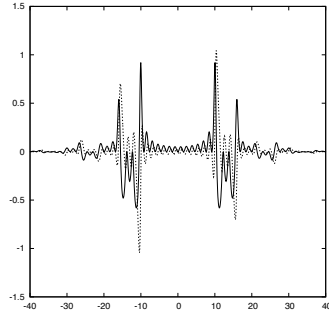


(c) The recovery error

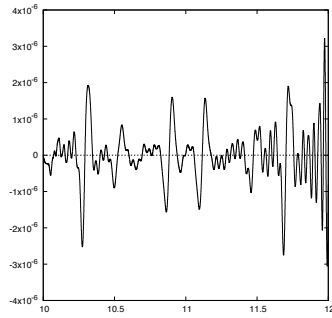
Figure 3: Numerical results for a Gaussian bump $q(r) = e^{-32(x-2)^2}$ with $n = 4$. The time to generate the data was 37 seconds, and the time to solve was 88 seconds. The solve was done using 470 frequencies in the range $[-240, 240]$, and a spatial step size of $1/40000$.



(a) The exact and recovered potential

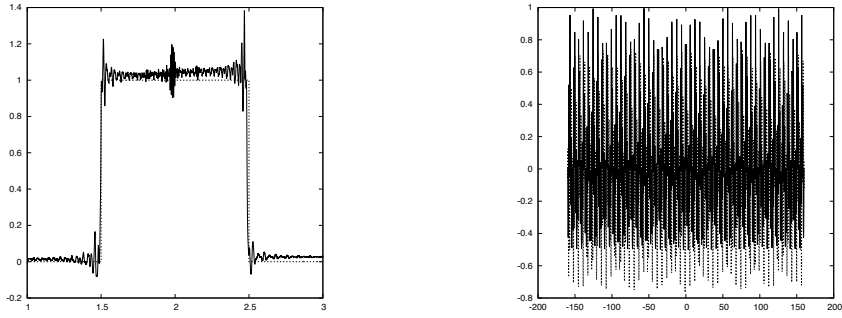


(b) The initial data
 $\phi_0 - (\sqrt{r}H_0(kr))' / (H_0(kr))$



(c) The recovery error

Figure 4: Numerical results for the potential $q(r) = \frac{1}{10} [\cos(a(r - 2)\pi) + 1] - \frac{a^2}{10b^2} [1 - \cos(b(r - 2)\pi)]$ with $a = 9, b = 10$ and $n = 0$. The time to generate the data was 62 seconds, and the time to solve was 130 seconds. The solve was done using 270 frequencies in the range $[-160, 160]$ and a spatial step size of $1/20000$.



(a) The exact (solid line) and recovered (dashed line) potential

(b) The initial data $\phi_0 - (\sqrt{r}H_0(kr))' / (H_0(kr))$

Figure 5: Numerical results for the potential $q(r)$ which is identically one on the interval $[1.5, 2.5]$ and zero otherwise. Here $n = 0$. The time to generate the data was 65 seconds, and the time to solve was 130 seconds. The solve was done using frequencies in the range $[-160, 160]$ and a spatial step size of $1/20000$.

and continuing to the outermost edge of the scatterer. The algorithm was implemented in Fortran and numerical results were presented illustrating its accuracy.

The method developed in this paper applies almost immediately to the three-dimensional radially-symmetric problem as well as waveguides with constant cross-sectional parameters. Details of the analysis and numerical implementation will be published at a later date. Finally, it should be possible to remove the requirement for radial symmetry, obtaining trace formulas involving the coupling of multiple modes for non-radially symmetric scatterers. This line of inquiry is being vigorously pursued.

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