Abstract. We propose a new method for the solution of the wide angle wave equation in three dimensions. In contrast with standard techniques, our approach requires only solutions of successive tridiagonal systems in the resulting finite difference parabolic equations. The method is based on a simple approximation to the square-root operator written formally as $\sqrt{I+X+Y}$ where X is a partial differential operator with respect to the depth z and Y is a partial differential operator with respect to the azimuthal angle θ. We exploit the fact that the partial derivative term Y with respect to the azimuthal angle is small, but not negligible, as compared with other terms. It is then natural to replace the square-root operator by an expansion which is of order 2 with respect to the X operator, and of order 1 with respect to the Y operator. An important feature of this approach is that it is then possible to derive a rational function approximation to the exponential of the square-root operator which has the property of being stable, and accurate. Moreover, the approximation decouples naturally as a product of a $(1,1)$ rational function of X times a $(1,1)$ rational function of Y. As a consequence, this will result in a solution technique that requires only two tridiagonal system solutions per step, namely one for the X operator and one for the Y operator. Numerical examples are reported that show the wide angle capability of this method.

An efficient method for solving the three-dimensional wide angle wave equation

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1. Introduction

The wide angle three-dimensional parabolic approximation technique developed by Siegmann, Kriegsmann, and Lee [5] has been proven capable of handling wide angle propagation in the vertical plane. This technique is based on a pseudo-differential three- dimensional parabolic wave equation of which the 3-D parabolic approximation introduced by Tappert [7] is a special case. Methods for solving this equation have been developed by Baer and Perkins [1], using the fast Fourier transform, but are only applicable to small angles of propagation. Moreover, these methods do not extend easily to equations with variable coefficients and do not handle rigid boundary conditions.

In order to be able to handle wide angle propagation in the general variable coefficient case and to easily treat rigid boundary conditions, we choose the three-dimensional parabolic wave equation as the representative equation. The main contribution of this paper is a method for efficiently solving this equation for wide angle propagations. A solution technique for the wide angle 3-D equation has been previously developed by Schultz, Lee, and Jackson [4] using the Crank-Nicolson scheme in conjunction with a preconditioned conjugate gradient method. Due to the resulting properties of the discretized finite difference equations, the operator is neither Hermitian nor positive real. Therefore no effective preconditioners are known for this case and the solution adopted by Schultz, Lee, and Jackson [4] was to precondition the normal equations. This squares the condition number of the initial matrix but gives satisfactory results as far as accuracy is concerned.

In this paper we propose a new method to solve the same 3-D wide angle parabolic approximation. What makes our technique so attractive as compared to other techniques is that each integration step requires solving only two successive tridiagonal systems. The method resembles alternating direction schemes but its foundation and analysis are different. The main goal of this paper is to derive this method and to discuss its validity and accuracy. The theory is then verified by performing two numerical tests, an azimuthally independent case and azimuthally dependent one. The first example is for testing the accuracy of the method and for determining how wide an angle it can accomodate. The second example is for verifying whether angular dependencies are well handled and for comparing the speed of our method with the speed of other methods. This last test shows an example where the new method is orders of magnitude faster than existing competing techniques.

2. Background

The standard wide angle 3-D wave equation was developed using the classical formulation of the Helmholtz equation in three dimensions, in cylindrical coordinates (r, θ, z):

$$\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} + k_0^2 n^2 p = 0. \tag{2.1}$$

In the above equation p represents the acoustic pressure, $k_0 = \omega/c_0$, where c_0 is a reference sound speed, $\omega = 2\pi f$, in which f is the frequency of the signal and finally $n = n(r, \theta, z) = c_0/c(r, \theta, z)$ is the index of refraction, in which $c(r, \theta, z)$ is the sound speed.

We make a standard transformation of the above equation by writing the pressure in the form [7]:

$$p(r, \theta, z) = u(r, \theta, z)v(r)$$

where the factor $v(r)$ represents a rapidly varying portion of the pressure and $u(r, \theta, z)$ is its modulation, a slowly varying function with respect to range. After neglecting small terms, making use of the far-field approximation ($k_0 r >> 1$), and rearranging the above equation we obtain:

$$\frac{\partial^2 u}{\partial r^2} + 2ik_0\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} + (n^2 - 1)k_0^2 u = 0. \tag{2.2}$$

This new equation has been at the origin of the very successful small angle parabolic approximation technique, which consists in simply dropping the second order derivative with respect to r and integrating the resulting parabolic equation.

It is convenient to define the operators:

$$X = \frac{1}{k_0^2}\frac{\partial^2}{\partial z^2} + (n^2 - 1), \tag{2.3}$$

$$Y = \frac{1}{k_0^2 r^2}\frac{\partial^2}{\partial \theta^2}, \tag{2.4}$$

after which the above equation reads as follows:

$$\frac{\partial^2 u}{\partial r^2} + 2ik_0\frac{\partial u}{\partial r} + k_0^2\,(X + Y)\,u = 0. \tag{2.5}$$

The standard wide angle PE technique [5] starts by approximately factoring the above operator as the product

$$\left[\frac{\partial}{\partial r} + ik_0 - ik_0 Q\right]\left[\frac{\partial}{\partial r} + ik_0 + ik_0 Q\right], \tag{2.6}$$

in which

$$Q \equiv \sqrt{1 + X + Y}. \tag{2.7}$$

Then the operator Q is approximated by a rational function in the form

$$Q \approx \frac{1 + p_1 X + p_2 Y}{1 + q_1 X + q_2 Y} \tag{2.8}$$

which yields the wide angle 'parabolic' equation

$$u_r = \left(-ik_0 + ik_0 \frac{1 + p_1 X + p_2 Y}{1 + q_1 X + q_2 Y}\right) u \tag{2.9}$$

To solve (2.9) Schultz, Lee and Jackson [4] applied the Crank-Nicolson scheme in conjunction with a preconditioned conjugate gradient method. The application of Crank-Nicolson reduces (2.9) to a sequence of systems of difference equations of the form

$$(I - \frac{1}{2}\Delta r L)u^{n+1} = (I + \frac{1}{2}\Delta r L)u^n, \tag{2.10}$$

where

$$L \equiv -ik_0 + ik_0 \frac{1 + p_1 X + p_2 Y}{1 + q_1 X + q_2 Y}. \tag{2.11}$$

By multiplying both members of (2.10) by the denominator of (2.11), we obtain a marching process, in which a large block-tridiagonal linear system must be solved at each step.

3. The new approach

Our approach starts with the equation (2.5) which is formally considered as an ordinary differential equation with respect to the variable r. For convenience the variables z and θ will therefore be dropped out in the remainder of the paper: $u(r)$ stands for $u(r, z, \theta)$. Locally, its formal solution has the form

$$u(r + \Delta r) = e^{-ik_0 \Delta r} e^{ik_0 \Delta r \sqrt{1+X+Y}} u^+(r) + e^{-ik_0 \Delta r} e^{-ik_0 \Delta r \sqrt{1+X+Y}} u^-(r)$$

where $u^+(r)$ and $u^-(r)$ are some initial conditions at the range r. The first term in the above solution is the outgoing wave and the second is the incoming wave. In this paper we will neglect back-scattering and therefore the second term will be dropped to yield the local solution

$$u(r + \Delta r) = e^{-\delta} e^{\delta\sqrt{1+X+Y}} u(r) \tag{3.1}$$

in which we have set

$$\delta \equiv ik_0 \Delta r.$$

Note that this is also a local solution of the one way wave equation

$$u_r = (-ik_0 + ik_0\sqrt{1 + X + Y})u$$

which is obtained by neglecting the second factor in (2.6).

The approach taken in this paper consists of approximating the term $e^{\delta\sqrt{1+X+Y}}$ in a convenient and accurate manner. An easy way in which this can be done is to use the approximation

$$\sqrt{1 + X + Y} \approx 1 + \frac{1}{2}(X + Y)$$

which yields the standard three-dimensional narrow angle parabolic equation:

$$u_r = \left(\frac{1}{2}ik_0(n^2 - 1) + \frac{i}{2k_0}\frac{\partial^2}{\partial z^2} + \frac{i}{2k_0 r^2}\frac{\partial^2}{\partial \theta^2}\right) u \equiv Lu.$$

This equation was solved by Baer and Perkins [1, 3] using a split-step Fourier algorithm. However this equation accurately represents only narrow angle propagation.

To accomodate wide angle propagation, we consider the higher order approximation

$$\sqrt{1 + X + Y} \approx 1 + \frac{1}{2}X - \frac{1}{8}X^2 + \frac{1}{2}Y. \tag{3.2}$$

The corresponding approximation to the wave equation becomes

$$u_r = \left(-ik_0 + ik_0\left[1 + \frac{1}{2}X - \frac{1}{8}X^2 + \frac{1}{2}Y\right]\right) u, \tag{3.3}$$

and formula (3.1) becomes:

$$u(r + \Delta r) = e^{-\delta} e^{\delta(1+\frac{1}{2}X-\frac{1}{8}X^2+\frac{1}{2}Y)} u(r) \tag{3.4}$$

Assuming that $n(r, \theta, z)$ varies slowly with respect to θ, the operators X and Y are nearly commutative, and equation (3.3) yields

$$u(r + \Delta r) = e^{-\delta} e^{\delta(1+\frac{1}{2}X-\frac{1}{8}X^2)} e^{\frac{\delta}{2}Y} u(r) \tag{3.5}$$

In the following we seek an approximation of the term

$$G(\delta, X) = e^{\delta(1+\frac{1}{2}X-\frac{1}{8}X^2)} \tag{3.6}$$

Here, the function G and its approximations should be regarded as functions of the real variable X while δ is an independent parameter.

The Taylor series expansion of (3.6) about $X = 0$ is

$$G(\delta, X) = e^{\delta} \left[1 + \frac{\delta}{2} X + \frac{1}{2!} \left(\frac{\delta^2}{4} - \frac{\delta}{4} \right) X^2 \right] + O(X^3). \tag{3.7}$$

We seek an approximation to the function G in the form

$$G(\delta, X) \approx e^{\delta} \frac{1 + pX}{1 + \bar{p}X}, \tag{3.8}$$

where p is a complex number to be determined.

Writing that the expansion (3.7) is equal to the right hand side of (3.8) up to $O(X^3)$ we get the equation

$$1 + \frac{\delta}{2} X + \frac{1}{2!} \left(\frac{\delta^2}{4} - \frac{\delta}{4} \right) X^2 = \frac{1 + pX}{1 + \bar{p}X}$$

from which it is easy to obtain p:

$$p = \frac{1}{4} + \frac{\delta}{4}$$

A similar development for the term

$$H(\delta, Y) = e^{\frac{\delta}{2}Y}$$

leads to the approximation

$$H(\delta, Y) \approx \frac{1 + qY}{1 + \bar{q}Y}, \tag{3.9}$$

with

$$q = \frac{\delta}{4}.$$

Therefore we get the final expression

$$u(r+\Delta r) = \left[\frac{1+(\frac{1}{4}+\frac{\delta}{4})X}{1+(\frac{1}{4}-\frac{\delta}{4})X}\right]\left[\frac{1+\frac{\delta}{4}Y}{1-\frac{\delta}{4}Y}\right]u(r). \tag{3.10}$$

This can be rewritten as

$$u(r+\Delta r) = L\, u(r)$$

where

$$L = L_X^{-*}L_X L_Y^{-*}L_Y, \tag{3.11}$$

in which

$$L_X = 1+\left(\frac{1}{4}+\frac{\delta}{4}\right)X, \quad L_Y = 1+\frac{\delta}{4}Y, \tag{3.12}$$

and where X^{-*} stands for the inverse of X^*, the adjoint of X. Equation (3.10) can formally regarded as an explicit marching scheme. It can easily be seen that the two operators in the denominator of (3.10) are nonsingular because δ is purely imaginary and X and Y are both self-adjoint. In the next section we analyze the accuracy and stability of this scheme and in Section 5, we will see how to discretize it and use it numerically.

4. Theoretical aspects

4.1. Stability

The operators X and Y defined earlier are self-adjoint and therefore their corresponding eigenvalues are real. Since the numerator and denominator of each term between brackets in equation (3.10) are the conjugate of each other, a modal expansion of (3.10) shows that with respect to each mode, the error induced by the marching scheme will not increase exponentially, i.e., the scheme (3.10) is stable. After discretization, the eigenvalues of X and Y will remain real provided boundary conditions are properly handled and the discretizations in the numerators and denominators are calculated at the same range r (see next section). Under these conditions the scheme is stable. Note that the second part of (3.10) is the usual Crank-Nicolson approximation applied here to the term $e^{(\delta/2)Y}$.

4.2. Accuracy

To analyze the local error of the integration scheme (3.10) we must attempt to find an estimate of the difference between the operator

$$e^{-\delta}e^{\delta\sqrt{1+X+Y}} \tag{4.1}$$

and the operator in the right hand side of (3.10). In the following we consider that X and Y are two independent real variables. On the one hand, we find after some calculation that the second order Taylor expansion of the operator (4.1) is

$$e^{-\delta}e^{\delta\sqrt{1+X+Y}} = 1 + \frac{\delta}{2}(X+Y) + \frac{\delta}{8}(\delta-1)\left[X^2 + 2XY + Y^2\right] + O(\|(X,Y)\|^3) \quad (4.2)$$

On the other hand, the second order Taylor expansion of the operator of the right hand side of (3.10) is given by

$$\left[\frac{1+(\frac{1}{4}+\frac{\delta}{4})X}{1+(\frac{1}{4}-\frac{\delta}{4})X}\right]\left[\frac{1+\frac{\delta}{4}Y}{1-\frac{\delta}{4}Y}\right] = \left[1+\frac{\delta}{2}X+\frac{\delta}{8}(\delta-1)X^2+....\right]\left[1+\frac{\delta}{2}Y+\frac{\delta^2}{8}Y^2+....\right]$$

$$= 1 + \frac{\delta}{2}(X+Y) + \frac{\delta^2}{4}XY + \frac{\delta}{8}(\delta-1)X^2 + \frac{\delta^2}{8}Y^2 + ... \quad (4.3)$$

The larger terms in the difference between the two (4.2) and (4.3) are

$$e^{-\delta}e^{\delta\sqrt{1+X+Y}} - \left[\frac{1+(\frac{1}{4}+\frac{\delta}{4})X}{1+(\frac{1}{4}-\frac{\delta}{4})X}\right]\left[\frac{1+\frac{\delta}{4}Y}{1-\frac{\delta}{4}Y}\right] = -\frac{\delta}{4}XY - \frac{\delta}{8}Y^2 + O(\|(X,Y)\|^3).$$

Thus, one can expect good accuracy when Y is very small and X is small. The above error is better than an error of the form $O(X^2)$, because the error expression is the product of three small terms namely δ, Y, and X. Moreover, we have assumed that the term Y is much smaller than the term X.

When using a marching scheme, an upper bound for the global error at some point can be derived from the above local truncation error by expressing the error

$$e_h^{j+1} \equiv u_h^{j+1} - u^{j+1}.$$

where u_h^{j+1} represents the computed solution at step j and u^{j+1} the exact solution at the same point. On the one hand, we have

$$u_h^{j+1} = L_h u_h^j$$

where L_h is the discretization of the operator L as defined by (3.11). On the other hand

$$u^{j+1} = L_h u^j + \epsilon_j$$

where ϵ_j is the truncation error incurred at step j. Hence,

$$e_h^{j+1} = L_h e_h^j + \epsilon_j$$

and since the operator L_h is unitary, we have

$$\|e_h^{j+1}\| \leq \sum_{i=0}^{j} \|\epsilon_j\|.$$

Since the number of steps in range is $0(\delta^{-1})$, the above analysis shows that the norm of the global error $\|e_h^{j+1}\|$ is $0(X^2)$.

5. Finite Difference Solution

To employ formula (3.10) numerically, we must discretize the operators X and Y by central differences and replace the corresponding operators by their discrete analogues. However, we first put equation (3.10) in the form

$$u^{j+1} = \left[1 + (\frac{1}{4} - \frac{\delta}{4})X\right]^{-1} \left[1 + (\frac{1}{4} + \frac{\delta}{4})X\right] \left[1 - \frac{\delta}{4}Y\right]^{-1} \left[1 + \frac{\delta}{4}Y\right] u^j. \tag{5.1}$$

There are several way of rewriting (5.1) exploiting the commutativity of the operators L_X and L_X^{-*} and of L_Y and L_Y^{-*}. The near commutativity of X and Y can also be exploited to derive alternative formulae. For example we may consider the scheme

$$\left[1 + (\frac{1}{4} - \frac{\delta}{4})X\right] \left[1 - \frac{\delta}{4}Y\right] u^{j+1} = \left[1 + (\frac{1}{4} + \frac{\delta}{4})X\right] \left[1 + \frac{\delta}{4}Y\right] u^j. \tag{5.2}$$

Although not obvious at first, the above scheme is also unconditionally stable. The reason for this is that the corresponding operator $\tilde{L} = L_Y^{-*} L_X^{-*} L_X L_Y$ is also unitary as is readily seen by forming $\tilde{L}\tilde{L}^*$ which is found to be the identity operator. The question as to which of the various schemes is to be preferred is certainly worth further investigation but we will not pursue it in the present paper.

Let us denote by A the finite difference approximation of the operator $L_X = I + (\frac{1}{4} - \frac{\delta}{4})X$ and by B that of $L_Y = I + \frac{\delta}{4}Y$. Both matrices are tridiagonal with the structure indicated in Table 1.

	Super diagonal	Diagonal	Sub diagonal
A	$(\frac{1}{4}-\frac{\delta}{4})\frac{1}{k_0^2h^2}$	$1+(\frac{1}{4}-\frac{\delta}{4})(n^2-1)-2(\frac{1}{4}-\frac{\delta}{4})\frac{1}{k_0^2h^2}$	$(\frac{1}{4}-\frac{\delta}{4})\frac{1}{k_0^2h^2}$
B	$-\frac{\delta}{4}\frac{1}{k_0^2r^2}\frac{1}{(\Delta\theta)^2}$	$1+\frac{\delta}{2}\frac{1}{k_0^2r^2}\frac{1}{(\Delta\theta)^2}$	$-\frac{\delta}{4}\frac{1}{k_0^2r^2}\frac{1}{(\Delta\theta)^2}$

Table 1: Coefficients of the two tridiagonal matrices A and B

When solving tridiagonal systems with the matrices A and B it is of interest to know whether these matrices are diagonally dominant or not. While the matrix B is always diagonally dominant, the situation is more complicated for A. In the simple case where $n(r,\theta,z) \equiv 1$, the matrix is conditionally, i.e., for $h \geq 1/k_o$, diagonally dominant in the sense that the modulus of the diagonal term is greater than or equal to the sum of the modulii of the off-diagonal terms in the same row. The more general case where n is arbitrary is not easy to analyze.

The scheme (5.1) becomes

$$u^{j+1} = A^{-*}AB^{-*}Bu^j. \tag{5.3}$$

Note that we evaluate the matrices A and B at mid distance between u^{j+1} and u^j, i.e., at range $r+\Delta r/2$. This is in order to ensure that the operators A^* and A, as well as B^* and B, form two pairs of operators that are conjugate of each other. This choice will guarantee stability as was seen in Section 4.1

To perform one step of (5.3) we must start by computing $w^j = Bu^j$ and solve the tridiagonal system

$$B^*v^{j+1} = w^j.$$

Then we compute $w^j = A^{-1}A^*u^{j+1}$ and solve the tridiagonal system

$$A^*u^{j+1} = Aw^j. \tag{5.4}$$

Thus there are two multiplications of a tridigonal system by a vector and two tridiagonal systems to solve at every step.

6. Test examples

The numerical scheme (5.3) has been implemented into a research computer code which is used to predict wave fields at required ranges. We used the marching scheme (5.2) instead of the original scheme (5.1). The resulting values are compared against a known solution to check the validity as well as the accuracy of our scheme. All computations are made on the VAX-11/780 computer using single precision complex arithmetic. We present two examples. The first one is an azimuthally independent case and the second is an azimuthally dependent one. The input parameters for both examples are shown in Table 2 for convenience.

6.1. An azimuthally independent case

To start the computation, the initial field is taken from the following formula borrowed from [6]:

$$u(r,z) = \frac{i}{2z_h} \sum_{j=0}^{\infty} sin\left(kz_0\sqrt{1-a_j^2}\right) sin\left(kz\sqrt{1-a_j^2}\right) H_0^{(1)}(ka_j r). \tag{6.1}$$

where a_j satisfies

$$a_j = \sqrt{1-\left(\frac{(j+1/2)\pi}{kz_h}\right)^2}. \tag{6.2}$$

We are concerned with the propagating modes namely those for which a_j remains real.

In the computation the sector is divided into 10 portions. The two extreme sectors represent sector boundaries where the solution is supplied from the exact solution. The inner sectors, because of the azimuthal independence have the same initial values. It is expected that at different sectors at the same depth, the computed wave fields should be identical. We examined this hypothesis. Another verification we did was to look at the size of the angle of propagation. To simulate this we took the mode index j to be 9 so that we could obtain an angle of propagation of around 52^o. Our method could handle such a wide angle without any major difficulty.

Our results are summarized in the two tables 3 and 4. The first table shows the accuracy achieved and the second shows the angle of propagation for the case of 8 modes and 9 modes. In the tables the values appearing in the first row represent the calculated

Input parameters	Problem 1	Problem 2
Source	300 m	10 m
Initial range	100 m	10 m
Source frequency	20 Hz	20 Hz
Bottom depth	400 m	20 m
Sound speed	1500 m/s	1500 m
Reference sound speed	1500 m/s	1500 m/s
Receiver depths	156 m, 312 m	5 m
Propagating sector	$-2\frac{1}{2}, +2\frac{1}{2}$	-5, 5
Depth increment	4 m	0.2 m
Range step size	1 m	0.001 m, 0.25 m
Angular increment	0.5^o	1^o
Maximum range	1 km	10.5 m
Surface condition	pressure release	Dirichlet
Bottom condition	Rigid	Dirichlet
Size of matrices A and B	1000	1000

Table 2: Parameters for the two test problems.

$z_r(m)$ \ θ	$-\frac{1}{2}$ degree	$+\frac{1}{2}$ degree
156	-0.59261E-04 0.20995E-04 -0.59224E-04 0.20954E-04	-0.59259E-04 0.20995E-04 -0.59224E-04 0.20954E-04
312	-0.96924E-04 0.34283E-04 -0.96908E-04 0.34287E-04	-0.96924E-04 0.34285E-04 -0.96908E-04 0.34287E-04

Table 3: Wave field results at 1 km range: accuracy test.

values by our new method, the values in the second row are the exact solutions. The first columns are the real parts and the second columns are the imaginary parts.

6.2. An azimuthally dependent case

This second example deals with a low frequency propagation in shallow water. To construct an azimuthally dependent case, we modified a reference solution tested by Chan, Shen and Lee [2] and used the same exact input parameters to derive a system of equations with the same dimension in order to compare the computation speed. An exact solution, after the modification, to the wide angle 3-dimensional wave equation (3.3) can

Mode j	Angle size (Degrees)	Results Complex values	Results dB
6	31.03^o	(0.13749E-04,-0.23722 E-05) (0.15624E-04,-0.12743 E-05)	97.107 96.095
7	37.54^o	(-0.96649E-05, 0.25102 E-05) (-0.68974E-05, 0.23300 E-05)	100.013 102.757

Table 4: Wave field results at 1 km range: angle of propagation measurements.

be expressed by

$$u(r,\theta,z) = e^{-\Omega z} e^{im\theta} e^{im^2/(2k_0 r)} \tag{6.3}$$

The expression is used to generate the initial field. The input parameters used are shown in the second column of Table 2. The surface condition is taken to be

$$u(r,\theta,0) = e^{im\theta} e^{im^2/(2k_0 r)} \tag{6.4}$$

and the bottom condition was

$$u(r,\theta,z_{max}) = e^{-\Omega z_{max}} e^{im\theta} e^{im^2/(2k_0 r)} \tag{6.5}$$

The scalar Ω in Equation (6.5) is chosen to be $2k_0$. The angular modal number m is taken to be 3. To obtain an accuracy of 10^{-2}, as in [2], their methods need to take a range step size of 0.001 m.

We tested two different step sizes: 0.001 m and 0.25 m. The experiment with the first range step-size is only done for a comparison with reference [2]. We should point out that such a small step size for the 5-point method of Chan, Shen and Lee is necessary because the scheme is explicit. In this computation, we found that our method was approximately 1.6 times faster than the 5-point explicit method of [2], and 17 times faster than Crank-Nicholson of [4]. Note that the Crank-Nicholson of [4] uses a stable version of the conjugate gradient method applied to the normal equations, called Craig's method. Using the second

Method	Δr	Relative error	CPU time (h-m-s)
Crank Nicolson 5-Point Explicit New method	0.001 0.001 0.001	(0.18E-01,-0.12E-01) (0.10E-01,-0.11E-01) (0.26E-02,-0.11E-02)	03-47-10 00-21-35 00-13-12
New method	0.25	(0.22E-02,-0.11E-01)	00-00-09

Table 5: Results for Problem 2.

step size of 0.25 m, we found that the same accuracy could be achieved by our method as with $\Delta r = 0.001$ but the excution was much faster. Here, our method is approximately 160 times faster than the 5-point method and 1600 times faster than Crank-Nicolson. Results are displayed in Table 5.

7. Conclusion

Obtaining solutions to ocean acoustic propagation in three dimensions can be very complicated and computationally expensive. Moreover, it is now becoming the general consensus that two-dimensional models are no longer sufficiently representative. Efficient methods and clever implementations for dealing with three-dimensonal wave propagation are therefore very important.

The new method proposed in this paper is not only a fast and accurate method, but also has the property of being as representative a model as other well known existing 3-D models. Our numerical results have demonstrated that the method is efficient and have confirmed the theory that it is also stable.

Our new approach is based on considering a form of the 3-D wave equation as an ordinary differential equation with respect to range. Then a formal expression of the solution is written in terms of the exponential of the square-root of some operator. The artifice used in this paper is to approximate this exponential in a clever way by the product of two rational functions of the type $(1,1)$. As a consequence the resulting ODE-integration process, requires only two successive tridiagonal system solutions. The theory shows that our method is unconditionally stable. Moreover, it is so accurate that larger step-sizes can be afforded resulting in substantial savings in computational times. This has been widely confirmed by the numerical tests. Moreover, angles of propagation as wide as 31 dgrees have been accurately handled.

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