The Positive Solution of a Certain Nonlinear

Parabolic System and It's Computation I.

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THE POSITIVE SOLUTIONS OF A CERTAIN NONLINEAR PARABOLIC SYSTEM AND ITS COMPUTATION I

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Abstract

In this paper we shall investigate a type of nonlinear parabolic equation, coupled equations and their positive solutions. The properties of maximum and the nonlinear instabilities will be discussed briefly; for obtaining its numerical solution a special scheme will be presented.

1. Introduction

In the present paper we shall investigate a type of nonlinear parabolic equation and coupled equations to which the similar one may come from physics, chemistry and ecology. Specifically, the coupled equations are from solid electronics and have been simplified to describe the dynamic distributions of two kinds of charge carriers so called carrier equations. The unknown

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functions in these problems represent the densities of certain particles or bodies. The positivity of those functions in their mathematical model will play essential role. Ignorance of this property often cause serious instability especially in nonlinear problems, which we call nonlinear instability. A mathematical model describing a natural phenomena which is reproducible must be properly defined and continuously dependent on a number of given conditions. We are concerned with the case in which there is a steady state solution and a transitional solution wherein the former is the limit of the latter as time increase to infinity.

In order to solve such a problem we suggest using a special scheme ,positive scheme , which will be used in computation of the positive solution. This assures us to avoid the nonlinear instability .

Many authors have studied this type of problem. Some of them refered to Lyapunov stability, some of them refered to the direct integration methods. But we prefer the positivity of the solution, and emphasis is placed on the construction of the computational scheme.

1. A Nonlinear Parabolic Equation

Let us begin with a single nonlinear parabolic equation as follows:

$$\frac{\partial U}{\partial t}=a\frac{\partial^2 U}{\partial x^2} - b(U-K(x))U \qquad \text{(A)}$$

$$a,b > 0,\ -1 < x < 1,\ t > 0.$$

where $U(x,0)=\phi(x)$, $U(\pm 1, t)$ are given positive functions and $K(x)$ is given as an arbitrary continuous function. A set of functions which consists of all positive continuous functions we call C^+; and we call such a solution a positive solution.

THEOREM 1. If $U(x,t)$ is a positive solution of (A) then the following inequality holds:

$$0 < U(x,t) \leq K_u = \text{Max}\ \{\underset{x}{\text{Max}}K(x), \underset{\text{or } x=\pm 1}{\underset{t=0}{\text{Max}}} U\} \qquad \text{(I)}$$

Proof. On the contrary there must be $x^* \varepsilon (-1,1)$ and $t^*>0$ such that

$$U(x^*,t^*)= \underset{t\leq t^*}{\text{Max}_*}\ U(x,t) > K, \frac{\partial U(x^*,t^*)}{\partial t} \geq 0, \frac{\partial^2 U(x^*,t^*)}{\partial x^2} \leq 0,$$

and $(U(x^*,t^*)-K(x^*))U(x^*,t^*) > 0$. It conflicts with the equation (A).

THEOREM 2. The positive solution of (A) is unique.

The proof of theorem 2 is omitted. Considering the steady state solution of (A) it is obvious that this solution is free of variable t and is the limit of the solution $U(x,t)$ during $t \longrightarrow \infty$. The steady state solution of (A) satisfies the nonlinear elliptic equation and boundary conditions which we call problem (A^*). First we divide the function $K(x)$ into two positive functions $K^+(x)$ and $K^-(x)$ such that $K(x)=K^+(x)-K^-(x)$ i.e.

$$K^+(x) = \{ \begin{array}{ll} K(x) + p(x) \ \ldots & \text{if } K > 0; \\ p(x) \ \ldots\ldots\ldots & \text{otherwise}; \end{array}$$

$$K^-(x) = \{ \begin{array}{ll} p(x) \ \ldots\ldots\ldots & \text{if } K > 0; \\ -K(x)+p(x) \ \ldots & \text{otherwise}, \end{array}$$

where $p(x)$ is an arbitrary non-negative function.

Next we suggest an iterative process which is very important in finding a solution of (A). Consider the following iterative process:

$$\frac{\partial U^m}{\partial t}=a\frac{\partial^2 U^m}{\partial x^2}-b[U^m U^{m-1}-K^+U^{m-1}+K^-U^m] \qquad m=1,2,\ldots. \tag{Am}$$

where $U^m(\pm1,t)=U(\pm1,t)$, $U^m(x,0)=\phi(x)$, but U^0 is any function satisfying (I)› Each step of the process is forwarded by solving a linear parabolic equation.

THEOREM 3. All iterative solutions of (Am) satisfy the following inequality

$$0 < U^m(x,t) \leq K_u = \mathrm{Max}[\underset{x}{\mathrm{Max}}K^+(x),\ \underset{t=0 \text{ or } x=\pm1}{\mathrm{Max}} U]$$

Proof. For m=0 theorem is true . We suppose inequality valid for all integers less than m . Then we check the inequalıry for integer m. On the contrary there must be (x^*,t^*) in the region such that function $U^m(x^*,t^*)$ as a maximum. Both of them conflict with equality of (A_m).

Now we discuss the convergence of the sequence $\{U^m\}$ in the functional space L^2. Set $\varepsilon^m = U^m - U^{m-1}$, then ε^m's satisfy the following equation with homogeneous initial and boundary conditions:

$$\frac{\partial e^m}{\partial t}=a\frac{\partial^2 e^m}{\partial x^2} - b(U^{m-1}e^m+U^{m-1}e^{m-1}-K^+e^{m-1}+K^-e^m)$$

Multiply both sides by e^m and integrate on whole interval (-1,1), using the following notations:

$$|v(x)|=[\int_{-1}^{+1}|v(x)|^2dx]^{1/2};$$

$$(u,v)=\int_{-1}^{+1} u\ v\ dx.$$

Then we have

$$(1/2)\frac{\partial}{\partial t}|e^m|^2 = -a|\frac{\partial}{\partial x}e^m|^2 - b|/\ U^{m-1}e^m|^2 -$$

$$-b|/\ K^- e^m|^2 + b((K^+ - U^{m-1})e^{m-1}, e^m) \leq$$

$$\leq -a|e^m| + bC|e^m||e^{m-1}|.$$

By using the well known inequality $|e| \leq |\frac{\partial}{\partial x}e|$ we have

$$\frac{\partial}{\partial t}|e^m|^2 \leq -2a|e^m|^2 + bC(|e^m|^2 + |e^{m-1}|^2) \leq$$

$$\leq -n^1|e^m|^2 + n^2|e^{m-1}|^2$$

where $n^1 = 2a - bC$, $n^2 = bC$, i.e.

$$\frac{\partial}{\partial t}(|e^m|^2 e^{n^1 t}) \leq n^2|e^{m-1}|^2 e^{n^1 t},$$

thus we have

$$|e^m|^2 e^{n^1 t} \leq n^2\int_0^t |e^{m-1}|^2 e^{n^1 t} dt \leq$$

$$\leq (n^2)^2\int_0^t\int_0^s |e^{m-2}|^2 e^{n^1 s'} ds' ds \leq \ldots$$

$$\leq [(n^2)^{m-1}/(m-2)!]\int_0^t (t-s)^{m-2}|e^1|^2 e^{n^1 s} ds \leq$$

$$\leq [(n^2 t)^{m-1}/(m-1)!]\ \mathrm{Max}\ |e^1|^2 e^{n^1 t})$$

Therefore the following theorem can be drived.

Theorem 4. Iterative sequence $\{U^m\}$ is convergent in any finite region in the sense of the L^2-norm.

Further results about the generalized solution of (A) are beyond our main purpose. We are interested in this iterative process (A_m) and its motivated scheme of computation.

2. Nonlinear Instability and A Computational Scheme

When the solutions of (A) are confined in C^+, then the positive solution of (A^*) could be obtained as a limit of (A) during $t \longrightarrow \infty$. We call stable stesdy state solution. But once we disregard this restriction, the unstable phenomena will often occur. For example , if $K=C=$const when $U_o < 0$; $a=0$, then as the explicit solution shows:

$$U=C/[1-\{[(U_o-C)/U_o]e^{-bCt}\}]\longrightarrow\infty \text{ as } t\longrightarrow[1/(bC)]Ln[(U_o-C)/U_o].$$

The instability becomes obvious. According to variety of the constants C and U_o the possibilities are listed in the table 1 below and show the stable (unstable) region in figure 1.

$C < 0$	$C > 0$	
stable	stable	$U > 0$
limited	unstable	$C < U < 0$
unstable	unstable	$U < C,0$

Table 1.

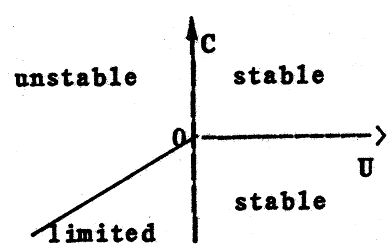

Figure 1.

The very unfavorably unstable situation will develop if one puts the initial value in the unstable region. So we must be careful to construct the approximate scheme so that once any initial function is selected in the stable region then all the intermidate results will never be out of the stable region. Thus the computation can be possibly carried on until a solution has been found.

Now we come to a computational scheme for problem (A). First of all we divide in terval $(-1,1)$ into 2N subintervals by nodes:

$$-1=x_{-N}<x_{-N+1}<\dots x_{-2}<x_{-1}<x_o<x_1<\dots x_{N-1}<x_N=1.$$

Let $h_i=x_i-x_{i-1}$, $i=-N+1,\dots,-1,0,1,\dots,N$ be lengths of these subintervals and use the notations:

$$U_i^m=U(x_i,t_m),\ t_m=m\Delta t,\ m=0,1,2,\dots;$$

$$\delta_x^2U_i^m=\frac{1}{h_ih_{i+1}}\left[\frac{2h_i}{h_i+h_{i+1}}U_{i+1}^m-2U_i^m+\frac{2h_{i+1}}{h_i+h_{i+1}}U_{i-1}^m\right],$$

where Δt is the length of the subinterval in t-axis.

We set the difference equation as follows:

$$(U_i^m-U_i^{m-1})/\Delta t=a\delta_x^2U_i^m-bU_i^{m-1}U_i^m+bK_i^+U_i^{m-1}-bK_i^-U_i^m \qquad \text{(Ah)}$$

where U_N^m, U_{-N}^m, U_i^o are given positive, $i=0,\pm1,\dots,\pm(N-1)$, $m=1,2,\dots$

Let Ch^+ be a set of positive functions defined on these nodes. We call a solution of (Ah), which belong to Ch^+, the positive solution.

Theorem 5. The coefficient matrix of the unknown functions U_i^m in (Ah) has a unique inverse with non-negative elements.

It is easy to see, if we remove all the terms of U_i^m to the left sides of the equations and $U_i^{m-1}/\Delta t$ to the right sides, then the coefficients of U_i^m form a special matrix with dominate diagonal elements negative, so there is a nonnegative inverse.

Theorem 6. Under the conditions of theorem 3 the equation (Ah) has the solution U_i^m in Ch^+ and the following inequality holds:

$$0 < U_i^m \le \mathrm{Max}\{\ \mathop{\mathrm{Max}}_i K_i^+,\ \mathop{\mathrm{Max}}_{\substack{m=0 \\ \text{or } i=\pm N}} U_i^m\}.$$

The proof is much the same as for section 1.

Theorem 7. If the solution of (A) has all continuous derivatives for x and t required, then the difference solution U_i^m approximates differential solutions in any finite section of the region in the sense of L_2 norm.

In fact, suppose we take e_i^m as the difference between the solution of (A) and the solution of (Ah). Then e_i^m suits the following difference equations:

$$(e_i^m-e_i^{m-1})/\Delta t=a\delta_x^2 e_i^m+$$
$$+bK_i^+e_i^{m-1}-bK_i^-e_i^m-bU_i^{m-1}e_i^m-bU_i^me_i^{m-1}+R_i^m$$

R_i^m-approximate error. Multiply the two sides of the equation by $e_i^m\ \bar{h}_i$, where $\bar{h}_i=(h_i+h_{i+1})/2$ and sum up from $i=-N+1$ to $N-1$. We introduce the notations about mesh functions in the same way as the continuous functions

$$|e^m|^2=\sum_{-N+1}^{N-1}\bar{h}_i(e_i^m)^2,$$

$$|(e_i^m)_x|_h^2=\sum_{-N}^{N-1}h_i[(e_{i+1}^m-e_i^m)/h_i]^2,$$

then we obtain

$$|e^m|_h^2+a\Delta t|e_x^m|_h^2+b\Delta t|(K^-)^{1/2}e^m|_h^2=$$

$$\sum_{-N+1}^{N-1}\bar{h}_i[e_i^me_i^{m-1}\{1+b\Delta t(K_i^+-U_i^m\}]+$$

$$+\Delta t \sum_{-N+1}^{N-1} e_i^m R_i^m \bar{h}_i \leq$$

$$\leq (1+bC\Delta t)|e^m|_h |e^{m-1}|_h + \Delta t |e^m|_h |R^m|_h;$$

i.e. $|e^m|_h^2 \leq (1+bC\Delta t)|e^{m-1}|_h^2 + \Delta t |R^m|_h{}^2$, and simply we have

$$|e^m|_h^2 \leq (1+bC\Delta t)^m |e^o|_h^2 + \Delta t \sum_{j=0}^{m-1} (1+bC\Delta t)^j |R^{m-j}|^2.$$

thus theorem 7 is obvious.

Other schemes may be available, but scheme (Ah) is special from the iterative process (Am) in section 1. Under a slightly strict condition we obtain the stability of the difference equations (Ah) with regard to initial values.

Theorem 8. If $a > b(\mathrm{Max}\ K^+)$, then the difference equations are stable with regard to initial values.

This means if U_i^m and V_i^m are two solutions of (Ah) with diff different initial values , then the difference between them does not increase. The proof is the same as for theorem 7 when omitting R_i^m.

3. A Coupled Nonlinear Parabolic System

Now we turn to the coupled nonlinear system:

$$\begin{aligned} \frac{\partial}{\partial t}U &= a_1\frac{\partial^2}{\partial x^2}U - b_1(U-V-K(x))U \\ \frac{\partial}{\partial t}V &= a_2\frac{\partial^2}{\partial x^2}V - b_2(V-U+K(x))V \end{aligned} \qquad \text{(B)}$$

$$a_1, a_2; b_1, b_2 > 0; \ -1 < x < 1; \ t>0;$$

where $U,V(\pm 1,t)$ and $U,V(x,0) \in C^+$ are given. To find the positive solution in

C^+ we still use the functions K^+ and K^- in section 1. and denote the inner maximum value as follows:

$$U_{max-in}=U(x^*,t^*)= \operatorname{Max}\{\operatorname{Max}_x U, \operatorname{Max}_x V\}$$
$$-1 < x^* < 1,\ t^*>0$$

the same to V_{max-in}.

Theorem 9. Only in region $K(x) \geq 0$ $U(x,t)$ could reach U_{max-in} and for which the following inequality holds:

$$0\leq U_{max-in}-V(x,t) \leq Kp = \operatorname{Max}_x K^+(x); \qquad \text{(II)}$$

only in region $K(x) \leq 0$ $V(x,t)$ could reach V_{max-in} and for which the following inequality holds:

$$0\leq V_{max-in}-U(x,t) \leq Km = \operatorname{Max}_x K^-(x), \qquad \text{(II)}$$

where the functions in each inequality are evaluated at the same point.

Using the same idea as in section 1 we suggest an iterative process:

$$\begin{aligned} \frac{\partial}{\partial t}U^m &= a_1\frac{\partial^2}{\partial x^2}U^m - b_1K^-U^m - b_1(U^m-V^m-K^+)U^{m-1} \\ \frac{\partial}{\partial t}V^m &= a_2\frac{\partial^2}{\partial x^2}V^m - b_2K^-V^m - b_2(V^m-U^m-K^-)V^{m-1} \end{aligned} \qquad \text{(Bm)}$$

where U^m,V^m $m=0,1,2,...$ have the same initial-boundary values as U,V and U^o,V^o are a pair of arbitrary functions satisfying (II) in C^+.

Theorem 10. Each pair of the functions in sequence $\{U^m,V^m\}$ generated by iterative process (Bm) remain in C^+ and the following inequalities hold:

$$\begin{aligned} 0 &\leq U^m_{max-in}-V^m \leq Kp; \\ 0 &\leq V^m_{max-in}-U^m \leq Km. \end{aligned} \qquad \text{(IIm)}$$

We leave the proofs of these two theorems to the readers. Even more results about convergence and generalized solutions could be obtained. But we are going to describe the problem of instability and a computational scheme in the next section.

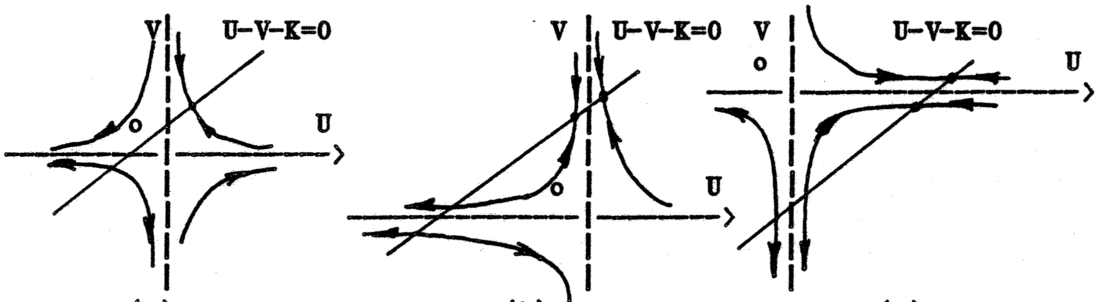

Figure 2.

4. Nonlinear Instability And Computational Scheme

First we investigate the instability of system (B). Assume $a_i = 0$, $B_i=1$, $i=1,2$, $K(x)=$ const divide the first equation by the other , then we obtain an integration of $U*V$ =const where the constant is decided by the initial value of U and V. All the possibilities about stability are shown in figure 2,(a),(b), (c). The arrows represent the behavior of the solutions U,V during $t \rightarrow \infty$. Only those points which have two arrows on both sides pointing to them represent the stable steady state solutions. For our purposes only the positive solutions are available , i.e. $U,V > 0$, otherwise nonlinear instability may arise as seriously as in section 2.

Our next task is to give the computational scheme of equations (B). Using the same division of time-space coordinates and the same notations as in section 2, we suggest the difference equations as follows:

$$(u_i^m-u_i^{m-1})/\Delta t=a_1\delta_x^2 u_i^m-b_1K_i^- u_i^m-$$
$$-b_1u_i^{m-1}(u_i^m-v_i^m)+b_1K_i^+u_i^{m-1};$$

$$(v_i^m-v_i^{m-1})/\Delta t=a_2\delta_x^2 v_i^m-b_2K_i^+ v_i^m- \qquad \text{(Bh)}$$
$$-b_2v_i^{m-1}(v_i^m-u_i^m)+b_2K_i^-v_i^{m-1}.$$

where $u_{\pm N}^m$, $v_{\pm N}^m$; u_i^o, v_i^o, $i=0,\pm1,\pm2,..$ are given, positive and satisfying (IIh). We call a solution of (Bh) which belong to Ch^+ positive solution.

We denote

$$u_{max-in}^m=u_{i^*}^m= \text{Max}(\ \underset{i}{\text{Max}}\ u_i^m,\ \underset{i}{\text{Max}}\ v_i^m)$$
$$v_{max-in}^m=v_{i^{**}}^m= \text{Max}(\ \underset{i}{\text{Max}}\ u_i^m,\ \underset{i}{\text{Max}}\ v_i^m)$$
$$-N < i^*,\ i^{**} < +N,$$

as inner maximums of the solutions.

Theorem 11. Problem (Bh) has a unique positive solution in Ch^+.

Proof. Remove all the terms containing u_i^m or v_i^m to the left sides of the equations and u_i^{m-1}, v_i^{m-1} to the right sides. Then all the terms on left sides are non-negative. The coefficient matrix of unknown functions u_i^m, v_i^m has dominant diagonal elements and negative non-zero-off-diagonal elements, so it has an unique non-negative inverse. Thus we get u_i^m, $v_i^m \geq 0$. Further we shall verify u_i^m, $v_i^m > 0$. If for some (m,i) u_i^m or $v_i^m = 0$, suppose $u_i^{m_o}=0$ and $u_i^m, v_i^m > 0$ for all $m < m_o$, then check the signs of every term in (Bh), we shall see the right side is positive which is conflict with left side.

Theorem 12. Only in region where $K(x) \geq 0$ could u_i^m reach u_{max-in}^m and to which we have

$$0 \le u^m_{max-in} - v^m_i \le K^+_i; \qquad \text{(IIh)}$$

only in region where $K(x) \le 0$ could v^m_i reach v^m_{max-in} and we have

$$0 \le v^m_{max-in} - u^m_i \le K^-_i. \qquad \text{(IIh)}$$

The proof of theorem 12 can be carried on in the same way as theorem 3 or 10.

Many other schemes have been found. Here we only mention the computational scheme (Bh). The reason is that does this scheme (Bh) keep almost same characteristics as the original problem (B). Besides the equations of problem (B) contain the main core of many more complex problems. We hope that it will be useful to those who are interested in fields we mentioned, and the method of constructing (Bh) could be easily used in solving some other related problems.

5. Discussion of The Algorithm

We write equations (Bh) in a matrix form as follows:

$$(\bar{U}^m_i - \bar{U}^{m-1}_i)/\Delta t = \begin{vmatrix} a_1 & 0 \\ 0 & a_2 \end{vmatrix} \delta^2_x \bar{U}^m_i - \begin{vmatrix} b_1 K^-_i & 0 \\ 0 & b_2 K^+_i \end{vmatrix} \bar{U}^m_i -$$

$$- \begin{vmatrix} b_1 u^{m-1}_i & -b_1 u^{m-1}_i \\ -b_2 v^{m-1}_i & b_2 v^{m-1}_i \end{vmatrix} \bar{U}^m_i + \begin{vmatrix} b_1 K^+_i & 0 \\ 0 & b_2 K^-_i \end{vmatrix} \bar{U}^m_i,$$

where $\bar{U}^m_i$ is a column vector $(u^m_i, v^m_i)^*$. Using notations:

$r_i=\Delta t/h_i h_{i+1}$, $p_i=h_i/(h_i+h_{i+1})$, $q_i=1-p_i$,

$$A_i=\begin{vmatrix}-a_1 q_i r_i & 0\\ 0 & -a_2 q_i r_i\end{vmatrix},\quad C_i=\begin{vmatrix}-a_1 p_i r_i & 0\\ 0 & -a_2 p_i r_i\end{vmatrix},$$

$$B_i^m=\begin{vmatrix}1+2a_1 r_i+b_1 K_i^-\Delta t+b_1 u_i^{m-1}\Delta t & -b_1 u_i^{m-1}\Delta t\\ -b_2 v_i^{m-1}\Delta t & 1+2a_2 r_i+b_2 K_i^+\Delta t+b_2 v_i^{m-1}\Delta t\end{vmatrix},$$

$$F_i^m=\begin{vmatrix}(1+b_1 K_i^+\Delta t)u_i^{m-1}\\ (1+b_2 K_i^-\Delta t)v_i^{m-1}\end{vmatrix}.$$

we get the following block-tridiagonal system:

$$A_i\bar{U}_{i-1}^m + B_i^m\bar{U}_i^m + C_i\bar{U}_{i+1}^m = F_i^m,$$

$$i=0,\pm1,\pm2,\dots,\pm(N-1) \qquad \text{(III)}$$

where $U_{\pm N}^m$ are given. The direct method can be used in solving (III). By eliminating A_i we transform (III) into (IV):

$$\bar{U}_i^m + C_i^*\bar{U}_{i+1}^m = F_i^{m*},$$

$$i=0,\pm1,\pm2,\dots,\pm(N-1), \qquad \text{(IV)}$$

$$C_{N+1}^* = (B_{-N+1})^{-1}*C_{N+1},\quad F_{-N+1}^{m*}= (B_{-N+1})^{-1}*F_{-N+1}^m;$$

$$C_{i+1}^* = (B_{i+1}-A_{i+1}C_i^*)^{-1}\ C_{i+1},$$

$$F_{i+1}^{m*} = (B_{i+1}-A_{i+1}C_i^*)^{-1}\ (F_{i+1}^m-A_{i+1}F_i^{m*})$$

$$i=-N+1,\dots,-2,-1,0,1,2,\dots,N-1.$$

We recall that processes (IV) and (V) are computable and the number of operations is about O(N) for each step.

We have worked on this coupled equations for a while. Notice 1) Theorem 12 is not sufficient for the stability; 2) Theorem 12 only provide a reasonable restriction on the difference solution as theorem 10 does; 3)because of nonlinear effect Δt should not be too big, so that a stable approximation can be used.

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